2023 秋季本科时间序列

第 8 次作业答案

12 月 3 日

1. 用数学归纳法证明, 当 $n \geq 1$,

$$J^n = \begin{bmatrix} \lambda^n & n\lambda^{n-1} & \frac{n(n-1)}{2}\lambda^{n-2} \\ 0 & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & \lambda^n \end{bmatrix}$$

由题已知，当 $n = 1$ 时，

$$J^1 = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

假设当 $n = k$ 时，

$$J^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{k(k-1)}{2}\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix}$$

则

$$J^{k+1} = J^k \times J = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \frac{k(k-1)}{2}\lambda^{k-2} \\ 0 & \lambda^k & k\lambda^{k-1} \\ 0 & 0 & \lambda^k \end{bmatrix} \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda^{k+1} & (k+1)\lambda^k & \frac{k(k+1)}{2}\lambda^{k-1} \\ 0 & \lambda^{k+1} & (k+1)\lambda^k \\ 0 & 0 & \lambda^{k+1} \end{bmatrix}$$

故得证

2. $A(\mathscr{L}) = I - \Phi\mathscr{L} = \begin{bmatrix} 1-\frac{1}{2}\mathscr{L} & -\frac{1}{2}\mathscr{L} \\ -\frac{1}{9}\mathscr{L} & 1-\frac{1}{3}\mathscr{L} \end{bmatrix}$，$A(z) = \begin{bmatrix} 1-\frac{1}{2}z & -\frac{1}{2}z \\ -\frac{1}{9}z & 1-\frac{1}{3}z \end{bmatrix}$

则

$$\det A(z) = (1-\frac{1}{2}z)(1-\frac{1}{3}z) - \frac{1}{18}z^2 = 1 - \frac{5}{6}z + \frac{1}{9}z^2$$

令 $\det A(z) = 0$，得 $z_1 = 6$，$z_2 = \frac{3}{2}$

由 $|z_1| > 1$，$|z_2| > 1$，

$$A^{-1}(z) = \frac{1}{\det A(z)} \times A^*(z) = \frac{1}{(1-\frac{1}{6}z)(1-\frac{2}{3}z)} \begin{bmatrix} 1-\frac{1}{3}z & \frac{1}{2}z \\ \frac{1}{9}z & 1-\frac{1}{2}z \end{bmatrix}$$

其级数展开表达式为：

$$A^{-1}(z) = \sum_{i=0}^{\infty}(\frac{1}{6})^i z^i \sum_{j=0}^{\infty}(\frac{2}{3})^j z^j \begin{bmatrix} 1-\frac{1}{3}z & \frac{1}{2}z \\ \frac{1}{9}z & 1-\frac{1}{2}z \end{bmatrix}$$