2018 秋季本科时间序列

第六次作业答案

2018 年 12 月 4 日

1. (a)

$$\boldsymbol{\theta} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \qquad \boldsymbol{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_T \end{bmatrix} \qquad \boldsymbol{X} = \begin{bmatrix} 1 & X_1 \\ 1 & X_2 \\ \vdots & \vdots \\ 1 & X_T \end{bmatrix} \qquad \boldsymbol{e} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_T \end{bmatrix}$$

(b) 令 $A = (X^T X)$ 有

$$A = (X^T X) = \begin{bmatrix} T & \sum_{i=1}^T X_i \\ \sum_{i=1}^T X_i & \sum_{i=1}^T X_i^2 \end{bmatrix}$$

由 Gramer 法则

$$A^{-1} = \frac{A^*}{|A|} = \frac{\begin{bmatrix} \sum_{i=1}^T X_i^2 & -\sum_{i=1}^T X_i \\ -\sum_{i=1}^T X_i & T \end{bmatrix}}{|A|}$$

其中 $|A| = T\sum_{i=1}^T X_i^2 - (\sum_{i=1}^T X_i)^2$, 从而得：

$$\hat{\boldsymbol{\theta}} = \begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = \boldsymbol{(X^T X)^{-1} X^T Y}$$

$$= \frac{1}{|A|} \begin{bmatrix} \sum_{i=1}^T X_i^2 & -\sum_{t=1}^T X_i \\ -\sum_{t=1}^T X_i & T \end{bmatrix} \begin{bmatrix} 1 & \cdots & 1 \\ X_1 & \cdots & X_T \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_T \end{bmatrix}$$

$$= \begin{bmatrix} \frac{(\frac{1}{T}\sum\limits_{j=1}^T Y_j)(\frac{1}{T}\sum\limits_{i=1}^T X_i^2) - (\frac{1}{T}\sum\limits_{j=1}^T Y_j X_j)(\frac{1}{T}\sum\limits_{i=1}^T X_i)}{\frac{1}{T}\sum_{i=1}^T X_i^2 - (\frac{1}{T}\sum_{i=1}^T X_i)^2} \\ \\ \frac{(\frac{1}{T}\sum\limits_{j=1}^T X_j Y_j) - (\frac{1}{T}\sum\limits_{j=1}^T Y_j)(\frac{1}{T}\sum\limits_{i=1}^T X_i)}{\frac{1}{T}\sum_{i=1}^T X_i^2 - (\frac{1}{T}\sum_{i=1}^T X_i)^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{T}\sum\limits_{j=1}^T Y_j - \frac{1}{T}\sum\limits_{i=1}^T X_i \frac{(\frac{1}{T}\sum\limits_{j=1}^T X_j Y_j) - (\frac{1}{T}\sum\limits_{j=1}^T Y_j)(\frac{1}{T}\sum\limits_{i=1}^T X_i)}{\frac{1}{T}\sum_{i=1}^T X_i^2 - (\frac{1}{T}\sum_{i=1}^T X_i)^2} \\ \\ \frac{(\frac{1}{T}\sum\limits_{j=1}^T X_j Y_j) - (\frac{1}{T}\sum\limits_{j=1}^T Y_j)(\frac{1}{T}\sum\limits_{i=1}^T X_i)}{\frac{1}{T}\sum_{i=1}^T X_i^2 - (\frac{1}{T}\sum_{i=1}^T X_i)^2} \end{bmatrix}$$

(c)

$$\mathbb{E}[Y_t] = \alpha + \beta\mathbb{E}[X_t] \qquad \mathbb{E}[X_tY_t] = \alpha\mathbb{E}[X_t] + \beta\mathbb{E}[X_t^2]$$

从而有

$$\beta = \frac{\mathbb{E}[X_tY_t] - \mathbb{E}[X_t]\mathbb{E}[Y_t]}{\mathbb{E}[X_t^2] - \mathbb{E}[X_t]^2} = \frac{\mathrm{cov}(X_t, Y_t)}{\mathrm{var}X_t}$$

进而有

$$\alpha = \mathbb{E}[Y_t] - \mathbb{E}[X_t]\frac{\mathrm{cov}(X_t, Y_t)}{\mathrm{var}X_t}$$

又

$$\frac{1}{T}\sum_{i=1}^{T} X_i \xrightarrow{a.s.} \mathbb{E}X_t \qquad \frac{1}{T}\sum_{i=1}^{T} Y_i \xrightarrow{a.s.} \mathbb{E}Y_t$$

$$\frac{1}{T}\sum_{i=1}^{T} X_i^2 - (\frac{1}{T}\sum_{i=1}^{T} X_i)^2 \xrightarrow{a.s.} \mathrm{var}X_t$$

$$(\frac{1}{T}\sum_{j=1}^{T} X_jY_j) - (\frac{1}{T}\sum_{j=1}^{T} Y_j)(\frac{1}{T}\sum_{i=1}^{T} X_i) \xrightarrow{a.s.} \mathrm{cov}(X_t, Y_t)$$

所以

$$\hat{\boldsymbol{\theta}} = \begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} \xrightarrow{a.s.} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \boldsymbol{\theta}$$

2. (a)

$$\begin{aligned}
\boldsymbol{Y} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_T \end{bmatrix} &= \begin{bmatrix} \phi_1 X_0 + \phi_2 X_{1-2} + \cdots + \phi_p X_{1-p} \\ \phi_1 X_1 + \phi_2 X_0 + \cdots + \phi_p X_{2-p} \\ \vdots \\ \phi_1 X_{T-1} + \phi_2 X_{T-2} + \cdots + \phi_p X_{T-p} \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_T \end{bmatrix} \\
&= \begin{bmatrix} X_0 & \cdots & X_{1-p} \\ X_1 & \cdots & X_{2-p} \\ \vdots & \ddots & \vdots \\ X_{T-1} & \cdots & X_{T-p} \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \end{bmatrix} + \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \vdots \\ \varepsilon_T \end{bmatrix} \\
&= \boldsymbol{X\phi} + \boldsymbol{\varepsilon}
\end{aligned}$$

(b)

$$\frac{1}{T}\boldsymbol{X}^T\boldsymbol{X} = \frac{1}{T}\begin{bmatrix} X_0 & \cdots & X_{T-1} \\ X_{1-2} & \cdots & X_{T-2} \\ \vdots & \ddots & \vdots \\ X_{1-p} & \cdots & X_{T-p} \end{bmatrix}\begin{bmatrix} X_0 & \cdots & X_{1-p} \\ X_1 & \cdots & X_{2-p} \\ \vdots & \ddots & \vdots \\ X_{T-1} & \cdots & X_{T-p} \end{bmatrix}$$

$$= \frac{1}{T}\begin{bmatrix} \sum_{t=1}^T X_{t-1}^2 & \cdots & \sum_{t=1}^T (X_{t-1}X_{t-p}) \\ \sum_{t=1}^T (X_{t-2}X_{t-1}) & \cdots & \sum_{t=1}^T (X_{t-2}X_{t-p})) \\ \vdots & \ddots & \vdots \\ \sum_{t=1}^T (X_{t-p}X_{t-1})) & \cdots & \sum_{t=1}^T X_{t-p}^2 \end{bmatrix}$$

因为

$$\frac{1}{T}\sum_{t=1}^T X_t X_{t-k} \xrightarrow{a.s.} \sigma(k)$$

所以

$$\frac{1}{T}\boldsymbol{X}^T\boldsymbol{X} \xrightarrow{a.s.} \begin{bmatrix} \sigma^2(0) & \cdots & \sigma^2(p-1) \\ \sigma^2(1) & \cdots & \sigma^2(p-2) \\ \vdots & \ddots & \vdots \\ \sigma^2(p-1) & \cdots & \sigma^2(0) \end{bmatrix}$$

(c) OLS 估计量为

$$\hat{\boldsymbol{\phi}} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y}$$

有

$$\boldsymbol{X}^T\boldsymbol{Y} = \begin{bmatrix} X_0 & \cdots & X_{T-1} \\ X_{1-2} & \cdots & X_{T-2} \\ \vdots & \ddots & \vdots \\ X_{1-p} & \cdots & X_{T-p} \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_T \end{bmatrix} = \begin{bmatrix} \sum_{t=1}^T X_{t-1}X_t \\ \sum_{t=1}^T X_{t-2}X_t \\ \vdots \\ \sum_{t=1}^T X_{t-p}X_t \end{bmatrix} \xrightarrow{a.s.} T\begin{bmatrix} \sigma^2(1) \\ \sigma^2(2) \\ \vdots \\ \sigma^2(p) \end{bmatrix}$$

综合 (b) 结论

$$(\frac{1}{T}\boldsymbol{X}^T\boldsymbol{X})^{-1}(\frac{1}{T}\boldsymbol{X}^T\boldsymbol{Y}) \xrightarrow{a.s.} \begin{bmatrix} \sigma^2(0) & \cdots & \sigma^2(p-1) \\ \sigma^2(1) & \cdots & \sigma^2(p-2) \\ \vdots & \ddots & \vdots \\ \sigma^2(p-1) & \cdots & \sigma^2(0) \end{bmatrix}^{-1}\begin{bmatrix} \sigma^2(1) \\ \sigma^2(2) \\ \vdots \\ \sigma^2(p) \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_p \end{bmatrix}$$

即

$$\hat{\boldsymbol{\phi}} = (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y} \xrightarrow{a.s.} \boldsymbol{\phi}$$

$\boldsymbol{\phi}$ 的 OLS 估计与 Yule-Walker 估计量一致