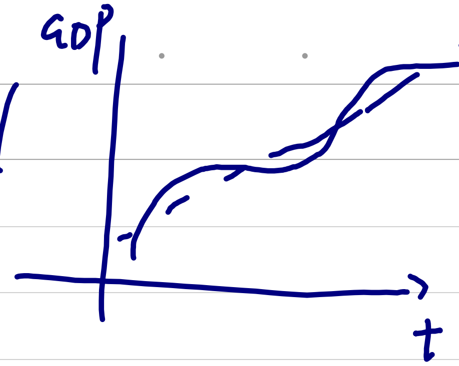
Stochastic Optimal $\underline{\text{Growth}}$ Model

↑ shocks

↓ { long-run. / short-term fluctuations }



+ labor-leisure trade-off ⇒ <u>Real</u> Business Cycle model (RBC)

$u(c, L)$

A standard for RBC

① general equilibrium model with well-defined micro-foundation.

② specification of shocks to explain both qualitatively and quantitatively the main features of macro-fluctuations

Process

①. set up model. ⇒ Economics

② Solve the model. { Dynamic Programming. / Dynamic Lagrangian }

③ steady-state $K_{t+1} = K_t = \bar{K}$ Comparative statics

④ linearized system ⇒ (log-linearization)

⑤ Solve policy function $\begin{cases} C_t = c(k_{t-1}, z_t) \\ k_t = k(k_{t-1}, z_t) \end{cases}$

Difference equations

* stability (Dynare)

* undetermined coefficient method. (Uhlig Toolkits)

⑥ Calibrations. variance/co-variance.

⇒ to match the stylized facts.

DSGE Extensions

① RBC + monetary factors (MIU, CIA)

cash-in-advance

money-in-the-utility

$(u(c_t, \frac{M_t}{P_t}, L_t)$

(real → nominal)

② RBC + monetary + $\begin{cases} \text{sticky price / wage} \\ \text{monopolistic competition} \end{cases}$

(New-Keynesian model)

③ . . ——— ——— ——— + open. economy

(Open New-Keynesian Model)

maths Tool.

log-linearization.

nonlinear

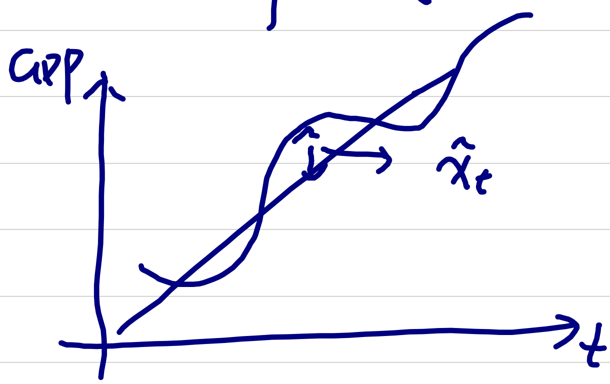
$$C_t^{-\sigma} = \beta C_{t+1}^{-\sigma}\left[r_{t+1} + 1 - \delta \right]$$

$\Rightarrow$ linear

$$\underline{\underline{-\sigma \hat{C}_t}} = -\sigma \hat{C}_{t+1} + \bar{\delta}\, \hat{r}_{t+1}$$

$$\underline{\underline{\hat{x}_t}} \equiv \log X_t - \log \bar{X}$$

$\bar{X}$: steady-state



Interpretation

$$\hat{x}_t \equiv \log \frac{X_t}{\bar{X}} = \log\left(1 + \frac{X_t - \bar{X}}{\bar{X}}\right) \approx \frac{X_t - \bar{X}}{\bar{X}} = \%\ \text{change}$$

$\Uparrow$

$$\left(Z_t \to 0, \quad Z_t \approx \log(1 + Z_t) \right)$$

$\hat{x}_t \times 100\%$. is approximately the percentage change. (diviation) w.r.t. the s.s.

Calculations (Taylor Expansion)

$$X_t = \bar{X}\left(\frac{X_t}{\bar{X}}\right) = \bar{X} e^{\log \frac{X_t}{\bar{X}}} = \underline{\bar{X} e^{\hat{x}_t}}$$

Taylor expansion around $\bar{X}$.

$$X_t = \bar{X} e^{\hat{x}_t} \approx \bar{X} e^0 + \bar{X} \cdot \left.\frac{\partial e^{\hat{x}_t}}{\partial \hat{x}_t}\right|_{\hat{x}_t = 0} (\hat{x}_t - 0)$$

$$= \bar{X}(1 + \hat{x}_t)$$

$$x_t \approx \bar{x}(1+\hat{x}_t) = \underset{\text{steady state}}{\bar{x}} + \underset{\text{deviation from s.s.}}{\bar{x}\cdot\hat{x}_t}$$

e.g.

$$X_t \cdot Y_t \approx \bar{x}(1+\hat{x}_t)\,\bar{Y}(1+\hat{Y}_t)$$

$$= \bar{x}\bar{Y}(1+\hat{x}_t+\hat{Y}_t + \underbrace{\hat{x}_t\cdot\hat{Y}_t}_{\approx 0})$$

$$\approx \bar{x}\bar{Y}(1+\hat{x}_t+\hat{Y}_t)$$

General formula

$$f(x_t) \approx f(\bar{x}) + f'(x_t)\Big|_{\bar{x}}(x_t - \bar{x})$$

$$\approx f(\bar{x}) + f'(x_t)\Big|_{\bar{x}}(\bar{x}(1+\hat{x}_t) - \bar{x})$$

$$= f(\bar{x}) + f'(x_t)\Big|_{\bar{x}}\,\bar{x}\,\hat{x}_t$$

$$= f(\bar{x})\cdot\left(1+\underbrace{\frac{f'(x)\big|_{\bar{x}}\cdot\bar{x}}{f(\bar{x})}}_{\eta_x}\cdot\hat{x}_t\right)$$

$$f(x_t) \approx f(\bar{x})\cdot(1+\eta_x\hat{x}_t) \qquad \eta_x = \frac{df(x)}{dx}\frac{\bar{x}}{f(\bar{x})}$$

$$f(x_t, y_t) \approx f(\bar{x},\bar{Y})(1+\eta_x\hat{x}_t + \eta_Y\hat{Y}_t)$$

$$\frac{X_t}{Y_t} = \frac{\bar{X}}{\bar{Y}} \cdot \left(1 + \frac{\frac{1}{\bar{Y}} \cdot \bar{X}}{\frac{\bar{X}}{\bar{Y}}} \hat{x}_t + \frac{-\frac{\bar{X}}{\bar{Y}^2} \cdot \bar{Y}}{\frac{\bar{X}}{\bar{Y}}} \cdot \hat{Y}_t \right)$$

$$= \frac{\bar{X}}{\bar{Y}} \left(1 + \hat{x}_t - \hat{Y}_t \right)$$

$$\underline{\underline{C_t^{-\sigma}}} = \bar{C}^{-\sigma} \left(1 + \frac{-\sigma \bar{C}^{-\sigma-1} \cdot \bar{C}}{\bar{C}^{-\sigma}} \hat{C}_t \right)$$

$$= \bar{C}^{-\sigma} \left(\underline{\underline{1 - \sigma \hat{C}_t}} \right)$$

New-Keynesian Model.

Common features New / Traditional Keynesian

* A framework allows demand to partially determine supply.

*. Phillips curve. generated by nominal rigidity (short-run output-inflation trade-off)

*. Inefficiency $\Leftarrow$ monopolistic competition which motive monetary policy.

unique to NK

*. based on Walrasian Equilibrium. (market clearing imposed)

*. Incorporate rational expectation. (forward-looking based on mathematical expectation)

Three agents → final goods → Price Index

① Household. C, L (Heterogeneous goods → intermediate goods)

② Firm { factor input. / price decision: } monopolistic competition → nominal rigidity { Calvo-type. / Rotemberg → adjustment cost. }

③ monetary authority
(Government, fiscal)

Four markets.

{ goods. Differentiated goods $j \in (0,1)$
labor. perfect competition / wage rigidities
bond.
money (cashless) }

Household problem

$$\max E_0 \sum_{t=0}^{\infty} \beta^t u(C_t, \underset{\text{labor}}{N_t})$$

Leisure: L_t

$L_t + N_t = \bar{L}$

$u_c > 0 \quad u_{cc} < 0 \quad u_N < 0 \quad u_{NN} > 0$

where C_t is consumption index given by.

$$C_t \equiv \left(\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}} \quad \text{(CES)}$$

$\varepsilon > 1$ - subs. elas.

Assumption: Two steps solution

$\begin{cases} \text{static fashion for } C_t(i) \\ \text{Dynamic choice for } C_t \end{cases}$

① Static. (Dixit & Stiglitz, 1977)

↳ Aggregation

$$\max_{C_t(i)} C_t \equiv \left(\int_0^1 C_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right)^{\frac{\varepsilon}{\varepsilon-1}}$$

s.t. $\int_0^1 P_t(i) C_t(i) \, di \equiv Z_t$ ← total nominal expenditure.

F.O.C.

$\Rightarrow \quad C_t(i) = C_t(j) \left(\frac{P_t(i)}{P_t(j)} \right)^{-\varepsilon}$ for any pair (i, j)

$\Rightarrow \quad C_t(i) = \left(\frac{P_t(i)}{P_t} \right)^{-\varepsilon} C_t$ iso-elastic demand for goods i

$P_t \equiv \left[\int_0^1 P_t(i)^{1-\varepsilon} di \right]^{\frac{1}{1-\varepsilon}}$: price index

Conclusion

$$P_t C_t = \int_0^1 P_t(i) \cdot C_t(i)\, di = Z_t$$

Price index × Consumption index = Total expenditure

Step 2. Intertemporal problem

$$\max\ E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t)$$

related to interest rate

$$\text{s.t. } \underbrace{\int_0^1 P_t(i) C_t(i)\, di}_{} + \underbrace{Q_t B_t}_{\text{Bond}} \le B_{t-1} + W_t N_t + \underset{\text{Profits.}}{D_t}$$

$$Q_t = \frac{1}{1+i_t} \qquad i_t.\ \text{interest rate.}$$

$$P_t \cdot C_t + Q_t B_t \le B_{t-1} + W_t N_t + D_t$$

$$\mathcal{L} = U(C_t, N_t) + \lambda_t \left[B_{t-1} + W_t N_t + D_t - P_t C_t - Q_t B_t \right]$$

$$+ E_t \beta \lambda_{t+1} \left[B_t + W_{t+1} N_{t+1} + D_{t+1} - P_{t+1} C_{t+1} - Q_{t+1} B_{t+1} \right]$$

F.O.C. $C_t,\ N_t,\ B_t$.

MRS C,N

real wage

C_t: $U_C = \lambda_t P_t$

N_t: $-U_N = \lambda_t W_t$

$$\Rightarrow \frac{-U_N}{U_C} = \frac{W_t}{P_t}$$

B_t: $-\lambda_t Q_t + \beta E_t \lambda_{t+1} = 0 \Rightarrow Q_t = \beta E_t \frac{\lambda_{t+1}}{\lambda_t}$

$$\Rightarrow Q_t = \beta E_t \left\{ \frac{U_c(C_{t+1})}{U_c(C_t)} \cdot \frac{P_t}{P_{t+1}} \right\}$$ Euler equation

(IS curve)

e.g. $$U(C_t, N_t) = \frac{C_t^{1-\sigma}}{1-\sigma} - \psi \frac{N_t^{1+\eta}}{1+\eta}$$

CRRA $(\sigma \neq 1 \quad \sigma > 0 \quad \eta > 0)$

↳ relative risk aversion $\equiv -\frac{u'}{u''\bar{C}} = \frac{1}{\sigma}$

$$Q_t = \beta E_t \left\{ \frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma}} \frac{P_t}{P_{t+1}} \right\}$$ ✓

log-linearization take log. on both sides.

$$\log Q_t = \log \beta - \sigma E_t \log C_{t+1} + \sigma \log C_t + \log P_t - E_t \log P_{t+1}$$

$$\Rightarrow \hat{q}_t = -\sigma E_t \hat{C}_{t+1} + \sigma \hat{C}_t + \underbrace{\hat{P}_t - E_t \hat{P}_{t+1}}_{-E_t \hat{\pi}_{t+1}}$$

$$\Rightarrow \hat{C}_t = E_t \hat{C}_{t+1} + \frac{1}{\sigma} [\hat{q}_t + E_t \hat{\pi}_{t+1}]$$

$$q_t = \frac{1}{1+i_t} = \frac{1}{R_t} \Rightarrow \hat{q}_t = -\hat{R}_t$$

$$\Rightarrow \hat{C}_t = E_t \hat{C}_{t+1} - \frac{1}{\sigma}[\underbrace{\hat{R}_t - E_t \hat{\Pi}_{t+1}}^{\text{real return.}}] \quad \checkmark$$

(log-linearized IS Curve.)

Firm: ① monopolistic competition Labor-only

② nominal rigidity

differentiated good i $\quad Y(i) = A_t N_t(i)^{1-\alpha}$

A_t: same for all i

$\alpha = 0$: $Y(i)$ constant returns-to-scale.

$Y(i) = A_t N_t(i)$ (CRS)

Actions: ① Cost minimization by choosing $N_t(i)$

② discounted profit maximization by choosing $P_t^*(i)$

1. Cost minimization

$$\min_{N_t(i)} \frac{W_t N_t(i)}{P_t} \quad \text{s.t.} \quad Y_t(i) = A_t N_t(i)$$

$$\Rightarrow \boxed{\underline{W_t} = \underline{MC_t A_t} = MC_t \frac{\partial Y_t(i)}{\partial N_t(i)}}$$

MC_t: marginal cost. (common for all firms)

2. Price settings.

① flexible price

$$W_t = MC_t \frac{\partial Y_t(i)}{\partial N_t(i)} = \frac{Y_t(i)}{N_t(i)} MC_t$$

$$\max_{P_t(i)} \frac{P_t(i) Y_t(i)}{P_t} - \frac{W_t \cdot N_t(i)}{P_t}$$

$$= \frac{P_t(i) Y_t(i)}{P_t} - \frac{Y_t(i)}{\cancel{N_t(i)}} \cdot \frac{MC_t \cdot \cancel{N_t(i)}}{P_t}$$

$$= \left(\frac{P_t(i)}{P_t} - \frac{MC_t}{P_t}\right) \underline{\underline{Y_t(i)}}$$

$$Y_t(i) = C_t(i) = \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} \cdot C_t$$

Demand

$$= \left(\frac{P_t(i)}{P_t} - \frac{MC_t}{P_t}\right)\left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} C_t$$

F.o.c.

$$\left[\frac{P_t(i)^{1-\varepsilon}}{P_t} - \frac{MC_t}{P_t} \cdot P_t(i)^{-\varepsilon}\right] \underline{\underline{P_t^{\varepsilon} C_t}}$$

$$(1-\varepsilon) \frac{(P_t(i))^{\cancel{-\varepsilon}}}{\cancel{P_t}} - \frac{MC_t}{\cancel{P_t}} (-\varepsilon) P_t(i)^{\cancel{-\varepsilon}-1} = 0$$

$$\Rightarrow P_t^*(i) = \frac{1}{1-\frac{1}{\varepsilon}} MC_t = \mu \cdot MC_t$$

$\varepsilon \to \infty \quad \mu \to 1 \Rightarrow P_t^*(i) \to MC_t$ perfect competition

$\varepsilon > 1 \Rightarrow \mu > 1 \quad P_t^*(i) > MC_t \Rightarrow$ inefficiency

2. Sticky prices (Calvo-setting)

P_t^* can adjust price with Prob. $1-\theta$

P_{t-1} cannot adjust price with Prob. θ

$\theta \uparrow \rightarrow$ stickness $\uparrow$.

Aggregate price index (CES)

$$P_t = \left[\theta \underline{\underline{P_{t-1}}}^{1-\varepsilon} + (1-\theta) \underline{\underline{P_t^*}}^{1-\varepsilon} \right]^{\frac{1}{1-\varepsilon}}$$

log-linearized. $\Rightarrow$ $\hat{p}_t = \theta \hat{p}_{t-1} + (1-\theta) \hat{p}_t^*$.

$-\hat{p}_{t-1}$ on both side

$$\Rightarrow \underbrace{\hat{p}_t - \hat{p}_{t-1}}_{\hat{\pi}_t} = (\theta - 1) \hat{p}_{t-1} + (1-\theta) \hat{p}_t^*$$

$$\hat{\pi}_t = (1-\theta) \underbrace{(\hat{p}_t^* - \hat{p}_{t-1})}_{\text{positive inflation comes from}}$$

Optimal price setting,

Firm sets prices given it will not be allowed to change its prices. in the future.

$$\left[\overset{Q_{t,t+1}}{P_t(i)}\, Y_t(i) - Cost_t(i)\right] + \underset{\substack{\uparrow \\ \text{Prob.}}}{\theta}\, E_t\left[\underbrace{\overset{Q_{t,t+1}}{P_t(i)}\, Y_{t+1}(i) - Cost_{t+1}(i)}_{t+1}\right]$$

$$+\ \theta^2 E_t\left[\underbrace{\overset{Q_{t,t+2}}{P_t(i)}\, Y_{t+2}(i) - Cost_{t+2}(i)}_{t+2}\right]$$

$$\Rightarrow \boxed{\begin{aligned} &\max \sum_{k=0}^{\infty} \theta^k E_t\left[Q_{t,t+k}\left(P_t^* \cdot Y_{t+k|t} - Cost(Y_{t+k|t})\right)\right] \\ &\text{s.t.} \quad \underline{Y_{t+k|t} = \left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon} C_{t+k}} \end{aligned}}$$

Demand for period $t+k$

F.O.C.

$$\Rightarrow \quad P_t^* = \frac{\varepsilon}{\varepsilon - 1} \cdot \frac{E_t \sum_{k=0}^{\infty} \theta^k Q_{t,t+k} Y_{t+k|t} Cost_{t+k|t}}{E_t \sum_{k=0}^{\infty} \theta^k Q_{t,t+k} Y_{t+k|t}}$$

Optimal price depends on future values of demand condition & future value of marginal cost

Log-linearization

$$\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \frac{(1-\theta)(1-\beta\theta)}{\theta} \hat{mc}^r_{t+1}$$

Equilibrium

1. goods market

$$Y_t(i) = C_t(i) \quad \forall i \in [0,1]$$

final goods. $$Y_t = \left[\int_0^1 Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}}\right]^{\frac{\varepsilon}{\varepsilon-1}} \Rightarrow Y_t = C_t$$

✓

Log-linearized ⇒ $$\hat{y}_t = \hat{c}_t$$

DIS $$\hat{y}_t = E_t \hat{y}_{t+1} - \frac{1}{\sigma}\left[\hat{R}_t - E_t \hat{\pi}_{t+1}\right]$$

2. labor market $$N_t = \int_0^1 N_t(i)\, di$$

[extension: sticky wage

$$N_t = \left[\int_0^1 N_t(i)^{\frac{\varepsilon_n-1}{\varepsilon_n}} di\right]^{\frac{\varepsilon_n}{\varepsilon_n-1}} \quad \varepsilon_n > 1$$

$$\Rightarrow Y_t(i) = A_t N_t(i)$$

$$\Rightarrow N_t = \int_0^1 \left(\frac{Y_t(i)}{A_t}\right) di = \int_0^1 \frac{\overbrace{\left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} C_t}^{C_t(i)}}{A_t}\, di$$

$$= \frac{Y_t}{A_t} \underbrace{\int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\varepsilon} di}_{\text{price dispersion}} \quad D_t = \exp(d_t) > 1$$

$$\Rightarrow Y_t = A_t N_t \cdot \underline{\exp(-d_t)} \quad \hat{d}_t \approx 0$$

(Gali, 2008)

Log-linearized ⇒ $$\hat{y}_t = \hat{a}_t + \hat{n}_t$$

real marginal cost. $MC_t^r \longrightarrow y_t$

Firm's optimization problem $\dfrac{W_t}{P_t} = MC_t^r \cdot \dfrac{\partial Y_t(i)}{\partial N_t(i)}$

$= MC_t^r \cdot A_t$

labor demand

$\Rightarrow \hat{MC}_t^r + \hat{a}_t = \hat{w}_t - \hat{p}_t$

L-L trade-off $\dfrac{W_t}{P_t} = -\dfrac{U_N}{U_C}$ $\quad U = \dfrac{C^{1-\sigma}}{1-\sigma} - \dfrac{N^{1+\eta}}{1+\eta}$

$= \dfrac{N_t^{\eta}}{C_t^{-\sigma}}$

labor supply

$\Rightarrow \hat{w}_t - \hat{p}_t = \eta \hat{N}_t + \sigma \hat{C}_t$ $\quad \begin{cases} \hat{y}_t = \hat{a}_t + \hat{n}_t \\ \hat{C}_t = \hat{y}_t \end{cases}$

$\Rightarrow \hat{MC}_t^r + \hat{a}_t = \eta \hat{N}_t + \sigma \hat{C}_t$

$= \eta (\hat{y}_t - \hat{a}_t) + \sigma \hat{y}_t$

$\Rightarrow$ (*) $\underline{\hat{mc}_t^r = (\eta+\sigma)\hat{y}_t - (1+\eta)\hat{a}_t}$

Phillips Curve $\hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + \dfrac{(1-\theta)(1-\theta\beta)}{\theta} [(\eta+\sigma)\hat{y}_t - (1+\eta)\hat{a}_t]$

$\hat{mc} = 0$

$\Rightarrow$ (**) $\underline{0 = (\eta+\sigma)\hat{y}_t^n - (1+\eta)\hat{a}_t} \Rightarrow \hat{y}_t^n = \dfrac{1+\eta}{\eta+\sigma}\hat{a}_t$

Natural level of output. MC is constant.

$$\hat{mc} = 0$$

$$\hat{mc}^r_t = (\eta+\sigma)\underbrace{(\hat{y}_t - \hat{y}^n_t)}_{\text{output gap} \equiv \tilde{y}_t}$$

$$\Rightarrow \quad \underline{\pi_t = \beta E_t \hat{\pi}_{t+1} + k\tilde{y}_t} \qquad k = \frac{(1-\theta)(1-\theta\beta)}{\theta}(\eta+\sigma)$$

Dynamic IS.

$$\hat{y}_t = E_t \hat{y}_{t+1} - \frac{1}{\sigma}(\hat{R}_t - E_t\hat{\pi}_{t+1})$$

$$\tilde{y}_t = \hat{y}_t - \hat{y}^n_t$$

$$\Rightarrow \quad \tilde{y}_t + \hat{y}^n_t = E_t\{\tilde{y}_{t+1} + \hat{y}^n_{t+1}\} - \frac{1}{\sigma}(\hat{R}_t - E_t\hat{\pi}_{t+1})$$

$$\Rightarrow \quad \boxed{\tilde{y}_t = E_t\tilde{y}_{t+1} - \frac{1}{\sigma}(\hat{R}_t - E_t\hat{\pi}_{t+1}) + (E_t\hat{y}^n_{t+1} - \hat{y}^n_t)}$$

$$y^n_t = \frac{1+\eta}{\eta+\sigma}\hat{a}_t$$

Define $\underline{r^n_t} = \sigma E_t(y^n_{t+1} - y^n_t)$

$$= \sigma\frac{1+\eta}{\eta+\sigma}E_t(\hat{a}_{t+1} - \hat{a}_t)$$

$$\Rightarrow \quad \tilde{y}_t = -\frac{1}{\sigma}(\hat{R}_t - E_t\hat{\pi}_{t+1} - \hat{r}^n_t) + E_t\tilde{y}_{t+1}$$

$(\hat{\pi}_t, \tilde{y}_t)$

$$\begin{cases} \text{DIS} & \tilde{y}_t = E_t \tilde{y}_{t+1} - \frac{1}{\sigma}(\hat{R}_t - E_t \hat{\pi}_{t+1}) + \underbrace{\varepsilon_t^{IS}}_{\text{shocks.}} \\ \text{AS} & \hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + k \tilde{y}_t \end{cases}$$

Monetary policy Let $\underline{\hat{R}_t = \phi_\pi \hat{\pi}_t + \phi_y \tilde{y}_t + \overset{\text{shock}}{\overset{\downarrow}{v_t}}}$

<u>Taylor principle:</u> $\phi_y = 0 \quad \phi_\pi > 1 \Rightarrow$ saddle path equilibrium

Proof: Stability of Difference equation system

$$(**) \begin{cases} \hat{R}_t = \phi_\pi \hat{\pi}_t \\ \tilde{y}_t = E_t \tilde{y}_{t+1} - \frac{1}{\sigma}(\hat{R}_t - E_t \hat{\pi}_{t+1}) \\ \hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + k \tilde{y}_t \end{cases}$$

$$\Rightarrow \begin{cases} \tilde{y}_t = E_t \tilde{y}_{t+1} - \frac{1}{\sigma}(\phi_\pi \hat{\pi}_t - E_t \hat{\pi}_{t+1}) \\ \hat{\pi}_t = \beta E_t \hat{\pi}_{t+1} + k \tilde{y}_t \end{cases}$$

$$\begin{pmatrix} 1 & \frac{1}{\sigma} \\ 0 & \beta \end{pmatrix} \begin{pmatrix} E_t \tilde{y}_{t+1} \\ E_t \hat{\pi}_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & \frac{\phi_\pi}{\sigma} \\ -k & 1 \end{pmatrix} \begin{pmatrix} \tilde{y}_t \\ \hat{\pi}_t \end{pmatrix}$$

0 no. of stable roots ?

$\Rightarrow \underline{\phi_\pi > 1}$

$$AX_{t+1} = B \cdot X_t \Rightarrow X_{t+1} = \underline{A^{-1} \cdot B} X_t$$

$$= \text{Jacobian}$$

$\begin{cases} \text{no. of predetermined variables.} & (\text{e.g. } k_t \cdot B_t) \\ & \quad k_0 \cdot B_0 \\ \text{no. of jump variable variables} & : \tilde{y}_t, \hat{\pi}_t \end{cases}$

y_0 & π_0 not given

$$J = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

characteristic roots $\underline{b_1, b_2}$

$$X_t = A_1 b_1^t + A_2 b_2^t$$

$$|J - bI|$$

$$= \begin{vmatrix} a_{11}-b & a_{12} \\ a_{21} & a_{22}-b \end{vmatrix}$$

convergent: $|b_1| < 1$ $|b_2| < 1$

$t \to \infty$ $b_1^t \to 0$

no. of convergent path A_1, A_2 can be determined.

X_0 is given $\Rightarrow A_1$

Y_0 is given $\Rightarrow A_2$

$\left.\begin{array}{l} 0 \text{ predetermined variable} \\ + \ |b_1| > 1, \ |b_2| > 1 \end{array}\right\} \Rightarrow \underline{\text{saddle}}$

$\underline{\text{Determinant.}}$

characteristic equation

$$|J-b\mathbb{1}| = (a_{11}-b)(a_{22}-b) - a_{12}\cdot a_{21}$$

$$= a_{11}a_{22} - (a_{11}+a_{22})b + b^2 - a_{12}a_{21}$$

$$= b^2 - \underbrace{(a_{11}+a_{22})}_{\text{Trace}=T}b + \underbrace{a_{11}a_{22} - a_{12}a_{21}}_{|J| \ \text{Determinant} = D}$$

$$= b^2 - T\cdot b + D \qquad (b_1\cdot b_2 = D, \ b_1+b_2 = T)$$

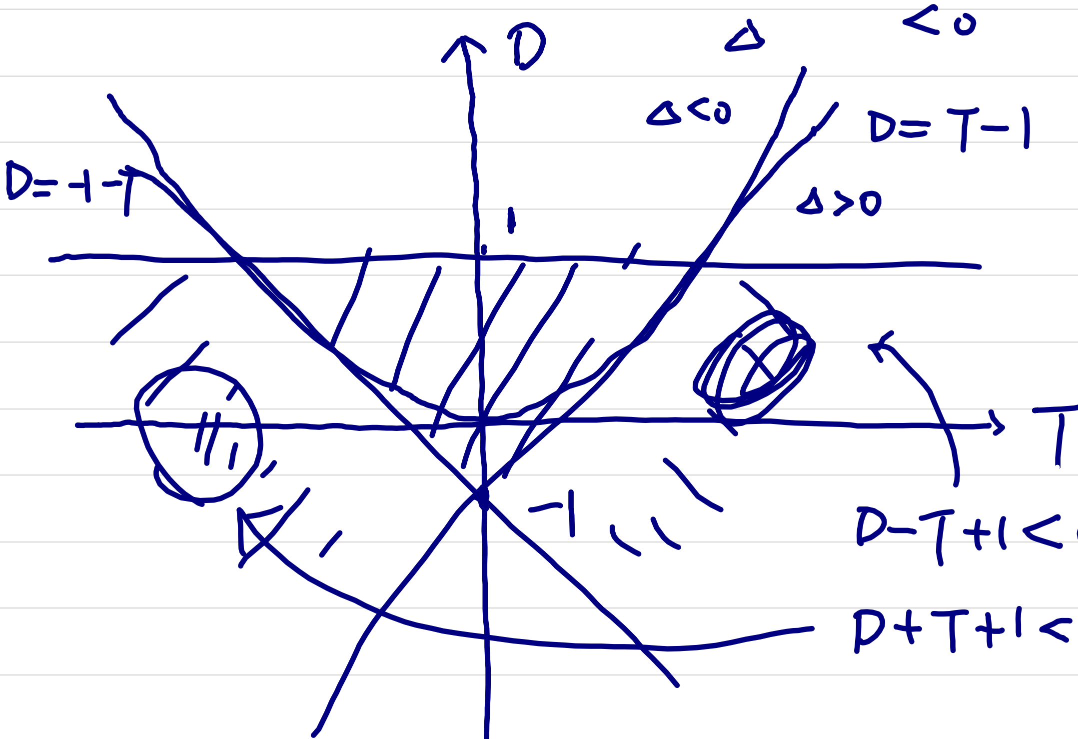
$$(b_1-1)(b_2-1) = b_1\cdot b_2 - (b_1+b_2) + 1 = D - T + 1$$

$$(b_1+1)(b_2+1) = b_1\cdot b_2 + (b_1+b_2) + 1 = D + T + 1$$

$-(-1)$

$\Delta = T^2 - 4D$ $\quad \Delta > 0 \Rightarrow$ real roots

$\quad\quad\quad\quad\quad\quad\quad\quad <0$ complex



stable Triangle

$D - T + 1 < 0 \Rightarrow b_1 > 1, \ b_2 < 1$

$D + T + 1 < 0 \Rightarrow b_1 < -1, \ b_2 > -1$

Extensions:

$$\begin{cases} \hat{R}_t = \phi_\pi E_t \hat{\pi}_{t+1} + \phi_y E_t \tilde{y}_{t+1} \Rightarrow \text{Indeter-} \quad \phi_\pi > 1 \\ \hat{R}_t = \phi_\pi \hat{\pi}_{t-1} + \phi_y \tilde{y}_{t-1} \end{cases}$$

Stability ? $\hat{R}_t = \phi_\pi E_t \hat{\pi}_{t+1} + \phi_y \tilde{y}_t$

$\Rightarrow$ Determinacy.

+ investment choice.

$$i_t = k_{t+1} - (1-\delta) k_t \quad \delta: \text{depreciation rate}$$

$$P_t (C_t + i_t) + Q_t B_t \leq B_{t-1} + W_t N_t + R_t^k K_t + \Pi_t$$

$$Y_t = A_t \cdot K_t^\alpha N_t^{1-\alpha}$$

+ sticky wage $\quad N_t = \left[\int_0^1 N_t(i)^{\frac{\varepsilon_n - 1}{\varepsilon_n}} di \right]^{\frac{\varepsilon_n}{\varepsilon_n - 1}}$

$$\Rightarrow N_t(i) = \left[\frac{W_t(i)}{W_t} \right]^{-\varepsilon_n} N_t$$

$$W_t(i) = \begin{cases} W_t^*(i) & \text{prob. } 1-\theta_w \\ W_{t-1}(i) & \text{prob. } \theta_w \end{cases}$$

sticky wage Phillips curve

$$\hat{\pi}_{w,t} = E_t \hat{\pi}_{w,t+1} + \lambda_w (\sigma \hat{C}_t + \eta \hat{N}_t - \hat{w}_t)$$

$$\lambda_w = \frac{(1-\theta_w)(1-\theta_w \beta)}{\theta_w} \frac{1}{1+\eta \varepsilon_n}$$

+ habit formation

$$u(C_t - bC_{t-1}, N_t)$$

C_{t-1} : external habit

c_{t-1} : internal habit

+ investment adjustment cost

$$I_t = I\left(\frac{K_{t+1}}{K_t}\right) \quad \text{or} \quad K_{t+1} = I\left(\frac{I_{t+1}}{I_t}\right)$$

$$\max_{P_t^*} E_t \sum_{k=0}^{\infty} \Big\{ Q_{t,t+k} \big(\underbrace{P_t^*}_{} \ldots$$

$$\text{s.t.} \quad \underbrace{Y_{t+k|t} = \big(\ldots}_{}$$

$$\underbrace{Y_{t+k|t} + P_t^* \frac{\partial Y_{t+k|}}{\partial P_t^*}}_{}$$

$$\frac{\partial Y_{t+k|t}}{\partial P_t^*} = (-\varepsilon) \ldots$$

$$\sum (A+B) = 0$$

$$\underbrace{Y_{t+k|t}} - \underline{\Psi_{t+k}(Y_{t+k|t})} \Big\}$$

$$\underbrace{\frac{p_t^*}{\;\;{}_{t+k}} \Big) - \varepsilon \quad C_{t+k}}$$

$$- \psi_{t+k|t} \cdot \underbrace{\frac{\partial Y_{t+k|t}}{\partial p_t^*}}$$

$$\frac{Y_{t+k|t}}{p_t^*}$$

$$\Rightarrow \quad \sum A = -\sum B$$

$$X_t \approx \bar{X}(1+\hat{x}_t) \qquad \hat{x}_t = \frac{X_t - \cdots}{\cdots}$$

$$X_t Y_t \approx \bar{X}\bar{Y}(1+\hat{x}_t+\hat{y}_t)$$

$$X_t^{\sigma} \approx \bar{X}^{\sigma}(1+\sigma\hat{x}_t)$$

$$LHS = P_t^{*} E_t \sum_{k=0}^{\infty} (\beta\theta)^k \tilde{Q}_{t,t+k}$$

$$\approx \cancel{\overline{LHS}} + E_t \sum_{k=0}^{\infty} (\beta\theta)^k \bar{Q}\,\bar{P} \cdots$$

$$RHS = \mu E_t \sum_{k=0}^{\infty} (\beta\theta)^k \tilde{Q}_{t,t+k} P_t \cdots$$

$$= \cancel{\overline{RHS}} + E_t \sum_{k=0}^{\infty} (\beta\theta)^k \bar{Q}\,\bar{P} \cdots$$

$$\frac{\bar{x}}{x} \approx \log x_t - \log \bar{x}$$

$$\underline{x_t \approx \bar{x} + \bar{x}\hat{x}_t}$$

?

$$\psi_{t+k|t} = P_{t+k} \cdot mc_{t+k|t}$$

$$P_{t+k}^{\varepsilon} Y_{t+k}$$

$$\underbrace{\quad}^{1+\varepsilon} \bar{Y}\left(\cancel{\hat{g}_{t,t+k}} + \cancel{\hat{P}_{t+k}} + \cancel{\hat{y}_{t+k}} + \hat{P}_t^{y}\right)$$

$$\varepsilon \quad Y_{t+k} \psi_{t+k|t}$$

$$\varepsilon \quad \bar{\Psi} \cdot M \left(\cancel{\hat{g}_{t,t+k}} + \varepsilon \cancel{\hat{P}_{t+k}} + \cancel{\hat{y}_{t+k}} + \hat{mc}_{t+k|t}\right) + \hat{P}_{t+k}$$

$$\sum (\beta\theta)^k p^* = \frac{p^*}{1-\beta\theta}$$

$$\frac{p_t^*}{1-\beta\theta} = E_t \sum_{k=0}^{\infty} \beta\theta^k (\hat{mc}_t$$

$$\hat{y}_t = \hat{a}_t + (1-\alpha)\hat{n}_t =$$

$$mc = \frac{W}{mpn}$$

$x = x(t)$

$y = y(t)$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$+k/t + \hat{p}_{t+k})$

$$\hat{r}_t = \frac{\hat{y}_t - \hat{a}_t}{1-\alpha}$$

$w = mc \cdot mpn$

$Y = AN^{1-\alpha}$

$$\hat{p}_t^* = (1-\beta\theta) E_t \sum_{k=0}^{\infty} (\beta\theta)^k \Big\{ \underbrace{\widehat{mc}^r_{t+k}}$$

$$(1-\beta\theta) \sum (\beta\theta)^k \left(-\frac{\alpha\varepsilon}{1-\alpha}\right) \hat{p}_t^*$$

$$= \cancel{(1-\beta\theta)} \cdot \frac{1}{\cancel{1-\beta\theta}} \cdot \left(-\frac{\alpha\varepsilon}{1-\alpha}\right) \hat{p}_t^*$$

$$- \frac{\alpha \Sigma}{1-\alpha} \left(\hat{p}_0^* - \hat{p}_{t+k} \right) + \hat{p}_{t+k} \Big\}$$

$$\hat{\tilde{p}}_t = (1-\beta\theta) E_t \sum_{k=0}^{\infty} (\beta\theta)^k \{$$

$$= (1-\beta\theta)(④ \hat{mc}_t + \hat{p}_t$$

$$+$$

$$(1-\beta\theta) E_t \sum_{k=1}^{\infty} (\beta\theta)^k \{ ④ \hat{mc}$$

$$= (1-\beta\theta) E_t \sum_{k=0}^{\infty} (\beta\theta)^{k+1} \{ ④ \hat{m}$$

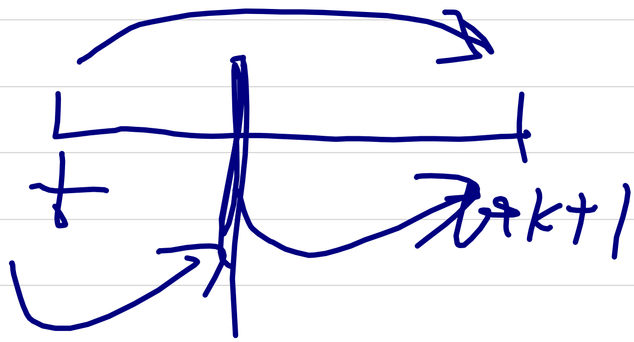
$$= \beta\theta (1-\beta\theta) E_t \sum_{k=0}^{\infty} (\beta\theta)^k \{ ④ \hat{m}$$

$$= \beta\theta (1-\beta\theta) E_{t+1} (\beta\theta$$

$$\theta \, \widehat{mc}_{t+k} + \hat{p}_{t+k} \}$$

$$)$$

$$k' = k+1$$



$$_{t+k} + \hat{p}_{t+k} \}$$

$$c_{t+k+1} + \hat{p}_{t+k+1} \}$$

$$\hat{p}^*_{t+1}$$

$$c_{(t+1)+k} + \hat{p}_{(t+1)+k} \}$$

$$^k \{ \theta \, \widehat{mc}_{(t+1)+k} + \hat{p}_{(t+1)+k} \}.$$

$$\hat{p}_t = \theta \hat{p}_{t-1} + (1-\theta)\hat{p}_t^*$$

$$= \theta \hat{p}_{t-1} + (1-\theta)(1-\beta\theta$$

$$+ (1-\theta)\beta\theta \,-$$

$$\pi_t = \beta E_t \pi_{t+1} + \lambda)\, \hat{mc}$$

$) \, (\Theta \, \hat{mc}_t + \hat{p}_t)$

$$\frac{E_t \hat{p}_{t+1} - \theta \hat{p}_t}{1-\theta}$$

$$\pi_t = \hat{p}_t - \hat{p}_{t-1}$$

t