2021 秋季本科时间序列

第 3 次作业答案

10 月 27 日

1. 极坐标形式下 $z = |z|e^{\mathrm{i}\theta}$, $\bar{z} = |z|e^{-\mathrm{i}\theta}$

$$\overline{z^a} = \overline{(|z|e^{\mathrm{i}\theta})^a} = \overline{|z|^a e^{a\mathrm{i}\theta}} = |z|^a e^{-a\mathrm{i}\theta}$$

$$\bar{z}^a = (|z|e^{-\mathrm{i}\theta})^a = |z|^a e^{-a\mathrm{i}\theta}$$

故得证 $\overline{z^a} = \bar{z}^a$

2. 根据滞后算子性质可得 $X_t = (1-\rho\mathscr{L})^{-1}Y_t = \sum_{i=0}^{\infty}(\rho^i\mathscr{L}^i)Y_t = \sum_{i=0}^{\infty}(\rho^i)Y_{t-i}$

则 X_t 期望为 $\mu_X = \mathbb{E}(X_t) = \mathbb{E}(\sum_{i=0}^{\infty}\rho^i Y_{t-i}) = \sum_{i=0}^{\infty}\rho^i\mathbb{E}(Y_{t-i}) = \sum_{i=0}^{\infty}\rho^i\mu_Y = \frac{\mu_Y}{1-\rho}$

X_t 自协方差为:

$$\begin{aligned}
\gamma_X(k) &= \mathrm{cov}(X_t, X_{t-k}) \\
&= \mathrm{cov}\left(\sum_{i=0}^{\infty}\rho^i Y_{t-i}, \sum_{j=0}^{\infty}\rho^j Y_{t-k-j}\right) \\
&= \sum_{i=0}^{\infty}\rho^i\sum_{j=0}^{\infty}\rho^j\mathrm{cov}(Y_{t-i}, Y_{t-k-j}) \\
&= \sum_{i=0}^{\infty}\rho^i\sum_{j=0}^{\infty}\rho^j\gamma_Y(j+k-i)
\end{aligned}$$

由 $|\gamma_Y(k)| \le \gamma_Y(0)$ 可得

$$\gamma_X(k) = \sum_{i=0}^{\infty}\rho^i\sum_{j=0}^{\infty}\rho^j\gamma_Y(j+k-i) \le \sum_{i=0}^{\infty}\rho^i\sum_{j=0}^{\infty}\rho^j\gamma_Y(0) = \frac{\gamma_Y(0)}{(1-\rho)^2}$$

$\{Y_t\}$ 为协方差平稳过程, 即 $\mu_Y, \gamma_Y(j+k-i)$ 与 t 无关

则 $\mu_X, \gamma_X(k)$ 也与 t 无关

又因为 $\mu_X, \gamma_X(k)$ 存在

故得证 $X_t = (1-\rho\mathscr{L})^{-1}Y_t$ 为协方差平稳过程。

3. $Y_t = \sum_{j=-k}^{k} \frac{1}{2k+1} X_{t-j} = \sum_{j=-k}^{k} \frac{1}{2k+1} \mathscr{L}^j X_t$

由滤波多项式可得 $C(\mathscr{L}) = \sum_{j=-k}^{k} \frac{1}{2k+1} \mathscr{L}^j$

$G_k(\omega) = \sqrt{(C(e^{\mathrm{i}\omega})C(e^{-\mathrm{i}\omega})}$, 其中

$$C(e^{\mathrm{i}\omega})C(e^{-\mathrm{i}\omega}) = \sum_{p=-k}^{k} \frac{1}{2k+1} e^{p\mathrm{i}\omega} \sum_{q=-k}^{k} \frac{1}{2k+1} e^{-q\mathrm{i}\omega}$$

利用欧拉公式 $e^{\mathrm{i}\theta} = \cos\theta + i\sin\theta$

$$C(e^{\mathrm{i}\omega})C(e^{-\mathrm{i}\omega}) = (\frac{1}{2k+1})^2 \sum_{p=-k}^{k} [\cos(p\omega) + i\sin(p\omega)] \sum_{q=-k}^{k} [\cos(q\omega) - i\sin(q\omega)]$$

又因为 $\cos\theta$ 为偶函数，$\sin\theta$ 为奇函数

$$C(e^{\mathrm{i}\omega})C(e^{-\mathrm{i}\omega}) = (\frac{1}{2k+1})^2 \sum_{p=-k}^{k} \cos(p\omega) \sum_{q=-k}^{k} \cos(q\omega) = (\frac{1}{2k+1})^2 [\sum_{p=-k}^{k} \cos(p\omega)]^2$$

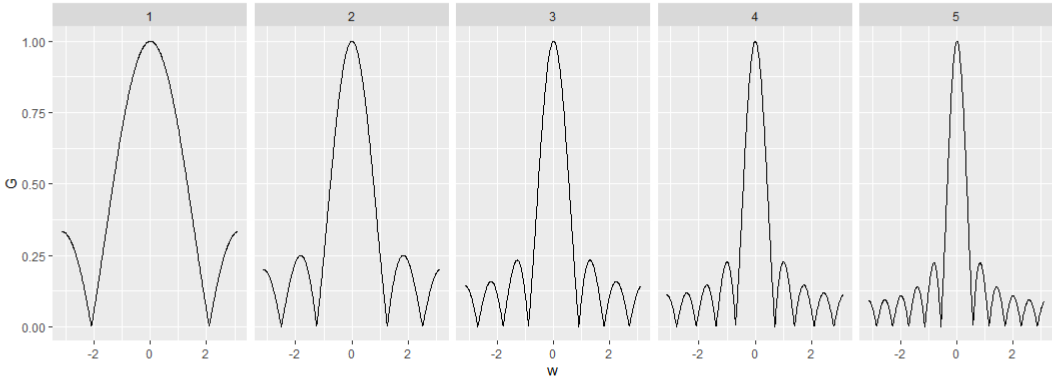
因此 $G_k(\omega) = \frac{1}{2k+1} |\sum_{p=-k}^{k} \cos(p\omega)|$, G_k 随着 k 的增加而减小

代码如下：

```r
library(tidyverse)
num<-matrix(0,3145,3)
i=1
for (k in 1:5)
  for (w in seq(-pi,pi,0.01))
    m=0
    for (j in 1:k){ m=m+cos(w*j)}
    G=abs(2*m+1)/(2*k+1)
    num[i,]=c(k,w,G)
    i=i+1

colnames(num)<-c("k","w","G")
df<-data.frame(num)
df$k<-factor(df$k)

ggplot(df)+
geom_line(aes(x=w,y=G))+
facet_wrap(~k,ncol=5)
```



1
2
3
4
5
1.00
0.75
0.50
0.25
0.00
G
-2
0
2
w