2018 秋季本科时间序列

第二次作业答案

2018 年 10 月 22 日

1. (a) 首先有

$$\mathbb{E}\hat{\mu}_T = \mathbb{E}\frac{1}{T}\sum_{t=1}^{T} X_t = \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}X_t = \mu$$

以及

$$Var(\hat{\mu}_T) = Var(\frac{1}{T}\sum_{t=1}^{T} X_t) = \frac{1}{T^2}\sum_{t=1}^{T} Var(X_t) = \frac{\sigma^2}{T}$$

从而有

$$\begin{aligned}
\mathbb{E}\hat{\sigma}_T^2 &= \frac{1}{T-1}\sum_{t=1}^{T}\mathbb{E}(X_t^2 - 2X_t\hat{\mu}_T + \hat{\mu}_T^2) \\
&= \frac{1}{T-1}\sum_{t=1}^{T}(\mathbb{E}X_t^2 - 2\mathbb{E}(X_t\hat{\mu}_T) + \mathbb{E}\hat{\mu}_T^2) \\
&= \frac{1}{T-1}\sum_{t=1}^{T}[(Var(X_t) + (\mathbb{E}X_t)^2) - 2\mathbb{E}\left(X_t\frac{1}{T}\sum_{j=1}^{T}X_j\right) + (Var(\hat{\mu}_T) + (\mathbb{E}\hat{\mu}_T)^2)] \\
&= \frac{1}{T-1}\sum_{t=1}^{T}[\frac{T+1}{T}\sigma^2 + 2\mu^2 - 2\left(\frac{1}{T}\sum_{j=1}^{T}\mathbb{E}(X_tX_j)\right)]
\end{aligned}$$

现计算 $\sum_{j=1}^{T}\mathbb{E}(X_tX_j)$

$$\sum_{j=1}^{T}\mathbb{E}(X_tX_j) = \sum_{j\neq t}^{T}[Cov(X_t, X_j) + \mathbb{E}X_t\mathbb{E}X_j] + \sigma^2 + \mu^2 = \sigma^2 + T\mu^2$$

从而有

$$\mathbb{E}\hat{\sigma}_T^2 = \frac{1}{T-1}\sum_{t=1}^{T}[\frac{T+1}{T}\sigma^2 + 2\mu^2 - 2\frac{\sigma^2}{T} - 2\mu^2] = \sigma^2$$

(b)

$$\begin{aligned}
\hat{\sigma}_T^2 &= \frac{1}{T}\sum_{t=1}^{T}(X_t - \hat{\mu}_T)^2 \\
&= \frac{1}{T}\left(\sum_{t=1}^{T} X_t^2 - 2\hat{\mu}_T \sum_{t=1}^{T} X_t + T\hat{\mu}_T^2\right) \\
&= \frac{1}{T}\left(\sum_{t=1}^{T} X_t^2 - 2\hat{\mu}_T T\hat{\mu}_T + T\hat{\mu}_T^2\right) \\
&= \frac{1}{T}\sum_{t=1}^{T} X_t^2 - (\hat{\mu}_T)^2.
\end{aligned}$$

另有

$$\begin{aligned}
\hat{\sigma}_{k,T}^2 &= \frac{1}{T-k}\sum_{t=k+1}^{T}(X_t - \hat{\mu}_T)(X_{t-k} - \hat{\mu}_T) \\
&= \frac{1}{T-k}[\sum_{t=k+1}^{T} X_t X_{t-k} - \hat{\mu}_T \sum_{t=k+1}^{T}(X_t + X_{t-k})] + \hat{\mu}_T^2 \\
&= \frac{1}{T-k}\sum_{t=k+1}^{T} X_t X_{t-k} - \hat{\mu}_T(\hat{\mu}_1 + \hat{\mu}_2 - \hat{\mu}_T)
\end{aligned}$$

其中，$\hat{\mu}_1$ 是 $(k+1)$ 项到 T 项的均值，$\hat{\mu}_2$ 是第 1 项到 $T-k$ 项的均值

(c) 由平稳性条件

$$\lim_{T\to\infty}\hat{\mu}_T = \mathbb{E}X_t = \mu$$

$$\begin{aligned}
\lim_{T\to\infty}\hat{\sigma}_T^2 &= \lim_{T\to\infty}\left(\frac{1}{T}\sum_{t=1}^{T} X_t^2 - (\hat{\mu}_T)^2.\right) = \lim_{T\to\infty}\frac{1}{T}\left(\sum_{t=1}^{T} X_t^2\right) - \mu^2 \\
&= \lim_{T\to\infty}\frac{1}{T}\left(\sum_{t=1}^{T} X_t^2\right) - 2\mu^2 + \mu^2 \\
&= \lim_{T\to\infty}\frac{1}{T}\left(\sum_{t=1}^{T} X_t^2\right) - 2\mu\lim_{T\to\infty}\frac{1}{T}\left(\sum_{t=1}^{T} X_t\right) + \mu^2 \\
&= \lim_{T\to\infty}\frac{1}{T}\left(\sum_{t=1}^{T}(X_t^2 - 2\mu X_t + \mu^2)\right) \\
&= \lim_{T\to\infty}\frac{1}{T}\left(\sum_{t=1}^{T}(X_t - \mu)^2\right) \\
&= \sigma^2
\end{aligned}$$

另有

$$\lim_{T\to\infty}\hat{\mu}_1 = \lim_{T\to\infty}\hat{\mu}_2 = \lim_{T\to\infty}\hat{\mu}_T = \mathbb{E}X_t = \mu$$

同理有

$$\begin{aligned}\lim_{T\to\infty}\hat{\sigma}_k^2 &= \lim_{T\to\infty}\frac{1}{T-k}\sum_{t=k+1}^{T} X_t X_{t-k} - \mu^2 \\ &= \lim_{T\to\infty}\frac{1}{T-k}\sum_{t=k+1}^{T}(X_t-\mu)(X_{t-k}-\mu) \\ &= \sigma_k^2\end{aligned}$$

2.

$$\rho_{XY}^2 = \left(\frac{\hat{\sigma}_{XY}^2}{\hat{\sigma}_X^2\hat{\sigma}_Y^2}\right) = \left(\frac{[\sum_{t=1}^{T}(X_t-\hat{\mu}_X)(Y_t-\hat{\mu}_Y)]^2}{\sum_{t=1}^{T}(X_t-\hat{\mu}_X)^2\sum_{t=1}^{T}(Y_t-\hat{\mu}_Y)^2}\right)$$

由 Cauchy-Buniakowsky-Schwarz 不等式:
对任意实数 $a_1, a_2 \ldots a_k$ 和 $b_1, b_2 \ldots b_k$ 有：

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \le (\sum_{k=1}^{n} a_k^2)(\sum_{k=1}^{n} b_k^2)$$

所以有

$$\left(\sum_{t=1}^{T}(X_t-\hat{\mu}_X)(Y_t-\hat{\mu}_Y)\right)^2 \le \sum_{t=1}^{T}(X_t-\hat{\mu}_X)^2\sum_{t=1}^{T}(Y_t-\hat{\mu}_Y)^2$$

即

$$-1 \le \rho_{XY} = \frac{\hat{\sigma}_{XY}}{\hat{\sigma}_X\hat{\sigma}_Y} \le 1.$$