On Positive Autocorrelations of a Markov Chain*

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Abstract

We provide two sufficient conditions for a finite state Markov chain to have positive stationary first-order autocorrelation. The first condition requires all diagonal entries of the transition matrix to be greater than one half. The second requires the transition probability to satisfy a monotonicity property, which is a particular case of a result for Markov processes with more general state space.

I Introduction

In most dynamic economic analyses, it is central to study the effects of stochastic shocks to the economic system under consideration, and it is almost customary for researchers to consider the case where these shocks are persistent over time, as long as doing so is technically feasible. Researchers often model a shock within a stationary environment by an ergodic Markov chain, which allows for a variety of patterns of serial dependence. Consequently, the stationary autocorrelations of a Markov chain, especially the first-order one, provide a convenient summary of the persistence of a shock. Since the persistence of a shock is typically captured by a positive (first-order) autocorrelation, it is of interest to know when will a Markov chain possesses such a feature. Knowing this is helpful in guiding researchers to choose an appropriate Markov chain for their modeling purposes.

We shall provide two simple sufficient conditions on the transition matrix of a finite state ergodic Markov chain such that the first-order autocorrelation ρ under the stationary distribution is positive. Let $\{X_t\}$ be a finite state ergodic Markov chain, with $\mathbf{x} = (x_1, \ldots, x_n)'$ and $\mathbf{P} = (p_{ij})_{1\le i,j\le n}$ denoting the state space vector and the transition matrix. The first condition requires that the diagonal entries of the transition matrix to be greater than one half, which we call diagonal dominance condition. This condition captures one aspect of the notion of persistence, namely that

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the current state of a shock tends to persist into the next period. The mathematical interpretation is $p_{ii} = \mathbb{P}(X_{t+1} = x_i|X_t = x_i) > \mathbb{P}(X_{t+1} \neq x_i|X_t = x_i) = \sum_{j\neq i} p_{ij}$, or equivalently $p_{ii} > \frac{1}{2}$. The second condition requires that the cumulative transition distributions satisfy a monotonicity property with respect to the current states, i.e., $F_{i,j} \geq F_{i+1,j}\ \forall i, j$ where $F_{i,j} = \sum_{k\leq j} p_{i,k}$ and $\{x_1, \ldots, x_n\}$ is such that $x_i < x_{i+1}$. This condition capture another aspect of the notion of persistence, namely that whenever the current state becomes higher then the expected value of the next period's state should also be higher. Mathematically this means that the conditional expectation $\mathbb{E}(X_{t+1}|X_t = x)$ should be increasing in x, which in turn is an implication of the monotonicity condition.

It is nontrivial to characterize the dependence of ρ on the primitives of a Markov chain, especially on $\mathbf{P}$. Because the calculation of ρ is greatly complicated by the need for calculating first the stationary distribution $\mathbf{u} = (u_1, \ldots, u_n)'$, a highly nonlinear function of $\mathbf{P}$. Nonetheless, the intuition that most probabilistic properties of a Markov chain—including persistence/autocorrelation as we would argue—are determined mainly by $\mathbf{u}$ and $\mathbf{P}$, hence $\mathbf{P}$ ultimately, suggests that certain conditions on $\mathbf{P}$ alone should imply positive ρ. The two sufficient conditions developed in this paper, both of which are imposed on $\mathbf{P}$ exclusively, confirm this intuition.

To prove the sufficiency of the diagonal dominance condition, we first show that ρ is closely related to the eigenvalues of a particular matrix consisting of $\mathbf{u}$ and $\mathbf{P}$ and preserving the diagonal dominance property of $\mathbf{P}$. Relying on powerful tools from matrix theory, notably the method of Geršgorin disks for bounding the eigenvalues of a diagonally dominant non-negative matrix, we am able to give a simple lower bound of ρ. The sufficiency of the diagonal dominance conditio then follows easily. Furthermore, some fine-tuning results in matrix theory allow me to further weaken the condition into a weaker form, which requires $p_{ii} > \frac{1}{2}$ for only one i while $p_{jj} = \frac{1}{2}$ for all $j \neq i$. Along the way, we also develop simple upper bounds for ρ; numerical examples show that both the lower and upper bounds are tight. Despite drawing on existing results in matrix theory, to my best knowledge, this paper is the first one to give bounds on ρ based on $\mathbf{P}$, and consequently uncover the diagonal dominance condition for $\rho > 0$. The closest paper in this strand of literature to my work is Diaconis and Stroock (1991), in which the authors develop bounds on functions that are closely related to ρ of a Markov chain.[1] However, their techniques only apply to Markov chains that are reversible, whereas we impose no such assumption on $\mathbf{P}$ a priori.

The proof of the sufficiency of the monotonicity condition rests on a simple lemma: $\mathrm{cov}(X_t, X_{t+1}) > 0$ if $\mathbb{E}(X_{t+1}|X_t = x)$ is increasing in x. It turns out that this is a general result applying for a Markov process with continuous state space like $\mathbb{R}^1$. Correspondingly, a general version of the monotonicity condition in the transition distribution, $F(y|x) = \mathbb{P}(X_{t+1} \leq y|X_t = x)$, which states that

[1]Bounding eigenvalues of a non-negative matrix is a classical topic in matrix theory, known as spectral localization problem. There is a large literature on this topic; for classical references, see Seneta (1981) and Horn and Johnson (1985). Rothblum and Tan (1985) contain an extensive survey of bounding the second largest (in modulus) eigenvalue of a non-negative matrix; as shown below, this is closely related to the upper bound of ρ of a Markov chain. Diaconis and Stroock (1991) adopt a variational characterization of the eigenvalues of a Markov transition matrix and proceed to develop bounds for both the second largest and smallest eigenvalue. Without explicit acknowledgement, the functionals they use essentially correspond to $1 - \rho$ and ρ. However, their objective is to derive spectral bounds of $\mathbf{P}$ from bounds on ρ, exactly the opposite of the approach in this paper.

$F(y|x) \geq F(y|x')$ whenever $x < x'$, implies that $\mathbb{E}(X_{t+1}|X_t = x)$ is increasing in x. This is nothing but the familiar first order stochastic dominance of $F(y|x)$ in x. For the one dimensional Markov chain (process) considered here, the monotonicity condition on the transition distribution is equivalent to the concept of monotone transition function defined in Stokey et al. (1989, p.220), which is fundamental to most dynamic economic analysis with persistent shocks.[2] The result we demonstrate therefore makes clear that many works in dynamic economic modeling implicitly exclude the possibility of incorporating shocks with negative autocorrelations by assuming monotone transition function for each shock in their models.

The organization of the paper is as follows. In Section 2, we lay down the basic setup and state necessary assumptions. We develop the diagonal dominance condition in Section 3 and proceed to the monotonicity condition in Section 4. Then we give some results regarding higher order autocorrelations in Section 5. Numerical examples on both conditions appear in Section 6 and Section 7 concludes.

II Setup

Let $\{X_t\}_{t\geq 0}$ be a discrete time, finite state Markov chain with a state space $\{x_1, \ldots, x_n\}$ and a transition matrix $\mathbf{P} = (p_{ij})_{1\leq i,j\leq n}$. Let $\mathbf{x} = (x_1, \ldots, x_n)'$ (prime denotes transpose). Each state x_i is a real number, and in general, all states are mutually different, hence $\mathbf{x} \neq \mathbf{0}$ ($n \times 1$ vector of 0s); but $\{x_i\}$ is not necessarily ordered in i.

Throughout this paper, we assume $\{X_t\}$ is ergodic, i.e., $\mathbf{P}$ is irreducible and aperiodic. Let $\{\lambda_i(\mathbf{P})\}_{1\leq i\leq n}$ denote the eigenvalues of $\mathbf{P}$ ordered by $|\lambda_1(\mathbf{P})| \leq \cdots \leq |\lambda_n(\mathbf{P})|$ in modulus. It is well known that 1 is an eigenvalue of $\mathbf{P}$ and $|\lambda_i(\mathbf{P})| \leq 1$ for all i, so, without loss of generality, let $\lambda_n(\mathbf{P}) = 1$. Irreducibility of $\mathbf{P}$ implies that there is a unique (left) eigenvector $\mathbf{u} = (u_1, \ldots, u_n)'$ associated with $\lambda_n(\mathbf{P})$ such that $\mathbf{u}$ is strictly positive (entry-by-entry) and $1 = \mathbf{u}'\mathbf{1}$ ($n\times 1$ vector of 1s). In other words, $\mathbf{u}$ is the unique stationary distribution of $\mathbf{P}$. On top of irreducibility, aperiodicity is equivalent to the fact that $\mathbf{P}$ has no eigenvalue of a modulus 1 other than $\lambda_n(\mathbf{P})$, i.e., $|\lambda_i(\mathbf{P})| < 1$ for all $i < n$. All these facts are standard in the literature (Horn and Johnson 1985, Ch. 8; Seneta 1981, Ch. 4). We remark here that an ergodic transition matrix is *primitive* as defined in Horn and Johnson (1985, p. 516); thus we use "ergodic" and "primitive" interchangeably in the rest of the paper. Lastly, we adopt the convention that whenever $\mathbb{E}(\cdot)$ denotes an unconditional expectation, it is under the stationary distribution $\mathbf{u}$.

To simplify the exposition, we assume $\mathbb{E}X_t = 0$; this is innocuous as the same results on autocorrelations hold for the transformed process $Y_t \equiv X_t - \mathbb{E}X_t$ when $\mathbb{E}X_t \neq 0$. Given $\mathbb{E}X_t = 0$, the stationary variance and autocovariances can be written succinctly as $\mathrm{var}X_t = \mathbb{E}X_t^2$ and $\mathrm{cov}(X_t, X_{t+k}) = \mathbb{E}X_tX_{t+k}$ for any $k \geq 1$. Let $\mathbf{U} = \mathrm{diag}(\mathbf{u})$, the diagonal matrix where u_i is the i'th

[2] Donaldson and Mehra (1983) give an early application of monotone transition function to study optimal growth model with correlated productivity shock. Hopenhayn and Prescott (1992) develop a widely applicable theoretical framework of Markov process based on this concept. See Stokey et al. (1989, Ch. 12–13) for an introductory treatment of monotone Markov process with extensive examples of applications.

diagonal entry. Some simple algebra yields following expressions:

$$\mathbb{E}X_t = \mathbf{u}'\mathbf{x}, \qquad \mathbb{E}X_t^2 = \mathbf{x}'\mathbf{U}\mathbf{x}, \qquad \mathbb{E}X_t X_{t+k} = \mathbf{x}'\mathbf{U}\mathbf{P}^k\mathbf{x}.$$

As a consequence, the k'th-order stationary autocorrelation $\rho(k)$ is the ratio of two quadratic forms

$$\rho(k) = \frac{\mathbf{x}'\mathbf{U}\mathbf{P}^k\mathbf{x}}{\mathbf{x}'\mathbf{U}\mathbf{x}}.$$

With slight abuse of notation, denote simply by ρ the first-order autocorrelation.

Since $\mathbf{u}$ can be written directly as a function of $\mathbf{P}$ according to the following expression

$$\mathbf{u}' = \mathbf{e}_n'\left[(\mathbf{P}-\mathbf{I})_{n\times(n-1)}\middle|\mathbf{1}\right]^{-1} = \mathbf{e}_n'\begin{bmatrix} p_{11}-1 & \cdots & p_{1n} & 1 \\ \vdots & & \vdots & \vdots \\ p_{n-11} & \cdots & p_{n-1n-1}-1 & 1 \\ p_{n1} & \cdots & p_{nn-1} & 1 \end{bmatrix}^{-1},$$

where $\mathbf{1}$ is the $n\times 1$ vector of 1s and $\mathbf{e}_n = (0,\ldots,0,1)$, it is clear that $\rho(k)$ is a complicated function, especially in $\mathbf{P}$. The principal objective of the rest of this paper is to develop sufficient conditions on $\mathbf{P}$ under which it becomes tractable to characterize $\rho(k)$ with minimal extra conditions on $\mathbf{x}$. We will focus on $\rho = \rho(1)$ in the next two sections while take as given $\mathbf{P}$ as an ergodic matrix.

III Diagonal Dominance Condition

The basic idea to derive bounds on ρ is to use the variational characterization of quadratic forms. Towards this end, first note that

$$\rho = \frac{\mathbf{x}'((\mathbf{U}\mathbf{P}+\mathbf{P}'\mathbf{U})/2)\mathbf{x}}{\mathbf{x}'\mathbf{U}\mathbf{x}} \quad \text{subject to} \quad \mathbf{u}'\mathbf{x} = 0,$$

which follows from $\mathbf{x}'\mathbf{U}\mathbf{P}\mathbf{x} = \mathbf{x}'\mathbf{P}'\mathbf{U}\mathbf{x}$. Next, let $\mathbf{v} = (\sqrt{u_1},\ldots,\sqrt{u_n})'$, $\mathbf{V} = \mathrm{diag}(\mathbf{v})$, and

$$\mathbf{Q} = (\mathbf{V}\mathbf{P}\mathbf{V}^{-1} + \mathbf{V}^{-1}\mathbf{P}'\mathbf{V})/2.$$

By letting $\mathbf{y} = \mathbf{V}\mathbf{x}$, we can further write ρ as

$$\rho = \frac{\mathbf{x}'\mathbf{U}\mathbf{P}\mathbf{x}}{\mathbf{x}'\mathbf{U}\mathbf{x}} = \frac{\mathbf{x}'((\mathbf{U}\mathbf{P}+\mathbf{P}'\mathbf{U})/2)\mathbf{x}}{\mathbf{x}'\mathbf{U}\mathbf{x}} = \frac{\mathbf{y}'\mathbf{Q}\mathbf{y}}{\mathbf{y}'\mathbf{y}} \quad \text{subject to} \quad \mathbf{v}'\mathbf{y} = 0,$$

since $\mathbf{U} = \mathbf{V}^2 = \mathbf{V}'\mathbf{V}$, $(\mathbf{U}\mathbf{P}+\mathbf{P}'\mathbf{U})/2 = \mathbf{V}'\mathbf{Q}\mathbf{V}$, and $\mathbf{u}'\mathbf{x} = 0$ is equivalent to $\mathbf{v}'\mathbf{y} = 0$. Denote by $\lambda_1(\mathbf{Q}) \le \cdots \le \lambda_n(\mathbf{Q})$ the n eigenvalues of $\mathbf{Q}$, all of which are real as $\mathbf{Q}$ is symmetric. In principle, one can use standard variational characterization upon $\mathbf{y}'\mathbf{Q}\mathbf{y}/\mathbf{y}'\mathbf{y}$ right away to derive bounds on ρ in terms of $\{\lambda_i(\mathbf{Q})\}$. However, it is imperative to first work out more qualitative information of $\{\lambda_i(\mathbf{Q})\}$.

For this, let

$$\mathbf{R} = \mathbf{V}^{-1}\mathbf{Q}\mathbf{V} = (\mathbf{P}+\mathbf{U}^{-1}\mathbf{P}'\mathbf{U})/2,$$

it follows that $\mathbf{R}$ has the same set of eigenvalues $\{\lambda_i(\mathbf{R})\}$ as $\mathbf{Q}$, and $\{\lambda_i(\mathbf{R})\}$ can be arranged such that $\lambda_1(\mathbf{Q}) = \lambda_1(\mathbf{R}) \leq \cdots \leq \lambda_n(\mathbf{Q}) = \lambda_n(\mathbf{R})$. Evidently, $\mathbf{R}$ is nonnegative. Since

$$\mathbf{R1} = (\mathbf{P1} + \mathbf{U}^{-1}\mathbf{P}'\mathbf{U1})/2 = (\mathbf{1} + \mathbf{U}^{-1}\mathbf{P}'\mathbf{u})/2 = (\mathbf{1} + \mathbf{U}^{-1}\mathbf{u})/2 = (\mathbf{1} + \mathbf{1})/2 = \mathbf{1},$$

$\mathbf{R}$ is actually a transition matrix, and consequently $\lambda_n(\mathbf{R}) = 1$. Given that $\mathbf{P}$ is primitive, there exists an integer $m \geq 1$ such that $\mathbf{P}^m$ is strictly positive (Horn and Johnson, 1985, Theorem 8.5.2). As a result, $\mathbf{R}^m$ is strictly positive, hence $\mathbf{R}$ is primitive, implying $|\lambda_i(\mathbf{R})| < 1$ for all $i < n$. The following lemma summarizes the results obtained so far.

Lemma 1. *Suppose* $\mathbf{P}$ *is ergodic and let* $\{\lambda_i(\mathbf{Q})\}$ *and* $\{\lambda_i(\mathbf{R})\}$ *denote the sets of eigenvalues of* $\mathbf{Q}$ *and* $\mathbf{R}$. *Then* $\mathbf{R}$ *is an ergodic transition matrix and*

$$-1 < \lambda_1(\mathbf{Q}) = \lambda_1(\mathbf{R}) \leq \cdots \leq \lambda_{n-1}(\mathbf{Q}) = \lambda_{n-1}(\mathbf{R}) < \lambda_n(\mathbf{Q}) = \lambda_n(\mathbf{R}) = 1.$$

Now, the standard results of variational characterization imply the following bounds on ρ.

Lemma 2. *Suppose* $\mathbf{P}$ *is ergodic and let* $\{\lambda_i(\mathbf{R})\}$ *denote the eigenvalues of* $\mathbf{R}$. *Then*

$$-1 < \lambda_1(\mathbf{R}) \leq \rho \leq \lambda_{n-1}(\mathbf{R}) < 1.$$

Proof. The standard variational characterization (Horn and Johnson 1985, Sec. 4.2; Magnus and Neudecker 2007, Sec. 11.7) yields

$$\lambda_1(\mathbf{R}) = \lambda_1(\mathbf{Q}) \leq \min_{\mathbf{y}\neq\mathbf{0}} \frac{\mathbf{y}'\mathbf{Q}\mathbf{y}}{\mathbf{y}'\mathbf{y}} \leq \min_{\mathbf{v}'\mathbf{y}=0,\mathbf{y}\neq\mathbf{0}} \frac{\mathbf{y}'\mathbf{Q}\mathbf{y}}{\mathbf{y}'\mathbf{y}} = \max_{\mathbf{v}'\mathbf{y}=0,\mathbf{y}\neq\mathbf{0}} \frac{\mathbf{y}'\mathbf{Q}\mathbf{y}}{\mathbf{y}'\mathbf{y}} \leq \lambda_{n-1}(\mathbf{Q}) = \lambda_{n-1}(\mathbf{R}),$$

where the last inequality follows from the observation that

$$\mathbf{Qv} = (\mathbf{VP1} + \mathbf{V}^{-1}\mathbf{P}'\mathbf{u})/2 = (\mathbf{V1} + \mathbf{V}^{-1}\mathbf{u})/2 = \mathbf{v},$$

i.e., $\mathbf{v}$ is the eigenvector associated with $\lambda_n(\mathbf{Q}) = 1$. *QED*

Remark 1. A transition matrix $\mathbf{P}$ is reversible if and only if $u_i p_{ij} = u_j p_{ji}$ for all i, j, or in matrix form, $\mathbf{UP} = \mathbf{P}'\mathbf{U}$. Thus $\mathbf{UP}$ is symmetric if and only if $\mathbf{P}$ is reversible, and in this case, variational characterization can be applied directly to $\rho = \mathbf{x}'\mathbf{UPx}/\mathbf{x}'\mathbf{Ux}$. In particular, $\mathbf{UP} = \mathbf{P}'\mathbf{U}$ implies $\mathbf{VPV}^{-1} = \mathbf{V}^{-1}\mathbf{P}'\mathbf{V}$ and hence $\mathbf{Q} = \mathbf{VPV}^{-1}$, therefore $\{\lambda_i(\mathbf{P})\} = \{\lambda_i(\mathbf{Q})\}$, i.e., all $\mathbf{P}$'s eigenvalues are real. Diaconis and Stroock (1991) develop various bounds on $\lambda_1(\mathbf{P})$ and $\lambda_{n-1}(\mathbf{P})$ via bounds on $\rho = \mathbf{x}'\mathbf{UPx}/\mathbf{x}'\mathbf{x}$, which then provides bounds on the rate of convergence to stationarity $\max(|\lambda_1(\mathbf{P})|, \lambda_{n-1}(\mathbf{P}))$. Fill (1991) extends this work to nonreversible Markov chains and shows that, for a general ergodic $\mathbf{P}$, the rate of convergence to stationarity is bounded by $\lambda_{n-1}(\mathbf{M})$ and $\lambda_{n-1}(\mathbf{R})$, where $\mathbf{M} = \mathbf{PU}^{-1}\mathbf{P}'\mathbf{U}$ and $\mathbf{R}$ is as defined above. It is easily shown that both $\mathbf{M}$ and $\mathbf{R}$ are reversible transition matrices with the common stationary distribution $\mathbf{u}$, the results of Diaconis and Stroock (1991) can thereby be applied to give bounds on $\lambda_{n-1}(\mathbf{M})$ and $\lambda_{n-1}(\mathbf{R})$.

For $\rho > 0$, it suffices to have $\lambda_1(\mathbf{R}) > 0$. Given $\mathbf{P}$, one can always compute numerically $\lambda_1(\mathbf{R})$ by first computing $\mathbf{u}$. However, since both $\mathbf{u}$ and $\lambda_1(\mathbf{R})$ are intricate functions of $\mathbf{P}$, to gain more tractability, it is crucial to get a more explicit relationship between $\lambda_1(\mathbf{R})$ and $\mathbf{P}$. The Geršgorin discs prove to be an elegant way for relating $\lambda_1(\mathbf{R})$ directly to $\mathbf{P}$.

Proposition 1. *Suppose* $\mathbf{P}$ *is an ergodic matrix. Then* $\rho \geq 2\min_i p_{ii} - 1$.

Proof. Lemma 1 shows that $\mathbf{R}$ is a transition matrix, so that each of its row sums equals to 1. Since the i'th diagonal entry of $\mathbf{R}$ equals to p_{ii}, it follows that the off-diagonal sum of the i'th row is $1 - p_{ii}$. Because all eigenvalues of $\mathbf{R}$ are real, the theorem of Goršgorin discs (Horn and Johnson, 1985, p. 344) asserts that

$$\{\lambda_1(\mathbf{R}), \ldots, \lambda_n(\mathbf{R})\} \subset \bigcup_{i=1}^{n} \{z \in \mathbb{R}^1 : |z - p_{ii}| \leq 1 - p_{ii}\},$$

therefore $\lambda_1(\mathbf{R}) \geq 2\min_i p_{ii} - 1$. Lemma 2 then implies that $\rho \geq \lambda_1(\mathbf{R}) 2\min_i p_{ii} - 1$. *QED*

Remark 2. For a complex matrix $\mathbf{A} = (a_{ij})_{1 \leq i,j \leq n}$, the Goršgorin discs refer to the discs on the complex plane defined by $G_i(\mathbf{A}) = \{z \in \mathbb{C} : |z - a_{ii}| \leq \sum_{j \neq i} |a_{ij}|\}$ for each i where $|\cdot|$ denotes modulus. The Goršgorin region refers to $G(\mathbf{A}) = \cup_i G_i(\mathbf{A})$, which contains all eigenvalues of $\mathbf{A}$.

Based on the lower bound in this proposition, the diagonal dominance condition follows as a corollary. Recall that a matrix $\mathbf{A} = (a_{ij})$ is strictly diagonally dominant if $|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for all i. For a transition matrix $\mathbf{P}$, this is equivalent to $p_{ii} > 1/2$ for all i as each row sum of $\mathbf{P}$ is 1.

Corollary 2 (Strong diagonal dominance). *Suppose* $\mathbf{P}$ *is strictly diagonally dominant. Then* $\rho > 0$.

Proof. This trivially follows from $2\min_i p_{ii} - 1 > 0$ and the lower bound stated in the preceding proposition. *QED*

Specifically, we label this condition as the strong version of the diagonal dominance condition since $p_{ii} > 1/2$ is required for all i. Strictly diagonal dominance is needed for the lower bound $\lambda_1(\mathbf{R}) \geq 2\min_i p_{ii} - 1$ to imply $\lambda_1(\mathbf{R}) > 0$. However, we shall show below that strictly diagonal dominance can be weakened into diagonal dominance and one still gets $\lambda_1(\mathbf{R}) > 0$. First recall that a real matrix $\mathbf{A} = (a_{ij})$ is diagonally dominant if $|a_{ii}| \geq \sum_{j \neq i} |a_{ij}|$ for all i and the strict inequality holds for at least one i. The proof of this strengthened result requires some fine-tuning fact in matrix theory regarding the Geršgorin region of an irreducible matrix.

Corollary 3 (Weak diagonal dominance). *Suppose* $\mathbf{P}$ *is diagonally dominant. Then* $\rho > 0$.

Proof. The same derivation in the proof of Proposition 1 shows that whenever $\mathbf{P}$ is diagonally dominant, $\mathbf{R}$ is also diagonally dominant. By assumption, $\mathbf{P}$ is irreducible. Since $\mathbf{U}^{-1}\mathbf{P}'\mathbf{U}$ is nonnegative, $\mathbf{R} = (\mathbf{P} + \mathbf{U}^{-1}\mathbf{P}'\mathbf{U})/2$ is irreducible as well. A straightforward application of the theorem of Taussky (Horn and Johnson, 1985, Corollary 6.2.7) then shows that $\lambda_1(\mathbf{Q}) > 0$. *QED*

Remark 3. For various characterizations of irreducible matrix, see Horn and Johnson (1985, Sec. 6.2). The nature of the Taussky's Theorem is as follows. Given an irreducible matrix $\mathbf{A}$, a point λ belonging to the boundary of the Goršgorin region $G(\mathbf{A})$ can be an eigenvalue of $\mathbf{A}$ only if every Goršgorin disc passes through λ. Thus 0 is excluded from the spectrum of $\mathbf{R}$ by the disc associated with the strict inequality $2p_{ii} - 1 > 0$ for some (at least one) i, despite the possibility of $\min_i\{2p_{ii} - 1\} = 0$.

The matrix theoretic arguments employed so far can also be used to develop useful upper bounds on ρ. The next proposition states that ρ is bounded from above by the second largest eigenvalue of $\mathbf{R}$.

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