2018 秋季本科时间序列

第七次作业答案

2018 年 12 月 30 日

1. 当 $\sigma_{12}^2 = \sigma_{21}^2 = 0$ 时

$$\Sigma = \begin{bmatrix} \sigma_{11}^2 & 0 \\ 0 & \sigma_{22}^2 \end{bmatrix}$$

易得

$$\Sigma^{-1} = \begin{bmatrix} \frac{1}{\sigma_{11}^2} & 0 \\ 0 & \frac{1}{\sigma_{22}^2} \end{bmatrix}$$

从而有

$$f(\mathbf{x}) = \frac{\mathbf{1}}{\sqrt{(\mathbf{2}\pi)^{\mathbf{2}} \det(\boldsymbol{\Sigma})}} \exp\{-\frac{\mathbf{1}}{\mathbf{2}}\mathbf{x}^{\mathbf{T}}\boldsymbol{\Sigma}^{-\mathbf{1}}\mathbf{x}\} = \frac{\mathbf{1}}{\mathbf{2}\pi\sigma_{\mathbf{11}}\sigma_{\mathbf{22}}} \exp\{-\left(\frac{\mathbf{x}_{\mathbf{1}}^{\mathbf{2}}}{\mathbf{2}\sigma_{\mathbf{11}}^{\mathbf{2}}} + \frac{\mathbf{x}_{\mathbf{2}}^{\mathbf{2}}}{\mathbf{2}\sigma_{\mathbf{22}}^{\mathbf{2}}}\right)\}$$

进而可以求得 x_i 的边缘分布

$$f(x_1) = \int_{-\infty}^{+\infty} f(x_1, x_2)\mathrm{d}x_2 = \frac{1}{\sqrt{2\pi}\sigma_{11}} e^{-\frac{x_1^2}{2\sigma_{11}^2}}$$

同理有

$$f(x_2) = \int_{-\infty}^{+\infty} f(x_1, x_2)\mathrm{d}x_1 = \frac{1}{\sqrt{2\pi}\sigma_{22}} e^{-\frac{x_2^2}{2\sigma_{22}^2}}$$

由上

$$f(x_1, x_2) = f(x_1)f(x_2)$$

所以 x_1 与 x_2 独立

2. (a) 因为

$$Y_t = \boldsymbol{X_t^T \beta} + \varepsilon_t \qquad Y_t = \boldsymbol{X_t^T \hat{\beta}_T} + \hat{\varepsilon}_t$$

从而有

$$\begin{aligned}
\hat{\varepsilon}_t^2 - \varepsilon_t^2 &= (\hat{\varepsilon}_t + \varepsilon_t)(\hat{\varepsilon}_t - \varepsilon_t) \\
&= (2Y_t - \boldsymbol{X_t^T \hat{\beta}_T} - \boldsymbol{X_t^T \beta})(\boldsymbol{X_t^T \hat{\beta}_T} - \boldsymbol{X_t^T \beta}) \\
&= [2(\boldsymbol{X_t^T \beta} + \varepsilon_t) - X_t^T(\boldsymbol{\hat{\beta}_T} + \boldsymbol{\beta})]X_t^T(\boldsymbol{\hat{\beta}_T} - \boldsymbol{\beta}) \\
&= 2\varepsilon_t \boldsymbol{X_t^T}(\boldsymbol{\beta} - \boldsymbol{\hat{\beta}_T}) + (\boldsymbol{\beta} - \boldsymbol{\hat{\beta}_T})^T \boldsymbol{X_t X_t^T}(\boldsymbol{\beta} - \boldsymbol{\hat{\beta}_T})
\end{aligned}$$

(b)

$$\begin{aligned}
\sum_{t=1}^{T}\left(\hat{\varepsilon}_t^2 - \varepsilon_t^2\right) &= \sum_{t=1}^{T}\left[2\varepsilon_t \boldsymbol{X_t^T}(\boldsymbol{\beta} - \boldsymbol{\hat{\beta}_T}) + (\boldsymbol{\beta} - \boldsymbol{\hat{\beta}_T})^T \boldsymbol{X_t X_t^T}(\boldsymbol{\beta} - \boldsymbol{\hat{\beta}_T})\right] \\
&= 2\sum_{t=1}^{T}(\varepsilon_t \boldsymbol{X_t^T})(\boldsymbol{\beta} - \boldsymbol{\hat{\beta}_T}) + (\boldsymbol{\beta} - \boldsymbol{\hat{\beta}_T})^T \sum_{t=1}^{T}(\boldsymbol{X_t X_t^T})(\boldsymbol{\beta} - \boldsymbol{\hat{\beta}_T})
\end{aligned}$$

$\hat{\beta}_T$ 为 β 的 OLS 估计，且 X_t 为平稳序列，有

$$\beta - \hat{\beta}_T \xrightarrow{a.s.} 0 \qquad \sum_{t=1}^{T}(\varepsilon_t \boldsymbol{X_t^T}) \xrightarrow{a.s.} 0$$

易得

$$\sum_{t=1}^{T}\left(\hat{\varepsilon}_t^2 - \varepsilon_t^2\right) \xrightarrow{a.s.} 0$$

从而有

$$\frac{1}{T}\sum_{t=1}^{T}\hat{\varepsilon}_t^2 \xrightarrow{a.s.} \frac{1}{T}\sum_{t=1}^{T}\varepsilon_t^2 \xrightarrow{a.s.} \sigma_\varepsilon^2$$