2019 秋季本科时间序列

第 2 次作业参考答案

2019 年 10 月 13 日

1. (a) 由已知得

$$
\begin{aligned}
\boldsymbol{\Sigma} &= [\sigma_{ij}]_{1\le i,j\le n} \\
&= \mathbb{E}\begin{bmatrix} (X_1-\mathbb{E}X_1)(X_1-\mathbb{E}X_1) & \cdots & (X_1-\mathbb{E}X_1)(X_n-\mathbb{E}X_n) \\ \vdots & \ddots & \vdots \\ (X_n-\mathbb{E}X_n)(X_1-\mathbb{E}X_1) & \cdots & (X_n-\mathbb{E}X_n)(X_n-\mathbb{E}X_n) \end{bmatrix} \\
&= \mathbb{E}\left[\begin{bmatrix} X_1-\mathbb{E}X_1 \\ \vdots \\ X_n-\mathbb{E}X_n \end{bmatrix}\begin{bmatrix} X_1-\mathbb{E}X_1 & \cdots & X_n-\mathbb{E}X_n \end{bmatrix}\right] \\
&= \mathbb{E}\left[(\boldsymbol{X}-\boldsymbol{\mu})(\boldsymbol{X}-\boldsymbol{\mu})^\intercal\right] \\
&= \mathbb{E}\left[(\boldsymbol{X}-\boldsymbol{\mu})(\boldsymbol{X}^\intercal-\boldsymbol{\mu}^\intercal)\right] \\
&= \mathbb{E}\left[\boldsymbol{X}\boldsymbol{X}^\intercal-\boldsymbol{X}\boldsymbol{\mu}^\intercal-\boldsymbol{\mu}\boldsymbol{X}^\intercal+\boldsymbol{\mu}\boldsymbol{\mu}^\intercal\right] \\
&= \mathbb{E}\boldsymbol{X}\boldsymbol{X}^\intercal-\mathbb{E}\boldsymbol{X}\boldsymbol{\mu}^\intercal-\boldsymbol{\mu}\mathbb{E}\boldsymbol{X}^\intercal+\boldsymbol{\mu}\boldsymbol{\mu}^\intercal \\
&= \mathbb{E}\boldsymbol{X}\boldsymbol{X}^\intercal-\mathbb{E}\boldsymbol{X}\mathbb{E}\boldsymbol{X}^\intercal-\mathbb{E}\boldsymbol{X}\mathbb{E}\boldsymbol{X}^\intercal+\mathbb{E}\boldsymbol{X}\mathbb{E}\boldsymbol{X}^\intercal \\
&= \mathbb{E}\boldsymbol{X}\boldsymbol{X}^\intercal-\mathbb{E}\boldsymbol{X}\mathbb{E}\boldsymbol{X}^\intercal
\end{aligned}
$$

故 $\boldsymbol{\Sigma} = \mathbb{E}\left[(\boldsymbol{X}-\boldsymbol{\mu})(\boldsymbol{X}-\boldsymbol{\mu})^\intercal\right] = \mathbb{E}\boldsymbol{X}\boldsymbol{X}^\intercal-\mathbb{E}\boldsymbol{X}\mathbb{E}\boldsymbol{X}^\intercal$ 得证

(b) 由已知

$$\begin{aligned}
\boldsymbol{Y} &= \boldsymbol{A}\boldsymbol{X} \\
&= [a_{ij}]\left[X_1, \dots, X_n\right]^\intercal \\
\mathbb{E}\boldsymbol{Y} &= \mathbb{E}\boldsymbol{A}\boldsymbol{X} \\
&= \mathbb{E}\left[\sum_{j=1}^{n} a_{1j}X_j, \dots, \sum_{j=1}^{n} a_{mj}X_j\right]^\intercal \\
&= \left[\sum_{j=1}^{n} a_{1j}\mathbb{E}X_j, \dots, \sum_{j=1}^{n} a_{mj}\mathbb{E}X_j\right]^\intercal \\
&= \boldsymbol{A}\mathbb{E}\boldsymbol{X}
\end{aligned}$$

原式得证

(c) 由 (a)(b) 问结论可知 $\boldsymbol{\Sigma_Y} = \mathbb{E}\boldsymbol{Y}\boldsymbol{Y}^\intercal, \mathbb{E}\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{\mu}, \boldsymbol{\Sigma_X} = \mathbb{E}\boldsymbol{X}\boldsymbol{X}^\intercal - \mathbb{E}\boldsymbol{X}\mathbb{E}\boldsymbol{X}^\intercal$, 因此

$$\begin{aligned}
\boldsymbol{\Sigma_Y} &= \mathbb{E}[\boldsymbol{A}\boldsymbol{X}\boldsymbol{X}^\intercal\boldsymbol{A}^\intercal] - \boldsymbol{A}\boldsymbol{\mu}\boldsymbol{\mu}^\intercal\boldsymbol{A}^\intercal \\
&= \boldsymbol{A}\mathbb{E}(\boldsymbol{X}\boldsymbol{X}^\intercal)\boldsymbol{A}^\intercal - \boldsymbol{A}\boldsymbol{\mu}\boldsymbol{\mu}^\intercal\boldsymbol{A}^\intercal \\
&= \boldsymbol{A}[\mathbb{E}(\boldsymbol{X}\boldsymbol{X}^\intercal) - \boldsymbol{\mu}\boldsymbol{\mu}^\intercal]\boldsymbol{A}^\intercal \\
&= \boldsymbol{A}\boldsymbol{\Sigma_X}\boldsymbol{A}^\intercal
\end{aligned}$$

2. (a) 矩阵 $\boldsymbol{\Sigma}$ 的特征方程

$$\begin{aligned}
det\left[\lambda\boldsymbol{I} - \boldsymbol{\Sigma}\right] &= \begin{vmatrix} \lambda - \sigma_x^2 & -\rho\sigma_x\sigma_y \\ -\rho\sigma_x\sigma_y & \lambda - \sigma_y^2 \end{vmatrix} \\
&= (\lambda - \sigma_x^2)(\lambda - \sigma_y^2) - (\rho\sigma_x\sigma_y)^2 \\
&= \lambda^2 - (\sigma_x^2 + \sigma_y^2)\lambda + (1 - \rho^2)\sigma_x^2\sigma_y^2 \\
&= 0
\end{aligned}$$

上述方程的判别式为

$$\begin{aligned}
\delta &= \sigma_x^2 + \sigma_y^2 + 2\sigma_x^2\sigma_x^2\sigma_y^2 - 4(1 - \rho^2)\sigma_x^2\sigma_y^2 \\
&= (\sigma_x^2 - \sigma_y^2)^2 + 4\rho^2\sigma_x^2\sigma_y^2 \\
&\geq 0
\end{aligned}$$

那么，令 $f(\lambda) = \lambda^2 - (\sigma_x^2 + \sigma_y^2)\lambda + (1 - \lambda)^2\sigma_x^2\sigma_y^2$

该函数对称轴为 $\frac{\sigma_x^2+\sigma_y^2}{2} > 0$, 又 $f(0) = (1 - \rho^2)\sigma_x^2\sigma_y^2, |\rho| < 1$, 故 f(0)>0

综上，$\boldsymbol{\Sigma}$ 的特征值为严格大于 0 的实数

(b) 若 $|\rho| = 1$, 则矩阵 $\boldsymbol{\Sigma}$ 的特征方程为 $|\lambda\boldsymbol{I} - \boldsymbol{\Sigma}| = \lambda^2 - (\sigma_x^2 + \sigma_y^2)\lambda$

当 $\lambda = 0$ 时，$|\lambda\boldsymbol{I} - \Sigma| = 0$, 故 Σ 有一个特征值为 0.

3. (a) 由已知得 $\boldsymbol{X}_t$ 的概率密度函数为 $f(x) = \frac{1}{a}, x \sim [0, a]$ 则总体二阶矩为

$$\begin{aligned}\mathbb{E}(\boldsymbol{X}^2) &= \int_{-\infty}^{+\infty} x^2 f(x)\mathrm{d}x \\ &= \int_0^{+a} x^2 \frac{1}{a}\mathrm{d}x \\ &= \frac{a^2}{3}\end{aligned}$$

$\boldsymbol{X}_t$ 的样本二阶矩为 $\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{X}_t^2$
由 $\frac{a^2}{3} = \frac{1}{T}\sum_{t=1}^{T}\boldsymbol{X}_t^2$ 可得

$$\hat{a}_T = \sqrt{\frac{3}{T}\sum_{t=1}^{T}\boldsymbol{X}_t^2}$$

由大数定律有

$$\begin{aligned}\lim_{T\to+\infty}(\hat{a}_T) &= \lim_{T\to+\infty}\left(\sqrt{\frac{3}{T}\sum_{t=1}^{T}\boldsymbol{X}_t^2}\right) \\ &= \sqrt{3\frac{a^2}{3}} \\ &= a\end{aligned}$$

故此时的参数估计值 $\hat{a}_T$ 满足一致性

(b) 由已知得 $\boldsymbol{X}_t$ 的概率密度函数为

$$f(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 1 & \text{其他} \end{cases}$$

故 $\boldsymbol{X}_t$ 的样本一阶矩为 $\frac{1}{T}\sum_{t=1}^{T}X_t$, 总体一阶矩为 $\mathbb{E}(\boldsymbol{X}) = \frac{a+b}{2}$
样本二阶矩为 $\frac{1}{T}\sum_{t=1}^{T}X_t^2$, 总体二阶矩为

$$\begin{aligned}\mathbb{E}(\boldsymbol{X}^2) &= \int_{-\infty}^{+\infty} x^2 f(x)\mathrm{d}x \\ &= \int_a^b x^2 \frac{1}{b-a}\mathrm{d}x \\ &= \frac{a^2+ab+b^2}{3}\end{aligned}$$

联立得

$$\begin{cases} \frac{a+b}{2} = \frac{1}{T}\sum_{t=1}^{T}X_t \\ \frac{a^2+ab+b^2}{3} = \frac{1}{T}\sum_{t=1}^{T}X_t^2 \end{cases}$$

解得

$$\begin{cases} \hat{a}_T = \frac{1}{T}\sum_{t=1}^{T} X_t - \sqrt{3[\frac{1}{T}\sum_{t=1}^{T} \boldsymbol{X}_t^2 - (\frac{1}{T}\sum_{t=1}^{T} X_t)^2]} \\ \hat{b}_T = \frac{1}{T}\sum_{t=1}^{T} X_t + \sqrt{3[\frac{1}{T}\sum_{t=1}^{T} \boldsymbol{X}_t^2 - (\frac{1}{T}\sum_{t=1}^{T} X_t)^2]} \end{cases}$$

4. (a) 随机向量 $[X, Y]^\intercal$ 服从 2-元正态分布 $N(\mathbf{0}, \boldsymbol{\Sigma})$, 故 $\mathbb{E}X = \mathbb{E}Y = 0$

$$\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix} \tag{1}$$

故 $\mathbb{E}(Z) = \mathbb{E}(X+Y) = \mathbb{E}(X) + \mathbb{E}(Y) = 0$
由第 1 题 (c) 问 $\boldsymbol{\Sigma_Y} = \boldsymbol{A}\boldsymbol{\Sigma_X}\boldsymbol{A}^\intercal$
又

$$Z = \boldsymbol{A}\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} X \\ Y \end{bmatrix}$$

则 Z 方差为

$$\begin{aligned} \text{var}\big(Z\big) &= \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ &= \sigma_x^2 + \sigma_y^2 + 2\rho\sigma_x\sigma_y \end{aligned}$$

(b) 由题已知 Z 服从正态分布，又由 (a) 问得到 Z 的期望及方差
故可得 Z 的密度函数为 $f(z) = \frac{1}{\sqrt{2\pi A^\intercal \Sigma A}} \exp \frac{-z^2}{2A^\intercal \Sigma A}$