2023 秋季本科时间序列

第 3 次作业答案

10 月 24 日

1. 由题已知，$U \sim U([-\pi, \pi])$，$X_t = \cos(\pi t + U)$

   则

$$\mathbb{E}(X_t) = \frac{1}{T}\sum_{t=1}^{T} X_t = \frac{1}{T}\sum_{t=1}^{T} \cos(\pi t + U) = 0$$

   由 $\sigma_k^2 = \operatorname{cov}(X_{t+k}, X_t)$ 可得

$$k = 0, \operatorname{cov}(X_t, X_t) = \operatorname{var}(X_t)$$
$$k = 1, \operatorname{cov}(X_{t+1}, X_t) = -\operatorname{var}(X_t)$$
$$k = 2, \operatorname{cov}(X_{t+2}, X_t) = \operatorname{var}(X_t)$$
$$k = 3, \operatorname{cov}(X_{t+3}, X_t) = -\operatorname{var}(X_t)$$

   故

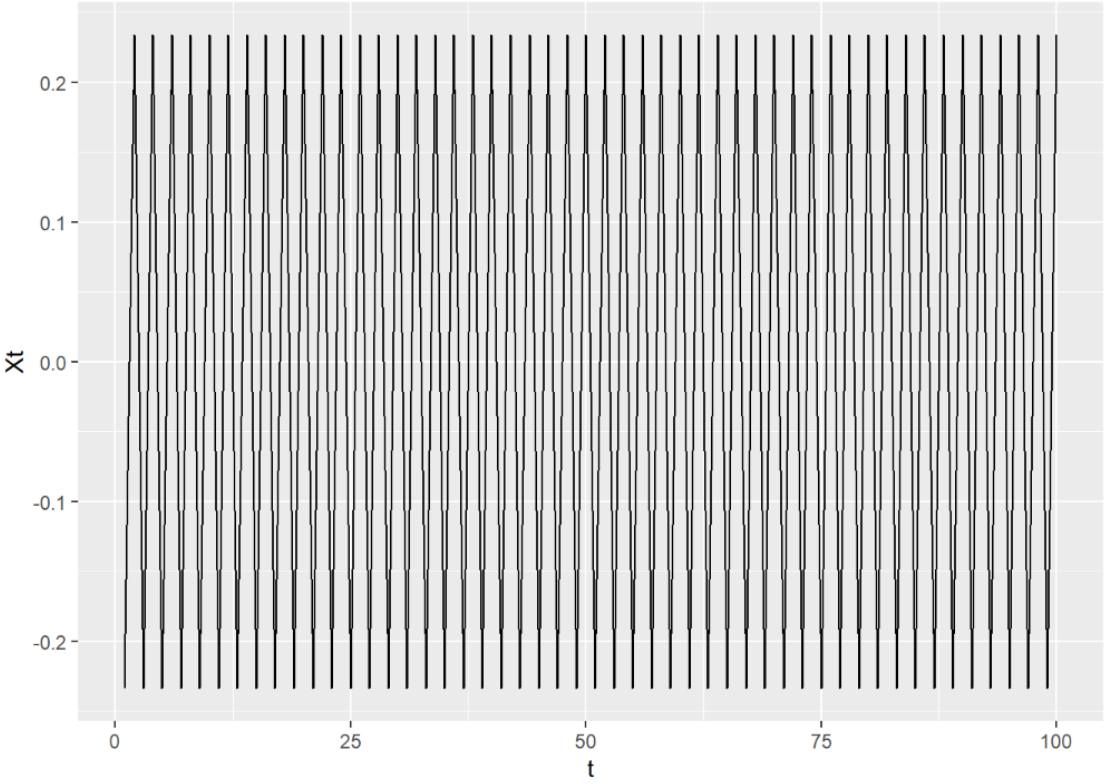
$$\sigma_k^2 = (-1)^k \operatorname{var}(X_t)$$
$$\operatorname{var}(X_t) = \int_{-\pi}^{\pi} \frac{1}{2\pi} \cos^2 U \mathrm{d}U = \frac{1}{2}$$
$$\sigma_k^2 = \frac{1}{2}(-1)^k$$

   故 σ_k^2 的周期为 2

   代码如下：

```
set.seed(123)
Xt <- 1:100
U <- runif(1,-pi,pi)
for (t in 1:100){
Xt[t] <- cos(pi*t+U)
}
X <- tibble(t = 1:100, Xt = Xt)
X
```

```r
ggplot(X, aes(x = t, y = Xt)) +
geom_line()
```



2. (a) 由 $X_t = 0.1 + 0.9X_{t-1} + \varepsilon_t$ 可得

$$X_0 = 0$$

$$X_1 = 0.1 + 0.9X_0 + \varepsilon_1$$

$$X_2 = 0.1 + 0.9X_1 + \varepsilon_2 = 0.1 + 0.9(0.1 + 0.9X_0 + \varepsilon_1) + \varepsilon_2$$

$$\cdots$$

$$X_t = 0.1 + 0.9X_{t-1} + \varepsilon_t = 0.1 \times (\sum 0.9^{t-1}) + \sum \varepsilon_t 0.9^{T-t}$$

则 $\mathbb{E}(X) = 1 - 0.9^t$，$\text{var}(X) = \frac{1-0.81^t}{1-0.81}$

当 $t \to \infty$，$\mathbb{E}(X) = 1$，$\text{var}(X) = \frac{100}{19}$

由 $X_t \sim N(1, \frac{100}{19})$ 易得 $X_{t+1} \sim N(1, \frac{100}{19})$

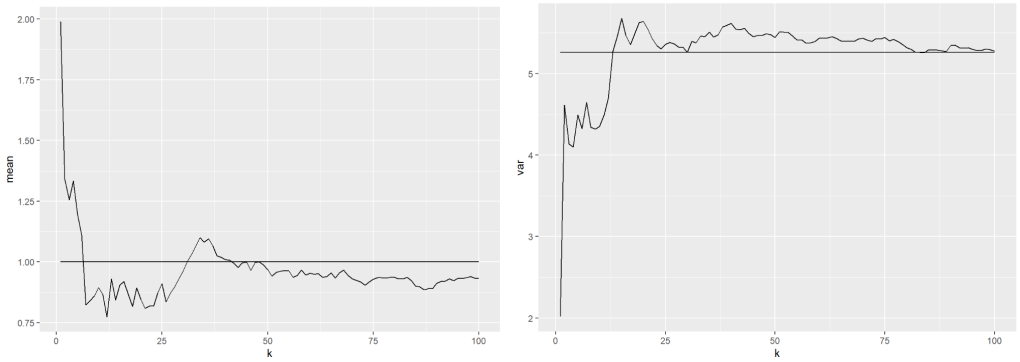
故 $N(1, \frac{100}{19})$ 是 X_t 的平稳分布

(b) 由下图可见，样本均值和样本方差分别收敛至期望 μ 和均值 σ^2

代码如下：

```r
set.seed(1)
n <- 10000
```

```r
x <- numeric(n)
x[1] <- rnorm(1, mean=1, sd=1/sqrt(0.19))
e <- rnorm(n)
for(t in 2:n){
 x[t] <- 0.1 + 0.9*x[t-1] + e[t]
}

mean_x <- numeric(100)
var_x <- numeric(100)
for(k in 1:100){
 mean_x[k] = mean(x[1:(100*k)])
 var_x[k] = var(x[1:(100*k)])
}

tbl <- tibble(k = 1:100,mean = mean_x,var = var_x)
tbl %>%
ggplot(aes(x = k,y = mean))+
geom_line()+
geom_line(aes(y = 1))

tbl %>%
ggplot(aes(x = k,y = var))+
geom_line()+
geom_line(aes(y = 100/19))
```



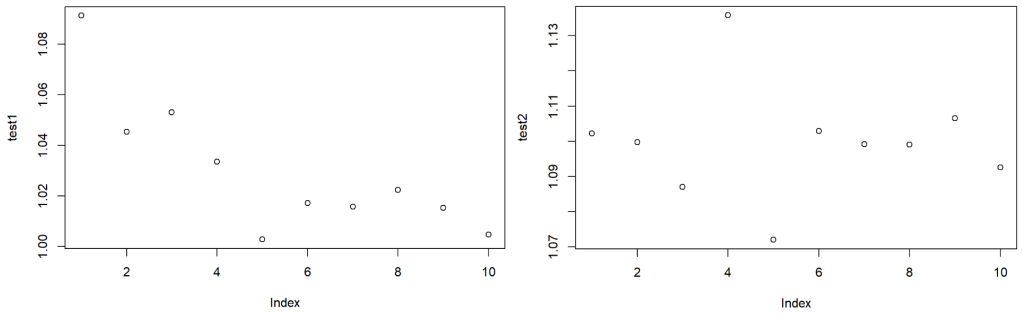
(c) i. 由题可得

$$
\begin{aligned}
\sigma_T^2 &= \frac{1}{T}\sum(X_t - \hat{\mu_T})^2 \\
&= \frac{1}{T}(\sum X_t^2 + T\hat{\mu}_T^2 - 2\hat{\mu}_T \sum X_t) \\
&= \frac{1}{T}\sum X_t^2 - \hat{\mu}_T^2 \\
\mathbb{E}(\sigma_T^2) &= \mathbb{E}(\frac{1}{T}\sum X_t^2 - \hat{\mu}_T^2) \\
&= \frac{1}{T}\sum \mathbb{E}(X_t^2) - \mathbb{E}(\hat{\mu}_T^2) \\
&= \frac{1}{T}\sum (\mathbb{E}^2(X_t) + \mathrm{var}(X_t)) - \mathbb{E}(\hat{\mu}_T^2) \\
&= \frac{1}{T}\sum (\mu^2 + \sigma^2) - E(\hat{\mu}_T^2) \\
&= \mu^2 + \sigma^2 - E(\hat{\mu}_T^2)
\end{aligned}
$$

ii. 两者不相等，代码如下：

```
mu_hat <- numeric(1000)

for(k in 1:1000){
 x <- numeric(1000)
 x[1] <- rnorm(1, mean=1, sd=1/sqrt(0.19))
 e <- rnorm(1000)
 for(t in 2:1000){
  x[t] <- 0.1 + 0.9*x[t-1] + e[t]
 }
 mu_hat[k] = mean(x)
}

mean(mu_hat^2)

test1 <- numeric(10)
for(i in 1:10){
 test1[i]=simu(rep = 1000,n=1000*i)
}
plot(test1)

test2 <- numeric(10)
for(i in 1:10){
 test2[i]=simu(rep = 1000*i,n=1000)
```

```
}
plot(test2)
```



3. ρ_1 和 ρ_2 下，X_t 平稳性存在差异，$T = 100$ 时代码如下：

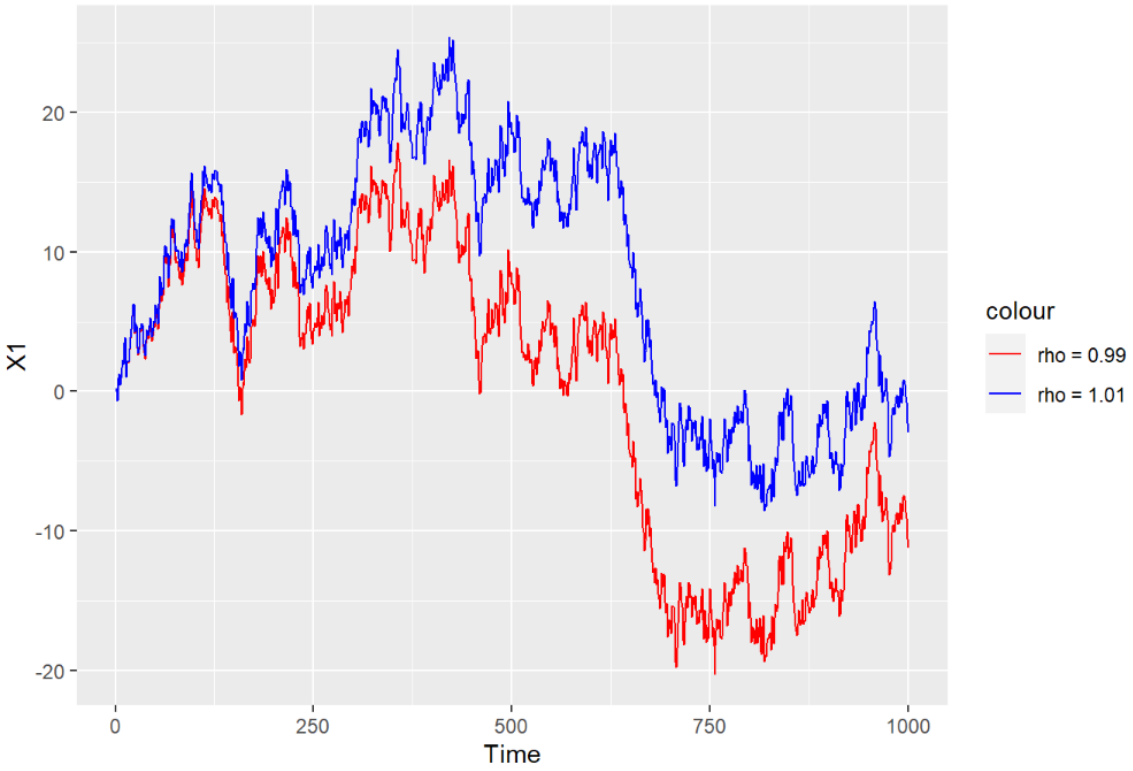
```
set.seed(1)
t <- 100
e <- rnorm(t)
r_1 <- 0.999
r_2 <- 1.001
x_1 <- e[1]

x_1 <- numeric(t)
for(i in 2:t){
 x_1[i] = r_1*x_1[i-1]+e[i]
}

x_2 <- numeric(t)
for(i in 2:t){
 x_2[i] = r_2*x_2[i-1]+e[i]
}

tbl <- tibble(Time = 1:t,X1 = x_1,X2 = x_2)
tbl %>%
ggplot(aes(x = Time))+
geom_line(aes(y = X1,color = "X1"))+
geom_line(aes(y = X2,color = "X2"))+
scale_color_manual(values = c("X1"="red","X2"="blue"),
label = c("X1"="rho␣=␣0.99","X2"= "rho␣=␣1.01"))+
```

```
labs(title = "T=100")
```

$T = 1000$ 时代码如下：

```
set.seed(1)
t <- 1000
e <- rnorm(t)
r_1 <- 0.999
r_2 <- 1.001
x_1 <- e[1]

x_1 <- numeric(t)
for(i in 2:t){
 x_1[i] = r_1*x_1[i-1]+e[i]
}

x_2 <- numeric(t)
for(i in 2:t){
 x_2[i] = r_2*x_2[i-1]+e[i]
}
```

```r

tbl <- tibble(Time = 1:t,X1 = x_1,X2 = x_2)
tbl %>%
ggplot(aes(x = Time))+
geom_line(aes(y = X1,color = "X1"))+
geom_line(aes(y = X2,color = "X2"))+
scale_color_manual(values = c("X1"="red","X2"="blue"),
label = c("X1"="rho = 0.99","X2"= "rho = 1.01"))+
labs(title = "T=1000")
```



$T = 10000$ 时代码如下：

```r
set.seed(1)
t <- 10000
e <- rnorm(t)
r_1 <- 0.999
r_2 <- 1.001
x_1 <- e[1]

x_1 <- numeric(t)
```

```
for(i in 2:t){
 x_1[i] = r_1*x_1[i-1]+e[i]
}

x_2 <- numeric(t)
for(i in 2:t){
 x_2[i] = r_2*x_2[i-1]+e[i]
}

tbl <- tibble(Time = 1:t,X1 = x_1,X2 = x_2)
tbl %>%
ggplot(aes(x = Time))+
geom_line(aes(y = X1,color = "X1"))+
geom_line(aes(y = X2,color = "X2"))+
scale_color_manual(values = c("X1"="red","X2"="blue"),
label = c("X1"="rho␣=␣0.99","X2"= "rho␣=␣1.01"))+
labs(title = "T=10000")
```