Dynamic Programming

Application: Consumption

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Introduction

From the perspective of business cycle theory, consumption is the largest component of total expenditures. One of the main aspects of consumption theory is the theme of consumption smoothing. This is evident in the data as the consumption of nondurables and services is not as volatile as income. In the GDP accounts durable expenditures is one of the relatively more volatile elements. Our theories and estimated models must confront these important facts.

Two-period Problem

- The consumer maximizes the discount present value of consumption over the two-period horizon. Assuming that preferences are separable across periods, we represent lifetime utility as:

$$\sum_{t=0}^{1} \beta^t u(c_t) = u(c_0) + \beta u(c_1)$$

- The consumer is endowed with some initial wealth at the start of period 0 and earns exogenous income y_t in period $t = 0, 1$.
- We assume that the agent can freely borrow and lend at a fixed interest rate between each of the two periods of life. The budget contraints:

$$a_1 = r_0(a_0 + y_0 - c_0) \text{ and } a_2 = r_1(a_1 + y_1 - c_1)$$

  a_t is the agent's wealth at the start of period t. r_t represents the gross return on wealth between period t and period t+1. We restrict consumption to be nonnegative and the stock of assets remaining at the end of the consumer's life (a_2) must be nonnegative.

Two-period Problem

- Combine the two budget contraints for each period, we get a lifetime budget contraint:
$$\frac{a_2}{r_1 r_0} + \frac{c_1}{r_0} + c_0 = (a_0 + y_0) + \frac{y_1}{r_0}$$
- The FOC with respect to c_1, c_2 yields:
$$u'(c_0) = \lambda = \beta r_0 u'(c_1)$$
where λ is the multiplier on the lifetime budget constraint.
- The FOC with respect to a_2:
$$0 < \lambda = \phi$$
where ϕ is the multiplier on the nonnegativity constraint for a_2.
- $a_2 = 0$ and the lifetime budget constraint is:
$$\frac{c_1}{r_0} + c_0 = a_0 + y_0 + \frac{y_1}{r_0} = w_0$$
where w_0 is lifetime wealth for the agent in terms of period 0 goods.

Two-period Problem

- As an example, suppose utility is quadratic in consumption:

$$u(c) = a + bc - \frac{d}{2}c^2$$

- In this case the Euler condition simplifies to:

$$b - dc_0 = \beta r_0(b - dc_1)$$

- With the further simplification that $\beta r_0 = 1$, we have constant consumption: $c_0 = c_1$.

Stochastic Income

- We now add uncertainty to the problem by supposing that income in period 1 y_1 is not known to the consumer in period 0. Further we use the result of $a_2 = 0$ and rewrite the optimization problem more compactly as

$$max_{c_0} E_{y_1|y_0}[u(c_0) + \beta u(R_0(a_0 + y_0 - c_0) + y_1)]$$

  where we have substituted for c1 using the budget constraint.

- we assume that $y_1 = \rho y_0 + \varepsilon_1$, here ε_1 is a shock to income that is not forecastable using period 0 information.
- The Euler equation for this problem is given by

$$u'(c_0) = E_{y_1|y_0} \beta R_0 u'(R_0(a_0 + y_0 - c_0) + y_1)$$

Stochastic Income

- The special case of quadratic utility and $\beta R_0 = 1$, for this case the Euler equation simplifies to

$$c_0 = E_{y_1|y_0} c_1 = R_0(a_0 + y_0 - c_0) + E_{y_1|y_0} y_1$$

- Solving for c_0 yields

$$c_0 = \frac{R_0 a_0}{1 + R_0} + y_0 \frac{R_o + \rho}{1 + R_0}$$

- We have

$$\frac{\partial c_0}{\partial y_0} = \frac{R_0 + \rho}{1 + R_0}$$

- In the extreme case of iid income shocks $\rho = 0$, consumers will save a fraction of an income increase and consume the remainder.
- In the opposite extreme of permanent shocks $\rho = 1$, current consumption moves one for one with current income. The sensitivity of consumption to income variations depends on the permanence of those shocks.

Portfolio Choice

- A second extension of the two-period problem is the addition of multiple assets.
- Assume that the household has no initial wealth and can save current income through these two assets. One is nonstochastic and has a one period gross return of R^s. The second asset is risky with a return denoted by $\tilde{R}^r$ and a mean return of $\bar{R}^r$.
- Let a^r and a^s denote the consumer's holdings of asset type $j = r, s$. Assets' prices are normalized at 1 in period 0.
- The consumer's choice problem can then be written as

$$max_{a^r, a^s} u(y_0 - a^r - a^s) + E_{\tilde{R}^r} \beta u(\tilde{R}^r a^r + R^s a^s + y_1)$$

Portfolio Choice

- Here we make the simplifying assumption that y_1 is known with certainty. The first-order conditions are

$$u'(y_0 - a^r - a^s) = \beta R^s E_{\tilde{R}^r} u'(\tilde{R}^r a^r + R^s a^s + y_1)$$

and

$$u'(y_0 - a^r - a^s) = \beta E_{\tilde{R}^r} \tilde{R}^r u'(\tilde{R}^r a^r + R^s a^s + y_1)$$

- Combine the above two equations, we have

$$R^s = \bar{R}^r + \frac{cov[\tilde{R}^r, u'(\tilde{R}^r a^r + R^s a^s + y_1)]}{E_{\tilde{R}^r} u'(\tilde{R}^r a^r + R^s a^s + y_1)}$$

- If the agent holds both the riskless and the risky asset ($a^r > 0$ and $a^s > 0$), then the strict concavity of $u(c)$ implies that the covariance must be negative. In this case, $\bar{R}^r$ must exceed R^s: the agent must be compensated for holding the risky asset.
- If $\bar{R}^r$ is less than R^s, the agent will sell the risky asset and buy additional units of the riskless asset.

Borrowing Restrictions

- A final extension of the two-period model is to impose a restriction on the borrowing of agents.
- To illustrate, consider a very extreme constraint where the consumer is able to save but not to borrow: $c_0 \leq y_0$.
- Thus the optimization problem of the agent is

$$max_{c_0 \leq y_0}[u(c_0) + \beta u(R_0(y_0 - c_0) + y_1)]$$

- Denote the multiplier on the borrowing constraint by μ, the first order condition is given by

$$u'(c_0) = \beta R_0 u'(R_0(y_0 - c_0) + y_1) + \mu$$

Borrowing Restrictions

- If the constraint does not bind, then the consumer has nonnegative savings and the familiar Euler equation for the two-period problem holds. However, if $\mu > 0$, then $c_0 = y_0$ and

$$u'(y_0) > \beta R_0 u'(y_1)$$

  The borrowing constraint is less likely to bind if βR_0 is not very large and if y_0 is large relative to y_1.
- An important implication of the model with borrowing constraints is that consumption will depend on the timing of income receipts and not just w_0.

Infinite Horizon Formulation

- Consider a household with a stock of wealth denoted by A, a current flow of income y, and a return on its investments over the past period given by R_{-1}.
- The state vector of the consumer's problem is (A, y, R_{-1}), and the associated Bellman equation is

$$v(A, y, R_{-1}) = max_c u(c) + \beta E_{y',R|R_{-1},y} v(A', y', R)$$

  for all (A, y, R_{-1})
- The transition equation for wealth is given by

$$A' = R(A + y - c)$$

Infinite Horizon Formulation: Stochastic Income

- The case we study is (fixed R)

$$v(A, y) = max_c u(c) + \beta E_{y'|y} v(A', y')$$

where $A' = R(A + y - c)$ for all (A, y).

- The Euler equation is

$$u'(c) = \beta R E_{y'|y} u'(c')$$

The interpretation of this equation is that the marginal loss of reducing consumption is balanced by the discounted expected marginal utility from consuming the proceeds in the following period.

Infinite Horizon Formulation: Stochastic Income

- As a leading example, consider the specification of utility

$$u(c) = \frac{c^{1-\gamma} - 1}{1 - \gamma}$$

  This is called the constant relative risk aversion case (CRRA), since $-cu''(c)/u'(c) = \gamma$.
- The Euler equation can be written as

$$1 = \beta R E(\frac{c'}{c})^{-\gamma}$$

- This equation is then used to estimate the parameters of the utility function, (β, γ).

Stochastic Returns: Portfolio Choice

- Assume that there are N assets available. Let R_{-1} denote the N-vector of gross returns between the current and previous period and let A be the current stock of wealth.
- Let s_i denote the share of asset $i = 1, 2, ..., N$ held by the agent.
- Normalizing the price of each asset to be unity, the current consumption of the agent is then

$$c = A - \sum_i s_i$$

- Substituting this into the Bellman equation, we have

$$v(A, y, R_{-1}) = max_{s_i} u(A - \sum_i s_i) + \beta E_{y', R|R_{-1}, y} v(\sum_i R_i s_i, y', R)$$

where R_i is the stochastic return on asset i.

Stochastic Returns: Portfolio Choice

- The first-order condition for the optimization problem holds for $i = 1, 2, ..., N$, and it is
$$u'(c) = \beta E_{R,y'|R_{-1},y} R_i v_A(\sum_i R_i s_i, y', R)$$
- The Euler equation is
$$u'(c) = \beta E_{R,y'|R_{-1},y} R_i u'(c')$$
for $i = 1, 2, ..., N$
- This system of Euler equations forms the basis for financial models that link asset prices to consumption flows.

Empirical Implementation

- The starting point for the analysis is the Euler equation for the household's problem with N assets.
- We rewrite that first-order condition here using time subscripts to show the timing of decisions and realizations of random variables:

$$u'(c_t) = \beta E_t R_{it+1} u'(c_{t+1})$$

  for $i = 1, 2, ..., N$ where R_{it+1} is defined as the real return on asset i between periods t and t+1.
- The expectation here is conditional on all variables observed in period t. Unknown t+1 variables include the return on the assets as well as period t+1 income.

Empirical Implementation

- The power of the GMM approach derives from this first-order condition.
- What the theory tells us is that while ex post this first order condition need not hold, any deviations from it will be unpredictable given period t information.
- That is, the period t+1 realization say, of income, may lead the consumer to increase consumption is period t+1, thus implying ex post that the first order condition does not hold. This deviation is consistent with the theory as long as it is not predictable given period t information.
- Formally, define $\varepsilon^i_{t+1}(\theta)$ as

$$\varepsilon^i_{t+1}(\theta) = \frac{\beta R_{it+1} u'(c_{t+1})}{u'(c_t)} - 1$$

for $i = 1, 2, ..., N$. Thus $\varepsilon^i_{t+1}(\theta)$ is a measure of the deviation for an asset i. We have added θ as an argument in this error to highlight its dependence on the parameters describing the household's preferences.

Empirical Implementation

- Household optimization implies that

$$E_t(\varepsilon^i_{t+1}(\theta)) = 0$$

  for $i = 1, 2, ..., N$
- Let z_t be a q-vector of variables that are in the period t information set. This restriction on conditional expectations implies that

$$E(\varepsilon^i_{t+1}(\theta) \otimes z_t) = 0$$

  for $i = 1, 2, ..., N$ where $\otimes$ is the Kronecker product.
- So the theory implies the Euler equation errors from any of the N first-order conditions ought to be orthogonal to any of the z_t variables in the information set. There are Nq restrictions created.
- The idea of GMM estimation is then to find the vector of structural parameters θ such that the above equation holds.

Empirical Implementation

- Of course, applied economists only have access to a sample, say of length T.
- Let $m_T(\theta)$ be an Nq vector where the component relating asset i to one of the variables in z_t, z_t^j, is defined by

$$\frac{1}{T}\sum_{t=1}^{T}(\varepsilon_{t+1}^{i}(\theta)z_t^j).$$

- The GMM estimator is defined as the value of θ that minimizes

$$J_T(\theta) = m_T(\theta)' W_T m_T(\theta)$$

where W_T is an $Nq \times Nq$ matrix that is used to weight the various moment restrictions.

Conclusion

This lecture has provided an application of nondurable consumption model.

Slides can be found at www.rongli.cc