2023 秋季本科时间序列

第 9 次作业答案

12 月 17 日

1. (a) 若 $\mathbf{\Phi}$ 的特征值模长小于 1，则 $\boldsymbol{X}_t$ 平稳。

$$|\mathbf{\Phi}-\lambda\boldsymbol{I}| = \begin{vmatrix} 0.5-\lambda & 0.1 \\ 0.4 & 0.8-\lambda \end{vmatrix} = \lambda^2 - 1.3\lambda + 0.36 = 0$$

解得 $\lambda_1 = 0.9, \lambda_2 = 0.4$, 故满足平稳性要求。$\mathbb{E}\boldsymbol{X}_t = \boldsymbol{c} + \mathbf{\Phi}\mathbb{E}\boldsymbol{X}_{t-1}$, 则有

$$\mathbb{E}\boldsymbol{X}_t = (\boldsymbol{I}-\mathbf{\Phi})^{-1}\boldsymbol{c} = \frac{50}{3}\begin{bmatrix} 0.2 & 0.1 \\ 0.4 & 0.5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \end{bmatrix}$$

(b) 把 $\lambda_1 = 0.9$ 代入得，

$$(\mathbf{\Phi}-\lambda\boldsymbol{I})\boldsymbol{x}_1 = \begin{bmatrix} -0.4 & 0.1 \\ 0.4 & -0.1 \end{bmatrix}\boldsymbol{x}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

同理，当 $\lambda = 0.4$ 时，解得:

$$(\mathbf{\Phi}-\lambda\boldsymbol{I})\boldsymbol{x}_1 = \begin{bmatrix} 0.1 & 0.1 \\ 0.4 & 0.4 \end{bmatrix}\boldsymbol{x}_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \boldsymbol{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

故 $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix}, \boldsymbol{A}^{-1} = \begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{4}{5} & -\frac{1}{5} \end{bmatrix}$

$$\mathbf{\Phi}^i = \boldsymbol{A}\mathbf{\Lambda}^i\boldsymbol{A}^{-1} = \begin{bmatrix} 1 & 1 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} 0.9^i & 0 \\ 0 & 0.4^i \end{bmatrix}\begin{bmatrix} \frac{1}{5} & \frac{1}{5} \\ \frac{4}{5} & -\frac{1}{5} \end{bmatrix} = \begin{bmatrix} \frac{1}{5}\times 0.9^i + \frac{4}{5}\times 0.4^i & \frac{1}{5}\times 0.9^i - \frac{1}{5}\times 0.4^i \\ \frac{4}{5}\times 0.9^i - \frac{4}{5}\times 0.4^i & \frac{4}{5}\times 0.9^i - \frac{1}{5}\times 0.4^i \end{bmatrix}$$

$\boldsymbol{X}_t$ 的 $MA(\infty)$ 展开为 $\boldsymbol{X}_t = (\boldsymbol{I} + \boldsymbol{\Phi} + \cdots + \boldsymbol{\Phi}^n)\boldsymbol{c} + \sum_{i=0}^{n} \boldsymbol{\Phi}^i \epsilon_{t-i}, n \to \infty$，则

$$\begin{aligned}
\text{var}\boldsymbol{X}_t &= \sum_{i=0}^{\infty} \boldsymbol{\Phi}^i \boldsymbol{\Omega} \boldsymbol{\Phi}^{Ti} \\
&= \sum_{i=0}^{\infty} \boldsymbol{A}\boldsymbol{\Lambda}^i \boldsymbol{A}^{-1} \boldsymbol{\Omega} (\boldsymbol{A}^{-1})^T \boldsymbol{\Lambda}^{Ti} \boldsymbol{A}^T \\
&= \sum_{i=0}^{\infty} \begin{bmatrix} \frac{2}{5} \times 0.9^i + \frac{2}{5} \times 0.4^i & 0.9^i \\ \frac{8}{5} \times 0.9^i - \frac{2}{5} \times 0.4^i & 4 \times 0.9^i \end{bmatrix} \begin{bmatrix} \frac{1}{5} \times 0.9^i + \frac{4}{5} \times 0.4^i & \frac{4}{5} \times 0.9^i - \frac{4}{5} \times 0.4^i \\ \frac{1}{5} \times 0.9^i - \frac{1}{5} \times 0.4^i & \frac{4}{5} \times 0.9^i - \frac{1}{5} \times 0.4^i \end{bmatrix} \\
&= \begin{bmatrix} \frac{2575}{1064} & \frac{12525}{2128} \\ \frac{12525}{2128} & \frac{6025}{266} \end{bmatrix}
\end{aligned}$$

2. (a) 代码如下：

```r
data<- read.xlsx("./hw9_data.xlsx")
library(forecast)
library(tseries)
```

构造 GDP 季节同比增速序列 y

```r
y<-c()
for (i in 1:120){
y[i]<-(exp(data$logrealGDP_nipa[i+4]-data$logrealGDP_nipa[i])-1)*100
}
y_a<-auto.arima(y,d=0,D=0,ic='aic',method='ML')
y_b<-auto.arima(y,d=0,D=0,ic='bic',method='ML')
y_a
y_b
```

```
## Series: y
## ARIMA(2,0,1) with non-zero mean
##
## Coefficients:
##          ar1      ar2      ma1    mean
##       1.3733  -0.3867  -0.7852  8.8039
## s.e.  0.1411   0.1339   0.0989  2.1333
##
## sigma^2 = 3.681:  log likelihood = -247.21
## AIC=504.42   AICc=504.94   BIC=518.35
```

```
## Series: y
## ARIMA(1,0,1) with non-zero mean
##
## Coefficients:
##          ar1      ma1    mean
##       0.8990  -0.3012  8.8755
## s.e.  0.0584   0.1249  1.1296
##
## sigma^2 = 3.746:  log likelihood = -248.55
## AIC=505.09   AICc=505.44   BIC=516.24
```

AIC 准则下最优为 ARMA(2, 0, 1)，BIC 准则下最优为 ARMA(1, 0, 1)

(b) 代码如下：

```
pi<-c()
for (i in 21:121){
 pi[i-20]=(data$CPI[i+4]/data$CPI[i]-1)*100
}
pi_a<-auto.arima(pi,d=0,D=0,ic='aic',method='ML')
pi_b<-auto.arima(pi,d=0,D=0,ic='bic',method='ML')
pi_a
pi_b
```

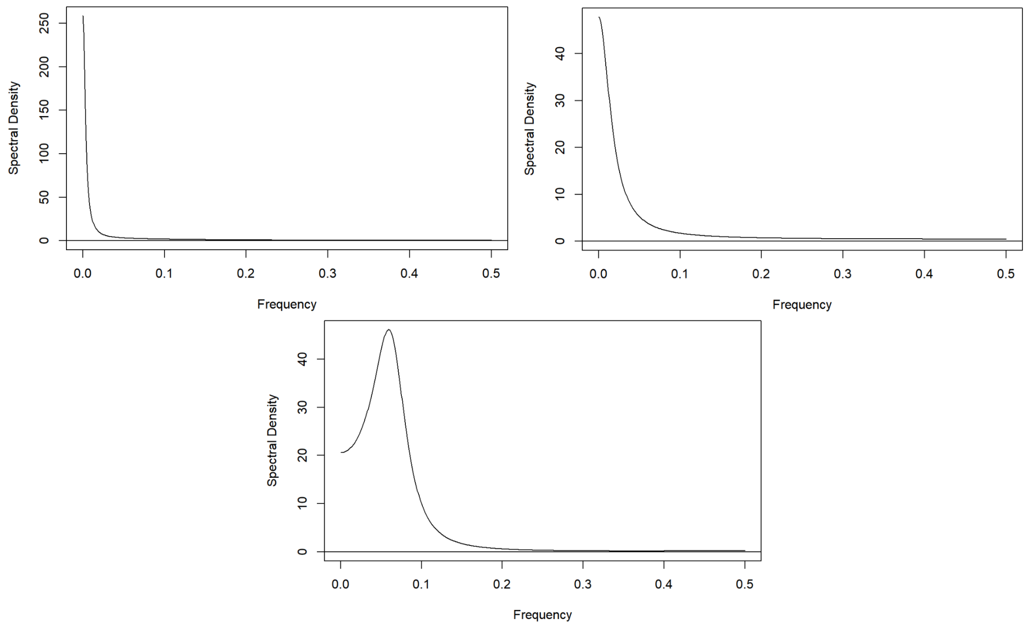
```
## Series: pi
## ARIMA(3,0,0) with non-zero mean
##
## Coefficients:
##          ar1      ar2      ar3    mean
##        1.170  -0.0999  -0.2905  1.9088
## s.e.   0.094   0.1497   0.0952  0.3463
##
## sigma^2 = 0.6173:  log likelihood = -118.07
## AIC=246.14   AICc=246.77   BIC=259.22
```

```
## Series: pi
## ARIMA(3,0,0) with non-zero mean
##
## Coefficients:
##          ar1      ar2      ar3    mean
##        1.170  -0.0999  -0.2905  1.9088
## s.e.   0.094   0.1497   0.0952  0.3463
##
## sigma^2 = 0.6173:  log likelihood = -118.07
## AIC=246.14   AICc=246.77   BIC=259.22
```

AIC 准则下最优为 ARMA(3, 0, 0)，BIC 准则下最优也为 ARMA(3, 0, 0)

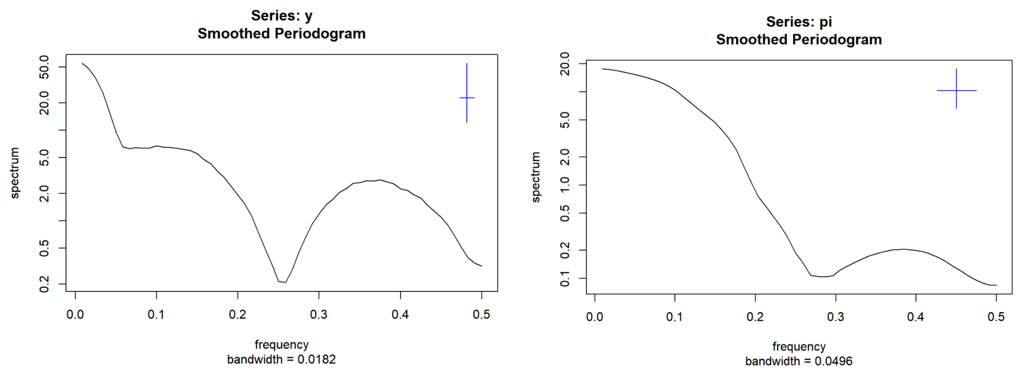
(c) 计算理论谱密度

```
ARMAspec(list(ar=c(y_a$coef[[1]],y_a$coef[[2]]),ma=y_a$coef[[3]]))
ARMAspec(list(ar=y_b$coef[[1]],ma=y_b$coef[[2]]))
ARMAspec(list(ar=c(pi_a$coef[[1]],pi_a$coef[[2]],pi_a$coef[[3]])))
```



估计样本谱密度

```
spec.pgram(y, kernel("daniell", c(1,3)),taper = 0.1)
spec.pgram(pi, kernel("daniell", c(5,7)),taper = 0.1)
```



比较理论谱密度与样本谱密度可知，两者存在较为明显的差别，因此前两问中估计所得 ARMA 模型不能充分刻画两个序列的周期波动特征。

(d) 对 y 建模

```
y_ols_a<-auto.arima(y,max.q=0,,d=0,D=0,ic='aic',method='CSS')
y_ols_b<-auto.arima(y,max.q=0,,d=0,D=0,ic='bic',method='CSS')
y_ols_a
```

```
y_ols_b
```

```
## Series: y
## ARIMA(2,0,0) with non-zero mean
##
## Coefficients:
##          ar1     ar2    mean
##       0.5905  0.2186  8.3106
## s.e.  0.0893  0.0889  0.9303
##
## sigma^2 = 3.637:  log likelihood = -247.23
```


```
## Series: y
## ARIMA(2,0,0) with non-zero mean
##
## Coefficients:
##          ar1     ar2    mean
##       0.5905  0.2186  8.3106
## s.e.  0.0893  0.0889  0.9303
##
## sigma^2 = 3.637:  log likelihood = -247.23
```


AIC 与 BIC 准则判断结果相同，均为 ARIMA(2, 0, 0)

```
y_ML<-arima(y,c(2,0,0),method = "ML")
y_ML
```

```
##
## Call:
## arima(x = y, order = c(2, 0, 0), method = "ML")
##
## Coefficients:
##          ar1     ar2  intercept
##       0.6132  0.2202     8.8886
## s.e.  0.0889  0.0908     0.9975
##
## sigma^2 estimated as 3.664:  log likelihood = -248.71,  aic = 503.42
```

对 π 建模

```
pi_ols_a<-auto.arima(pi,max.q=0,,d=0,D=0,ic='aic',method = 'CSS')
pi_ols_b<-auto.arima(pi,max.q=0,,d=0,D=0,ic='bic',method = 'CSS')
pi_ols_a
pi_ols_b
pi_ML<-arima(pi,c(3,0,0),method = "ML")
pi_ML
```

AIC 与 BIC 准则判断结果相同，均为 ARIMA(3, 0, 0)

```
## Series: pi
## ARIMA(3,0,0) with non-zero mean
##
## Coefficients:
##          ar1      ar2      ar3    mean
##       1.1566  -0.0905  -0.2906  2.0057
## s.e.  0.0947   0.1488   0.0940  0.3416
##
## sigma^2 = 0.599:  log likelihood = -116.91
```


```
## Series: pi
## ARIMA(3,0,0) with non-zero mean
##
## Coefficients:
##          ar1      ar2      ar3    mean
##       1.1566  -0.0905  -0.2906  2.0057
## s.e.  0.0947   0.1488   0.0940  0.3416
##
## sigma^2 = 0.599:  log likelihood = -116.91
```


```
## Call:
## arima(x = pi, order = c(3, 0, 0), method = "ML")
##
## Coefficients:
##         ar1      ar2      ar3  intercept
##       1.170  -0.0999  -0.2905     1.9088
## s.e.  0.094   0.1497   0.0952     0.3463
##
## sigma^2 estimated as 0.5929:  log likelihood = -118.07,  aic = 244.14
```

同模型的 OLS 估计与极大似然估计的结果基本相同

(e) 由样本谱密度图像可知，y 和 π 具有明显的高频波动特征，需要加入 MA 项捕捉高频波动特征。

(f)

```r
y2<-c()
for (i in 1:108){
 y2[i]<-(exp(data$logrealGDP_nipa[i+4]-data$logrealGDP_nipa[i])-1)*100
}
y2_a<-auto.arima(y,d=0,D=0,ic='aic',method='ML')
y2_b<-auto.arima(y,d=0,D=0,ic='bic',method='ML')
y2_a
y2_b
```

AIC 准则下最优为 ARMA(2, 0, 1)，BIC 准则下最优也为 ARMA(1, 0, 1)

```
## Series: y
## ARIMA(2,0,1) with non-zero mean
##
## Coefficients:
##          ar1      ar2      ma1    mean
##       1.3733  -0.3867  -0.7852  8.8039
## s.e.  0.1411   0.1339   0.0989  2.1333
##
## sigma^2 = 3.681:  log likelihood = -247.21
## AIC=504.42   AICc=504.94   BIC=518.35
```

```
## Series: y
## ARIMA(1,0,1) with non-zero mean
##
## Coefficients:
##          ar1      ma1    mean
##       0.8990  -0.3012  8.8755
## s.e.  0.0584   0.1249  1.1296
##
## sigma^2 = 3.746:  log likelihood = -248.55
## AIC=505.09   AICc=505.44   BIC=516.24
```

```r
pi2<-c()
for (i in 21:108){
 pi2[i-20]=(data$CPI[i+4]/data$CPI[i]-1)*100
}
pi2_a<-auto.arima(pi2,d=0,D=0,ic='aic',method='ML')
pi2_b<-auto.arima(pi2,d=0,D=0,ic='bic',method='ML')
pi2_a
pi2_b
```

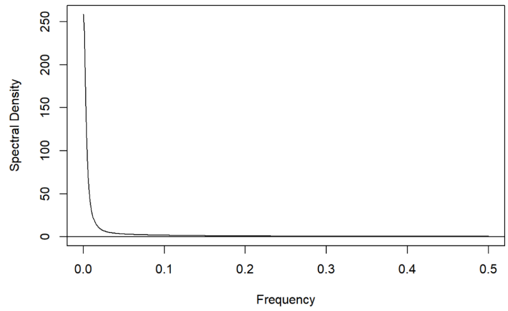
AIC 与 BIC 准则判断结果相同，均为 ARIMA(3, 0, 0)

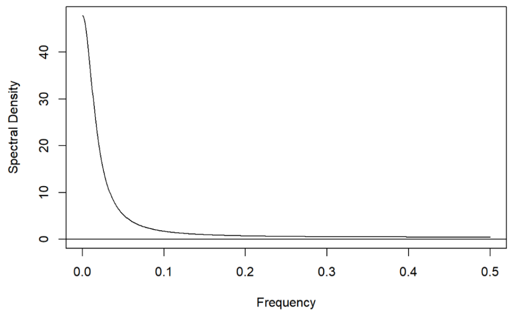
```
## Series: pi2
## ARIMA(3,0,0) with non-zero mean
##
## Coefficients:
##          ar1      ar2      ar3    mean
##       1.2381  -0.1529  -0.2839  2.0411
## s.e.  0.1028   0.1688   0.1040  0.3912
##
## sigma^2 = 0.5601:  log likelihood = -98.63
## AIC=207.26   AICc=207.99   BIC=219.65
```

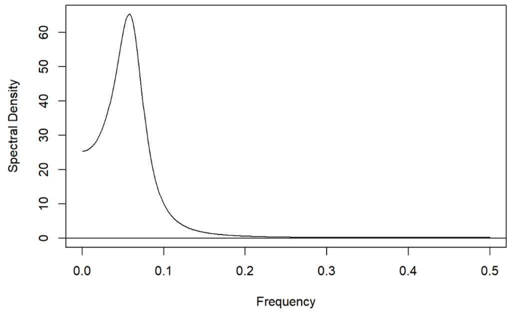
```
## Series: pi2
## ARIMA(3,0,0) with non-zero mean
##
## Coefficients:
##          ar1      ar2      ar3    mean
##       1.2381  -0.1529  -0.2839  2.0411
## s.e.  0.1028   0.1688   0.1040  0.3912
##
## sigma^2 = 0.5601:  log likelihood = -98.63
## AIC=207.26   AICc=207.99   BIC=219.65
```

比较理论和样本谱密度函数

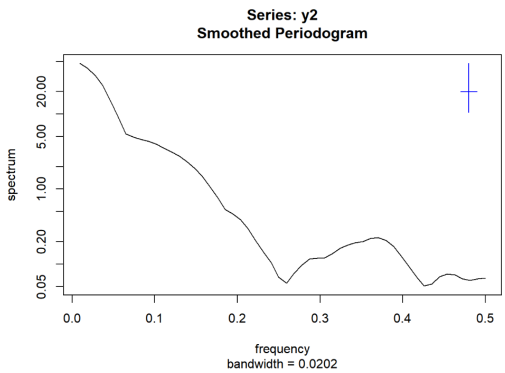
```r
ARMAspec(list(ar=c(y2_a$coef[[1]],y2_a$coef[[2]]),ma=y2_a$coef[[3]]))
ARMAspec(list(ar=y2_b$coef[[1]],ma=y2_b$coef[[2]]))
ARMAspec(list(ar=c(pi2_a$coef[[1]],pi2_a$coef[[2]],pi2_a$coef[[3]])))
```

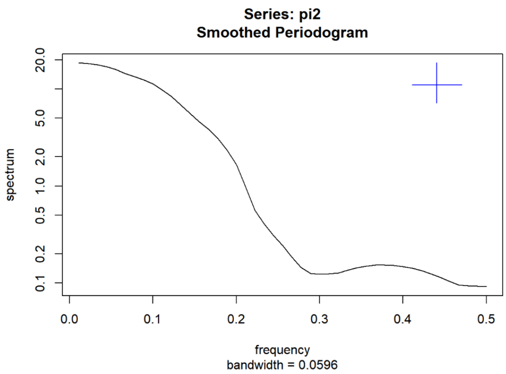







```
spec.pgram(y2, kernel("daniell", c(1,3)),taper = 0.1)
spec.pgram(pi2, kernel("daniell", c(5,7)),taper = 0.1)
```





此时理论与样本谱密度虽仍略有差异，但相比纳入疫情期间数据的情况更为相近

```
y2_ols_a<-auto.arima(y2,max.q=0,,d=0,D=0,ic='aic',method = 'CSS')
y2_ols_b<-auto.arima(y2,max.q=0,,d=0,D=0,ic='bic',method = 'CSS')
y2_ols_a
y2_ols_b
y2_ML<-arima(y2,c(2,0,0),method = "ML")
y2_ML
```

```
## Series: y2
## ARIMA(2,0,0) with non-zero mean
##
## Coefficients:
##          ar1      ar2    mean
##       1.2199  -0.2999  8.8127
## s.e.  0.0915   0.0898  0.8703
##
## sigma^2 = 0.492:  log likelihood = -114.44
```


```
## Series: y2
## ARIMA(2,0,0) with non-zero mean
##
## Coefficients:
##          ar1      ar2    mean
##       1.2199  -0.2999  8.8127
## s.e.  0.0915   0.0898  0.8703
##
## sigma^2 = 0.492:  log likelihood = -114.44
```


```
## Call:
## arima(x = y2, order = c(2, 0, 0), method = "ML")
##
## Coefficients:
##          ar1      ar2  intercept
##       1.2569  -0.3155     9.6406
## s.e.  0.0914   0.0933     1.0736
##
## sigma^2 estimated as 0.5057:  log likelihood = -117.76,  aic = 241.51
```

```
pi2_ols_a<-auto.arima(pi2,max.q=0,,d=0,D=0,ic='aic',method = 'CSS')
pi2_ols_b<-auto.arima(pi2,max.q=0,,d=0,D=0,ic='bic',method = 'CSS')
pi2_ols_a
pi2_ols_b
pi2_ML<-arima(pi2,c(3,0,0),method = "ML")
pi2_ML
```

```
## Series: pi2
## ARIMA(3,0,0) with non-zero mean
##
## Coefficients:
##          ar1      ar2      ar3    mean
##       1.2237  -0.1419  -0.2838  2.1540
## s.e.  0.1039   0.1677   0.1027  0.3894
##
## sigma^2 = 0.5416:  log likelihood = -97.37
```


```
## Series: pi2
## ARIMA(3,0,0) with non-zero mean
##
## Coefficients:
##          ar1      ar2      ar3    mean
##       1.2237  -0.1419  -0.2838  2.1540
## s.e.  0.1039   0.1677   0.1027  0.3894
##
## sigma^2 = 0.5416:  log likelihood = -97.37
```


```
## Call:
## arima(x = pi2, order = c(3, 0, 0), method = "ML")
##
## Coefficients:
##          ar1      ar2      ar3  intercept
##       1.2381  -0.1529  -0.2839     2.0411
## s.e.  0.1028   0.1688   0.1040     0.3912
##
## sigma^2 estimated as 0.5346:  log likelihood = -98.63,  aic = 205.26
```

纳入疫情期间的数据样本并不合理，因为疫情以及后续防疫政策等外生冲击显著改变 DGP，使得基于此前情况的模型不适合描述包括疫情期间样本的整体数据特征。