Dynamic Programming

Application: Stochastic Growth

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July 8th, 2024

Introduction

We begin our exploration of applications of dynamic programming problems in macroeconomics with the stochastic growth model. The stochastic growth model provides our first opportunity to review the techniques of dynamic programming, numerical methods and estimation methodology. We begin with a review of the nonstochastic model to get some basic concepts straight, and then we enrich the model to include shocks and other relevant features.

Nonstochastic Growth Model

- Consider the dynamic optimization problem of a very special household.
- This household is endowed with one unit of leisure each period and supplies this inelastically to a production process.
- The household consumes an amount c_t each period that it evaluates using a utility function $u()$. Assume that $u()$ is strictly increasing and strictly concave.
- The household's lifetime utility is given by

$$\sum_{1}^{\infty} \beta^{t-1} u(c_t)$$

- The household has access to a technology that produces output y from capital k, given its inelastically supplied labor services. Let $y = f(k)$ be the production function. Assume that $f(k)$ is strictly increasing and strictly concave.

Nonstochastic Growth Model

- The household faces a resource constraint that decomposes output into consumption and investment i_t:
$$f(k_t) = y_t = c_t + i_t$$
- The capital stock accumulates according to
$$k_{t+1} = (1-\delta)k_t + i_t$$
- Essentially the household's problem is to determine an optimal savings plan by splitting output between these two competing uses.
- we use the dynamic programming approach and consider the following functional equation:
$$V(k) = max_{k'} u(f(k) + (1-\delta)k - k') + \beta V(k')$$
for all k.

Nonstochastic Growth Model

- With $f(k)$ strictly concave, there will exist a maximal level of capital achievable by this economy given by $\bar{k}$ where

$$\bar{k} = (1-\delta)\bar{k} + f(\bar{k})$$

- This provides a bound on the capital stock for this economy and thus guarantees that our objective function, $u(c)$, is bounded on the set of feasible consumption levels, $[0, f(\bar{k}) + (1-\delta)\bar{k}]$.
- The first order condition is given by

$$u'(c) = \beta V'(k')$$

- The B-S condition is given by

$$V'(k) = u'(c)(f'(k) + (1-\delta))$$

- The Euler equation is given by

$$u'(c) = \beta u'(c')(f'(k') + 1 - \delta)$$

Nonstochastic Growth Model: An Analytical Example

- Suppose $u(c) = ln(c), f(k) = k^{\alpha}, \delta = 1$.
- We guess a value function $V(k) = A + Bln(k)$ for all k.
- We can solve for $B = \frac{\alpha}{1-\beta\alpha}$ and A.
- We then solve for the policy function $k' = \beta\alpha k^{\alpha}$ and $c = (1-\beta\alpha)k^{\alpha}$.

Stochastic Growth Model

- Fluctuations in the economy are created by shocks to the process of producing goods.
- Thus "good times" represent higher productivity of both labor and capital inputs.
- The production function is expressed as

$$Y_t = A_t F(K_t, N_t)$$

- We assume that labor is inelastically supplied at one unit per household. In this case we use the constant returns to scale assumption on $F(K, N)$ to write per capita output y_t as a strictly concave function of the per capita capital stock k_t:

$$y_t = A_t F(\frac{K_t}{N}, 1) = A_t f(k_t)$$

- Bellman's equation for the infinite horizon stochastic growth model is specified as

$$V(A, k) = max_{k'} u(Af(k) + (1 - \delta)k - k') + \beta E_{A'|A} V(A', k')$$

for all (A,k).

Stochastic Growth Model

- For the growth model it is important to be sure that the problem is bounded. For this, let $\bar{k}$ solve

  $$k = A^{+} f(k) + (1 - \delta)k$$

  where A^{+} is the largest productivity shock.
- Further we know that there is a policy function given by $k' = \phi(A, k)$.
- We guess the policy function is $\phi(A, k) = \lambda A k^{\alpha}$ and can solve for λ using the first order conditions, $\lambda = \beta\alpha$.

A Stochastic Growth Model with Endogenous Labor Supply

- The planner's problem

$$V(A,k) = max_{k',n} u(Af(k,n) + (1-\delta)k - k', 1-n) + \beta E_{A'|A} V(A',k')$$

  for all (A,k).
- Here the variables are measured in per capita terms: k and n are the capital and labor inputs per capita.
- In addition given k', the problem has a "static" choice of n. This distinction is important when we turn to a discussion of programming the solution to this functional equation.

A Stochastic Growth Model with Endogenous Labor Supply

- For given (A, k, k'), define $\sigma(A, k, k')$ from

$$\sigma(A, k, k') = max_n u(Af(k, n) + (1 - \delta)k - k', 1 - n)$$

  and let $n = \hat{\phi}(A, k, k')$ denote the solution to the optimization problem.

- The first-order condition for this problem is given by

$$u_c(c, 1 - n)Af_n(k, n) = u_l(c, 1 - n)$$

  .

- Thus, for the current productivity shock and current capital stock and for a level of future capital, $n = \hat{\phi}(A, k, k')$ characterizes the employment decision. We can think of $\sigma(A, k, k')$ as a return function given the current state (A, k) and control k'.

A Stochastic Growth Model with Endogenous Labor Supply

- The return function from this choice of the labor input can then be used to rewrite the functional equation as

$$V(A, k) = max_{k'} \sigma(A, k, k') + \beta E_{A'|A} V(A', k')$$

for all (A, k).

- This has the same structure as the stochastic growth model with a fixed labor supply, though the return function, $\sigma(A, k, k')$, is not a primitive object.
- The Euler equation

$$-\sigma_{k'}(A, k, k') = \beta E_{A'|A} \sigma_{k'}(A', k', k'')$$

- Using the labor decision problem, we can rewrite the Euler equation as

$$u_c(c, 1 - n) = \beta E_{A'|A}[u_c(c', 1 - n')(A' f_k(k', n') + 1 - \delta)]$$

Numerical Analysis

- The program should be structured to focus on solving $V(A,k) = max_{k'}\sigma(A,k,k') + \beta E_{A'|A}V(A',k')$ through the value function iteration.
- The problem is that the return function is derived and thus must be solved for inside of the program.
- The researcher can obtain an approximate solution to the employment policy function, given previously as $\hat{\phi}(A,k,k')$. This is achieved by specifying grids for the shocks, the capital state space and the employment space.
- As noted earlier, this is the point of approximation in the value function iteration routine: finer grids yield better approximations but are costly in terms of computer time.

Numerical Analysis

- Once $\hat{\phi}(A, k, k')$ is obtained, then

$$\sigma(A, k, k') = u(Af(k, \hat{\phi}(A, k, k')) + (1 - \delta)k - k', 1 - \hat{\phi}(A, k, k'))$$

  can be calculated and stored. This should all be done prior to starting the value function iteration phase of the program.
- The output of the program is then the policy function for capital accumulation, $k' = h(A, k)$, and a policy function for employment, $n = \phi(A, k)$, where $\phi(A, k) = \hat{\phi}(A, k, h(A, k))$.
- Hence both of these policy functions ultimately depend only on the state variables, (A, k).

Confronting the Data: Calibration

- The parameter vector is $\Phi = (\alpha, \delta, \beta, \xi, \rho, \sigma)$. Calibration is a way to matching moments.

Observed and predicted moments

Moments	U.S. data	KPR calibrated model
Std relative to output		
Consumption	0.69	0.64
Investment	1.35	2.31
Hours	0.52	0.48
Wages	1.14	0.69
Cross correlation with output		
Consumption	0.85	0.82
Investment	0.60	0.92
Hours	0.07	0.79
Wages	0.76	0.90

Some Extensions

- Technological Complementarities $y = Ak^{\alpha} n^{\phi} Y^{\gamma} Y_{-1}^{\varepsilon}$
- Multiple Sectors $y_j = A^j f(k^j, n^j)$
- Taste Shocks $V(A, S, k) = max_{k', n} u(c, 1 - n, S) + \beta E_{A', S' | A, S} V(A', S', k')$
- Taxes $c_t + i_t = (1 - \tau_t^k) r_t k_t + (1 - \tau_t^n) w_t n_t + \delta \tau_t^k k_t + T_t$

Conclusion

This lecture has provided an application of stochastic growth model.

Slides can be found at www.rongli.cc