2024 秋季本科时间序列

第 1 次作业答案

10 月 8 日

1. 利用协方差的对称双线性特征，展开 $\mathrm{var}(aX+Y)$

$$\begin{aligned}\mathrm{var}(aX+Y) &= \mathrm{cov}(aX+Y, aX+Y) \\ &= \mathrm{cov}(aX, aX) + \mathrm{cov}(aX, Y) + \mathrm{cov}(Y, aX) + \mathrm{cov}(Y, Y) \\ &= a^2\mathrm{var}(X) + 2a\mathrm{cov}(X, Y) + \mathrm{var}(Y)\end{aligned}$$

将其视作关于 a 的函数，令

$$f(a) = a^2\mathrm{var}(X) + 2a\mathrm{cov}(X, Y) + \mathrm{var}(Y)$$

由于对于任意实数 a，方差总是非负的，即 $\mathrm{var}(aX+Y) \geq 0$，所以函数 $f(a) \geq 0$ 对所有 $a \in \mathbb{R}$ 成立，即二次函数 $f(a)$ 的判别式为

$$\Delta = [2\mathrm{cov}(X, Y)]^2 - 4\mathrm{var}(X)\mathrm{var}(Y) = 4\left[\mathrm{cov}(X, Y)^2 - \mathrm{var}(X)\mathrm{var}(Y)\right] \leq 0$$

即

$$\mathrm{cov}(X, Y)^2 \leq \mathrm{var}(X)\mathrm{var}(Y)$$

相关系数的定义为

$$\rho_{XY} = \frac{\mathrm{cov}(X, Y)}{\sqrt{\mathrm{var}(X)}\sqrt{\mathrm{var}(Y)}}$$

所以

$$\rho_{XY}^2 = \left(\frac{\mathrm{cov}(X, Y)}{\sqrt{\mathrm{var}(X)}\sqrt{\mathrm{var}(Y)}}\right)^2 = \frac{\mathrm{cov}(X, Y)^2}{\mathrm{var}(X)\mathrm{var}(Y)} \leq 1$$

即

$$-1 \le \rho_{XY} \le 1$$

2. 定义中心化的样本向量 x,y 如下

$$\mathrm{x} = \begin{bmatrix} X_1 - \hat{\mu}_X \\ X_2 - \hat{\mu}_X \\ \vdots \\ X_N - \hat{\mu}_X \end{bmatrix}$$

$$\mathrm{y} = \begin{bmatrix} Y_1 - \hat{\mu}_Y \\ Y_2 - \hat{\mu}_Y \\ \vdots \\ Y_N - \hat{\mu}_Y \end{bmatrix}$$

因此，样本方差和协方差可以表示为向量内积的形式

$$\begin{aligned} \hat{\sigma}_X^2 &= \frac{1}{N-1}\mathrm{x}^\intercal \mathrm{x} \\ \hat{\sigma}_Y^2 &= \frac{1}{N-1}\mathrm{y}^\intercal \mathrm{y} \\ \hat{\sigma}_{XY} &= \frac{1}{N-1}\mathrm{x}^\intercal \mathrm{y} \\ &= \frac{1}{N-1}\mathrm{y}^\intercal \mathrm{x} \end{aligned}$$

与上题类似，考虑任意实数 a，考察线性组合 $a\boldsymbol{X} + \boldsymbol{Y}$ 的样本方差

$$\begin{aligned} \hat{\sigma}_{aX+Y}^2 &= \frac{1}{N-1}\sum_{i=1}^{N}(aX_i + Y_i - a\hat{\mu}_X - \hat{\mu}_Y)^2 \\ &= \frac{1}{N-1}(a\mathrm{x} + \mathrm{y})^\intercal(a\mathrm{x} + \mathrm{y}) \\ &= \frac{1}{N-1}\left(a^2\mathrm{x}^\intercal\mathrm{x} + 2a\mathrm{x}^\intercal\mathrm{y} + \mathrm{y}^\intercal\mathrm{y}\right) \\ &= a^2\hat{\sigma}_X^2 + 2a\hat{\sigma}_{XY} + \hat{\sigma}_Y^2 \end{aligned}$$

由于样本方差总是非负的，即

$$\hat{\sigma}_{aX+Y}^2 \ge 0, \quad \forall a \in \mathbb{R}$$

因此，关于 a 的二次函数

$$f(a) = a^2\hat{\sigma}_X^2 + 2a\hat{\sigma}_{XY} + \hat{\sigma}_Y^2 \geq 0, \quad \forall a \in \mathbb{R}$$

计算判别式 Δ

$$\begin{aligned}\Delta &= [2\hat{\sigma}_{XY}]^2 - 4\hat{\sigma}_X^2\hat{\sigma}_Y^2 \\ &= 4\hat{\sigma}_{XY}^2 - 4\hat{\sigma}_X^2\hat{\sigma}_Y^2 \\ &= 4(\hat{\sigma}_{XY}^2 - \hat{\sigma}_X^2\hat{\sigma}_Y^2) \leq 0\end{aligned}$$

即

$$\left(\frac{\hat{\sigma}_{XY}}{\hat{\sigma}_X\hat{\sigma}_Y}\right)^2 \leq 1$$

$$-1 \leq \frac{\hat{\sigma}_{XY}}{\hat{\sigma}_X\hat{\sigma}_Y} \leq 1$$

$$\therefore \hat{\rho}_{XY} \in [-1, 1]$$

3.

$$\because X \sim U\left[a, b\right], f_x(x) = \frac{1}{b-a}$$

$$\therefore \mathbb{E}(X) = \frac{a+b}{2}, \mathbb{E}(X^2) = \int_{-\infty}^{\infty} f(x)x^2 \mathrm{d}x = \frac{a^2+b^2+ab}{3}$$

由

$$\mathbb{E}(X) = \frac{a+b}{2} = \frac{1}{N}\sum_{i=1}^{N} x_i = \overline{x}$$

$$\mathbb{E}(X^2) = \frac{a^2+ab+b^2}{3} = \frac{1}{N}\sum_{i=1}^{N} x_i^2$$

可得

$$\hat{a} = \overline{x} - \sqrt{3} \times \sqrt{\frac{N-1}{N}}S$$

$$\hat{b} = \overline{x} + \sqrt{3} \times \sqrt{\frac{N-1}{N}}S$$

由 $\{X_t\}$ 是独立同分布序列可知

$$\bar{x} \xrightarrow{a.s.} \mathbb{E}(X) = \frac{a+b}{2}, S \xrightarrow{a.s.} \mathrm{var}(X) = \frac{(a-b)^2}{12}$$

故

$$\hat{a} \xrightarrow{a.s.} \mathbb{E}(X) - \sqrt{3}\mathrm{var}(X) = a, \hat{b} \xrightarrow{a.s.} \mathbb{E}(X) + \sqrt{3}\mathrm{var}(X) = b$$

故该矩估计具有一致性

4. 样本对应的似然函数为

$$L(\lambda|X) = \lambda^n e^{-\lambda \sum_{i=1}^n x_i}$$

两端同时取自然对数，可得

$$\ln L(\lambda) = n\ln(\lambda) - \lambda \sum_{i=1}^n x_i$$

等式两边取一阶导数，可得

$$\frac{\mathrm{d}\ln L(\lambda)}{\mathrm{d}\lambda} = \frac{n}{\lambda} - \sum_{i=1}^n x_i$$

等式两边取二阶导数，可得

$$\frac{\mathrm{d}^2\ln L(\lambda)}{\mathrm{d}\lambda^2} = -\frac{n}{\lambda^2} < 0$$

则似然函数在一阶导数等于 0 时取最大值

$$\frac{n}{\lambda} - \sum_{i=1}^n x_i = 0 \Longrightarrow \hat{\lambda} = \frac{n}{\sum_{i=1}^n x_i}$$

X 为指数分布，则

$$\mathbb{E}(x) = \int_{-\infty}^{\infty} x f(x)\mathrm{d}x = \int_{-\infty}^{\infty} x\lambda e^{-\lambda x}\mathrm{d}x = \frac{1}{\lambda} \Longrightarrow \lambda = \frac{1}{\mathbb{E}(X)}$$

由 $\{X_t\}$ 是独立同分布序列可知

$$\bar{x} \xrightarrow{a.s.} \mathbb{E}(X) \Longrightarrow \frac{1}{\bar{x}} \xrightarrow{a.s.} \frac{1}{\mathbb{E}(X)}$$

$$\hat{\lambda} = \frac{n}{\sum_{i=1}^n x_i} \xrightarrow{a.s.} \frac{1}{\mathbb{E}(X)} = \lambda$$

故该估计具有一致性

5. (a)

$$\mathbb{E}(\hat{S}_N^2) = \frac{1}{N}\mathbb{E}\left[\sum_{i=1}^N (X_i - \mu)^2)\right] = \frac{1}{N}\sum_{i=1}^N \mathbb{E}(X_i - \mu)^2 = \frac{1}{N} \times N \times \sigma^2 = \sigma^2$$

$$\therefore \mathbb{E}(\hat{S}_N^2) = \sigma^2$$

$$\lim_{N\to\infty} \hat{S}_N^2 = \lim_{N\to\infty} \left[\frac{1}{N}\sum_{i=1}^{N}(X_i-\mu)^2)\right] = \lim_{N\to\infty}\left[\frac{\sum_{i=1}^{N} X_i^2}{N} - 2\mu\frac{\sum_{i=1}^{N} X_i}{N} + \mu^2\right]$$

由大数定律可以得出

$$\frac{1}{N}\sum_{i=1}^{N} X_i \xrightarrow{a.s.} \mathbb{E}(X)=\mu, \frac{1}{N}\sum_{i=1}^{N} X_i^2 \xrightarrow{a.s.} \mathbb{E}(X^2)$$

$$\lim_{N\to\infty}\hat{S}_N^2 = \mathbb{E}(X^2)-\mu^2=\sigma^2 \Longrightarrow \hat{S}_N^2 \xrightarrow{a.s.} \sigma^2$$

(b)

$$\mathbb{E}(\hat{\sigma}_N^2) = \frac{1}{N-1}\mathbb{E}\left[\sum_{i=1}^{N}\left(X_i - \frac{1}{N}\sum_{i=1}^{N} X_i\right)^2\right] = \frac{1}{N-1}\mathbb{E}\left[\sum_{i=1}^{N} X_i^2 - 2\hat{\mu}_N\sum_{i=1}^{N} X_i + N\hat{\mu}_N^2\right]$$

$\mathbb{E}\left(\hat{\mu}_N\right) = \mathbb{E}\left(\frac{\sum X_i}{N}\right) = \frac{N\times\mu}{N} = \mu$

$\mathbb{E}(\hat{\mu}_N^2) = Var(\hat{\mu}_N) + [\mathbb{E}(\hat{\mu}_N)]^2 = \frac{1}{N^2}\sum_{i=1}^{N} Var(X_i) + \mu^2 = \frac{1}{N}\sigma^2 + \mu^2$

$$\begin{aligned}
\mathbb{E}(\hat{\sigma}_N^2) &= \frac{1}{N-1}\left[\mathbb{E}\left(\sum_{i=1}^{N} X_i^2\right) - 2N\mathbb{E}(\hat{\mu}_N^2) + N\mathbb{E}(\hat{\mu}_N^2)\right] \\
&= \frac{1}{N-1}\left[\sum_{i=1}^{N}\mathbb{E}(X_i^2) - N\mathbb{E}(\hat{\mu}_N^2)\right] \\
&= \frac{1}{N-1}\left[N(\sigma^2+\mu^2) - \sigma^2 - N\mu^2\right] \\
&= \frac{1}{N-1}\times(N-1)\times\sigma^2 \\
&= \sigma^2
\end{aligned}$$

$$\begin{aligned}
\lim_{N\to\infty}\hat{\sigma}_N^2 &= \lim_{N\to\infty}\frac{1}{N-1}\left(\sum_{i=1}^{N} X_i^2 - 2\hat{\mu}_N\sum_{i=1}^{N} X_i + N\hat{\mu}_N^2\right) \\
&= \lim_{N\to\infty}\frac{1}{N-1}\sum_{i=1}^{N}\left(X_i^2 - N\hat{\mu}_N^2\right) \\
&= \lim_{N\to\infty}\frac{N}{N-1}\frac{\sum_{i=1}^{N} X_i^2}{N} - \lim_{N\to\infty}\frac{N}{N-1}\hat{\mu}_N^2
\end{aligned}$$

由

$$\frac{\sum_{i=1}^{N} X_i^2}{N} \xrightarrow{a.s.} \mathbb{E}(X^2), \hat{\mu}_N^2 \xrightarrow{a.s.} \mu$$

可得

$$\lim_{N\to\infty}\hat{\sigma}_N^2 = \mathbb{E}(X^2)-\mu^2=\sigma^2$$