2018 秋季本科时间序列

第一次作业答案

2018 年 10 月 24 日

1. (a)

$$
\begin{aligned}
&\mathbb{P}(A_x) = \mathbb{P}(X \le x, Y \le x) = F(x,x) = \frac{1}{2}x^2 + \frac{1}{2}x^2 = x^2 \\
&\mathbb{P}(A_x \cup B_x) = \mathbb{P}(X \le x, Y \le 1) = F(x,1) = \frac{1}{2}x^2 + \frac{1}{2} \\
&\mathbb{P}(B_x) = \mathbb{P}(A_x \cup B_x) - \mathbb{P}(A_x) = \frac{1}{2} - \frac{1}{2}x^2 \\
&\mathbb{P}(A_x \cup C_x) = \mathbb{P}(X \le x, Y \le x) = F(1,x) = \frac{1}{2}x^2 + \frac{1}{2} \\
&\mathbb{P}(C_x) = \mathbb{P}(A_x \cup C_x) - \mathbb{P}(A_x) = \frac{1}{2} - \frac{1}{2}x^2 \\
&\mathbb{P}(D_x) = \mathbb{P}(X \le 1, Y \le 1) - \mathbb{P}(A_x) - \mathbb{P}(B_x) - \mathbb{P}(C_x) = 0
\end{aligned}
$$

   (b)

$$
\begin{aligned}
\mathbb{P}(X \le 1, Y \le 1) =& \mathbb{P}\{X \in [0,1], Y = 0\} + \mathbb{P}\{X \in (0,1), Y \in (0,1)\} \\
&+ \mathbb{P}\{X = 0, Y \in [0,1]\} - \mathbb{P}\{X = 0, Y = 0\}
\end{aligned}
$$

   另有

$$
\mathbb{P}\{X \in (0,1), Y \in (0,1)\} = \lim_{x \to 0} \mathbb{P}(D_x) = 0
$$

   所以

$$
\mathbb{P}\{X \le 1, Y \le 1\} = \mathbb{P}\{X \in [0,1], Y = 0\} + \mathbb{P}\{X = 0, Y \in [0,1]\}
$$

   (x, y) 的概率集中在两条边界上

2. (a)

$$
X \sim U([0,1])
$$

$$
f(x) = \begin{cases} 0 & x < 0 \text{ 或 } x > 1 \\ 1 & 0 \le x \le 1 \end{cases}
$$

$$F(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

(b) Y 与 X 有相同的分布函数

$$F(x, y) = \mathbb{P}\{X < x, Y < y\} = \mathbb{P}\{X < x, X < y\}$$

有

$$F(x) = \begin{cases} 0 & x < 0, y < 0 \\ min(x, y) & x \in [0, 1] \text{ 或 } y \in [0, 1] \\ 1 & x > 1, y > 1 \end{cases}$$

(c)

$$F(x, y) = \mathbb{P}\{X < x, Y < y\} = \mathbb{P}\{X < x, -X < y\} = \mathbb{P}\{-y < X < x\}$$

$$F(x) = \begin{cases} 0 & -y < x < 0 \\ x & -y < 0 < x < 1 \\ x + y & 0 < -y < x < 1 \\ 1 + y & 0 < -y < 1 < x \\ 1 & 1 < -y < x \end{cases}$$

3. (a)

$$\begin{aligned} \mathbb{E}(X) &= 1 \times \sum_{j=1}^{2} p_{1j} + 2 \times \sum_{j=1}^{2} p_{2j} \\ &= 1 \times (\frac{1}{2}\rho + \frac{1}{2}(1 - \rho)) + 2 \times (\frac{1}{2}\rho + \frac{1}{2}(1 - \rho)) \\ &= \frac{3}{2} \\ \mathbb{E}(Y) &= 1 \times \sum_{i=1}^{2} p_{i1} + 2 \times \sum_{i=1}^{2} p_{i2} \\ &= 1 \times (\frac{1}{2}\rho + \frac{1}{2}(1 - \rho)) + 2 \times (\frac{1}{2}\rho + \frac{1}{2}(1 - \rho)) \\ &= \frac{3}{2} \end{aligned}$$

$$Var(X) = \mathbb{E}(X^2) - (\mathbb{E}X)^2 = 1^2 \times \sum_{j=1}^{2} p_{1j} + 2^2 \times \sum_{j=1}^{2} p_{2j} - (\frac{3}{2})^2 = \frac{1}{4}$$

同理有 $Var(Y) = \frac{1}{4}$

(b)

$$
\begin{aligned}
Cov(X,Y) &= \mathbb{E}(XY) - \mathbb{E}(XY) \\
&= 1\times 1\times p_{11} + 1\times 2\times (p_{12}+p_{21}) + 2\times 2\times p_{22} - (\frac{3}{2})^2 \\
&= \frac{1}{2}\rho - \frac{1}{4}
\end{aligned}
$$

从而有

$$
\rho_{XY} = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = 2\rho - 1
$$

(c) 有

$$
\mathbb{E}(X) = x_1(\frac{1}{2}\rho + \frac{1}{2}(1-\rho)) + x_2(\frac{1}{2}\rho + \frac{1}{2}(1-\rho)) = \frac{1}{2}(x_1+x_2)
$$

同理 $\mathbb{E}(Y) = \frac{1}{2}(y_1+y_2)$, 则

$$
\begin{aligned}
Var(X) &= \mathbb{E}(X^2) - (\mathbb{E}(X))^2 \\
&= x_1^2(\frac{1}{2}\rho + \frac{1}{2}(1-\rho)) + x_2^2(\frac{1}{2}\rho + \frac{1}{2}(1-\rho)) - (\mathbb{E}(X))^2 \\
&= \frac{(x_1-x_2)^2}{4}
\end{aligned}
$$

同理 $Var(Y) = \frac{(y_1-y_2)^2}{4}$ 类似 (b) 题有

$$
\begin{aligned}
Cov(X,Y) &= \mathbb{E}(XY) - \mathbb{E}(XY) \\
&= x_1y_1p_{x_1y_1} + x_1y_2p_{x_1y_2} + x_2y_1p_{x_2y_1} + x_2y_2p_{x_2y_2} - \mathbb{E}(XY) \\
&= (\frac{\rho}{2} - \frac{1}{4})(x_1y_1 + x_2y_2 - x_1y_2 - x_2y_1)
\end{aligned}
$$

从而有

$$
\rho_{XY} = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} = 2\rho - 1
$$