Online Appendix for

# "On the optimal design of a Financial Stability Fund"

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## A Proofs

### A.1 Preliminary Lemmas

**Lemma 3.** $\partial_x FV(x,s) = V^{bf}(x,s) + x\partial_x V^{bf}(x,s) + \partial_x V^{lf}(x,s) = V^{bf}(x,s)$. *This implies the 'efficient risk-sharing' property:* $x\partial_x V^{bf}(x,s) = -\partial_x V^{lf}(x,s)$.

*Proof.* Given the optimization problem (8), the envelope condition — which is satisfied since the solution is unique (Marimon and Werner, 2021) — implies that $\partial_x FV = V^{bf}(x,s)$. At the same time, the decomposition in (10) implies that $\partial_x FV(x,s) = V^b(x,s) + x\partial_x V^{bf}(x,s) + \partial_x V^{lf}(x,s)$. Combining these two equations delivers the main result. □

Note that we use the 'efficient risk-sharing' property, $\big[x\partial_x V^b(x,s) + \partial_x V^l(x,s)\big] = 0$ when deriving the FOC with respect to $e$ in (16), by letting

$$\partial_e \mathbb{E}\big[FV(x',s')\big|s,e\big] = M(s)\sum_{s'|s}\partial_e^2\pi(x'_{xs}(s'),s')V^b(x'_{xs}(s'),s'), \tag{A.1}$$

where $M(s)$ summarizes the components of $FV(x',s')$ which do not depend on $s'$ or $e$.

For convenience, we will use the following notation in the next two lemmas: given any $s=(\theta,g)$ we will denote it as $s(i)$ if $g=g_i$, i.e. $s(i)=(\theta,g_i)$. Given our Assumption 2, a statement about the monotonicity of $s(i)$ in $i$ applies to all $\theta$ in $(\theta,g_i)$; in particular, since effort only affects the distribution of $g$, $\partial_e\pi(s'(i)|s,e) = \partial_e\pi^g(g'=g_i|g,e)$.

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**Lemma 4.** *i) Given Assumption 2, the law of motion $x'_{xs}(s'(i))$ is nondecreasing in $i$ and it is constant in $i$ if $\varrho(x,s)=0$. ii) $V^b(x'_{xs}(s'(i)),s'(i))$ and $FV(x'_{xs}(s'(i)),s'(i))$ are not decreasing in $i$.*

*Proof.* *i*) Assumption 2 implies that the monotone likelihood-ratio condition: $\frac{\partial_e \pi(s'(i)|s,e)}{\pi(s'(i)|s,e)}$ is nondecreasing in $i$ for every $e$. Therefore, we only need to recall (9):

$$x'_{xs}(s'(i)) = \frac{1+\nu^b+\varphi(s'(i)|s,e)}{1+\nu_l}\eta x \text{ and } \varphi(s'(i)|s,e) = \varrho\frac{\partial_e \pi(s'(i)|s,e)}{\pi(s'(i)|s,e)}.$$

*ii*) $V^b$ is either increasing or constant in $x$ (see the proof of Lemma 2) and by Lemma 3 $FV$ is also increasing in $x$; furthermore, given $x$, a higher $s$ means a higher surplus and therefore a higher $FV$ and, through risk-sharing, a higher $V^b$; in sum, both value functions are non-decreasing in $i$. □

It will be convenient to introduce some additional notation for the following lemmas. Let $\omega^h_{xs}(e) \equiv \sum_{s'|s}\pi(s'|s,\, e)V^{hf}(x'_{xs}(s'),s')$, for $h=b,l$, and $f\omega_{xs}(e) \equiv x\omega^b_{xs}(e)+\omega^l_{xs}(e)$. Note that, for example $\omega^h_{xs}(e)$, can also be written as:

$$\omega^h_{xs}(e) \equiv \sum_{i=1}^{N_g}\Delta V^h_{xs}(s'(i))\left[\sum_{j=i}^{N_g}\pi(s'(j)|s,\, e)\right],$$

where

$$\Delta V^h_{xs}(s'(i)) = \begin{cases}\sum_{\theta'|\theta}\pi^\theta(\theta'|\theta,\, e)[V^{hf}(x'_{xs}(s'(i))) - V^{hf}(x'_{xs}(s'(i-1)))], & i>1,\\ \sum_{\theta'|\theta}\pi^\theta(\theta'|\theta,\, e)V^{hf}(x'_{xs}(s'(1))), & i=1,\end{cases} \tag{A.2}$$

and we can similarly rewrite $f\omega_{xs}(e)$. Furthermore, let

$$\overline{\omega}^{h\prime}_{xs}(e) \equiv \sum_{i=1}^{N_g}\Delta V^h_{xs}(s'(i))\left[\sum_{j=i}^{N_g}\frac{\partial \pi(s'(j)|s,\, e)}{\partial e}\right]; \tag{A.3}$$

that is, function $\overline{\omega}^h_{xs}(e)$ is the function $\omega^h_{xs}(e)$ when taking derivatives with respect to $e$, and only the direct effect of $e$ on the distribution of $s$ is taken into account. We can similarly define $\overline{f\omega}_{xs}(e)$. We use $\overline{\omega}^h_{xs}(e)$ in the derivations that follow, since we are accounting from the fact that the solution to the Fund contract problem satisfies the *efficient risk-sharing* property by Lemma 3, which in this notation is: $x\omega^{b\prime}_{xs}(e) = -\omega^{l\prime}_{xs}(e)$.

**Lemma 5.** *The functions $\overline{\omega}^b_{xs}(e)$ and $\overline{f\omega}^l_{xs}(e)$ are non decreasing and concave. The saddle-point Lagrangean $\mathcal{L}(x,s)$ (i.e. of the saddle-point Bellman equation) is also concave in $e$.*

*Proof.* By Lemma 4 the value of the borrower and the Fund contract are non-decreasing in $s$ and, by Assumption 2, $\partial_e F_n(e,s) \leq 0$, which implies that all the terms within brackets in (A.3) are non-negative — i.e. $\overline{\omega}^{b\prime}_{xs}(e) \geq 0$ and $\overline{f\omega}'_{xs}(e) \geq 0$. Note that, using the latter definition of $\overline{\omega}^{h}_{xs}(e)$,

$$\overline{\omega}^{h\prime\prime}_{xs}(e) = \sum_{i=1}^{N_g} \Delta V^h_{xs}(s'(i)) \left[ \sum_{j=i}^{N_g} \frac{\partial^2 \pi(s'(j)|s,\, e)}{\partial e^2} \right]. \tag{A.4}$$

Similarly, by Assumption 2, $\partial^2_e F_n(e,s) \geq 0$ , which implies that all the terms within brackets in (A.4) are non-positive — i.e. $\overline{\omega}^{b\prime\prime}_{xs}(e) \leq 0$ and $\overline{f\omega}''_{xs}(e) \leq 0$.

To see that the above conditions guarantee that, given our assumptions, the Lagrangean $\mathcal{L}(x,s)$ is concave, note that

$$\partial_e \mathcal{L}(x,s) = -\, x(1+\nu_b)v'(e) - x\varrho v''(e) + \frac{1+\nu_l}{1+r}\overline{f\omega}'_{xs}(e) + x\varrho\beta\overline{\omega}^{b\prime\prime}_{xs}(e).$$

Note that $\partial_e \mathcal{L}(x,s) = 0$ is (18) expressed in this more synthetic notation. Therefore,

$$\partial^2_e \mathcal{L}(x,s) = \ -\, x(1+\nu_b)v''(e) - x\varrho v'''(e) + \frac{1+\nu_l}{1+r}\overline{f\omega}''_{xs}(e) + x\varrho\beta\overline{\omega}^{b\prime\prime\prime}_{xs}(e).$$

By assumption, the first two terms are negative and we have just shown that the third is also non-positive; finally, by Assumption 2, $\partial^3_e F_n(e,s) \geq 0$ and therefore $\overline{\omega}^{b\prime\prime\prime}_{xs}(e) \leq 0$. □

### A.2 Characterization of the Fund Solution

**Proof of Lemma 1.**

*Proof.* (a) *i*) It follows from equations (14) — $u'(c(x,s)) = 1/x$ — and (9) — $x'_{xs}(s'(i)) = \varphi(s'(i)|s,e)\eta x$ — and Lemma 4. *ii*) It follows from equation (15) and the monotonicity of $c(x,s)$ on $x$, i.e., *i*). *iii*) If $V^{bf}(x,s)$ is increasing and concave in $x$ — see *iv*) next —, the assumed convexity of $v$ implies, by equation (17), that $e$ is decreasing in $x$; with respect to $-g$ one would also expect it to be monotone (for example, decreasing if the likelihood-ratio is non-increasing in $-g$ since then $-g$ is a wealth effect); however, if next period there is a positive probability that the limited enforcement constraint of the lender binds, then this monotonicity can be distorted and effort can be higher when $-g$ is lower. *iv*) It follows from the monotonicity of $c(x,s), n(x,s)$ and $e(x,s)$ in $x$, and our assumptions on $U(c,n,e)$ imply that $V^{bf}(x,s)$ is increasing and strictly concave in $x$. Then, by Lemma 3, $\partial_x V^{lf}(x,s) < 0$ so that $V^{lf}(x,s)$ is decreasing in $x$.

(b) *i*) It follows from the fact that the policies, value functions and multipliers are evaluated when the constraints are binding as solutions to the saddle-point problem (SPFE); *ii*) It follows from the fact that $s$ may have a separate effect on the outside values (e.g. it does on $V^o(s)$), and *iii*) It follows from the *constrained qualification* constraints, (19) and (20) and *i*).

□

### A.3 Proof of Proposition 1

The proof parallels and extends the proof of Marcet and Marimon (2019) Theorem 3.

*Proof.* **Step 1**: *Checking that the necessary assumptions are satisfied.*

Given Assumptions 1, 2 and 3 and our assumptions on preferences, $U(c, n, e)$, and technologies, $f(n), \pi^g(\cdot; g, e)$, our economies satisfy the Marcet and Marimon (2019) assumptions. In particular, A2 on the functions (continuous in $(c, n, e)$ and measurable in $s$); A3 on the non-empty feasible sets; A5 on convex technologies.[1] Regarding concavity, A6, a clarification is in order. They consider SPFE value functions which are concave in the endogenous state variable and homogeneous of degree one in the co-state Pareto weights. Instead, we merge these conditions in our endogenous co-state $x$: $FV$ is homogeneous of degree one when we consider both Pareto weights $(x, 1)$, and concave in $x$, given our concavity assumptions, — making $V^{bf}$ strictly concave in $x$, i.e satisfying A6b — and Lemma 3. The uniform boundedness assumption A4 is also satisfied, since feasible $n$ and effort $e$ are bounded, and consumption $c$ is bounded by the technology and the lender's limited enforcement constraint. Therefore, the rewards ($U(c, n, e)$ and $c_l$) are bounded as well and, by Assumption **??**, the finiteness of $V^o(s)$ and $Z$ imply that the constraint functions are uniformly bounded as well. Finally, our *interiority* assumption is a version of A7b.

**Step 2**: *'Relaxed Fund Contract problem' and the existence of solutions to this problem.*

We will show first that a solution exists to a '*Relaxed Fund Contract problem*', which is the same as the *Fund Contract problem* except that constraint (3) is replaced by a weak inequality version:

$$\beta \sum_{s^{t+1}|s^t} \frac{\partial \pi(s^{t+1}|s_t,\, e(s^t))}{\partial e(s^t)} V^b(s^{t+1}|s^t) - v'(e(s^t)) \geq 0, \tag{A.5}$$

which can also be written as

$$\beta \overline{\omega}^b_{xs}{}'(e) - v'(e) \geq 0.$$

In particular, given our assumptions,

$$\beta \overline{\omega}^b_{xs}{}'''(e) - v'''(e) \leq 0.$$

Therefore (A.5) defines a convex set of feasible efforts.

Next, we will show that any solution to the 'Relaxed Fund Contract problem' is also a solution of the original problem. Let $\mathfrak{A} = \{(c, n, e) \in \mathbb{R}^3_+ : n \leq 1, e \leq 1\}$. This set is obviously compact and convex. Note that the pay-off of the fund $c_l(s) = \theta f(n) - g - c$ is concave given our concavity assumption on $f$.

We first decompose the saddle-point recursive contract problem into the choice of actions,

[1] Referring to $\{n(s)|f(n(s)) - g(s) \geq 0\}$, $e \in [0, 1]$ and Assumption 3.

$\mathfrak{a} = (c, n, e)$, and multipliers, $\gamma = (\nu_b, \nu_l, \varrho)$, given $FV(x, s)$, as follows:

$$
\begin{aligned}
SP^{\mathfrak{a}}(\gamma; x, s) = \Bigg\{ & \mathfrak{a} \in \mathfrak{A} : \text{ for all } \widetilde{\mathfrak{a}} \in \mathfrak{A}, \\
& x\Big[U(\mathfrak{a}) + \beta \sum_{s'|s} \pi(s'|s, e) V^b(x'(s'), s')\Big] \\
& + \Big[c_l(s) + \frac{1}{1+r} \sum_{s'|s} \pi(s'|s, e) V^l(x'(s'), s')\Big] \\
& + x\nu_b\Big[U(\mathfrak{a}) + \beta \sum_{s'|s} \pi(s'|s, e) V^b(x'(s'), s') - V^o(s)\Big] \\
& + \nu_l\Big[c_l(s) + \frac{1}{1+r} \sum_{s'|s} \pi(s'|s, e) V^l(x'(s'), s') - Z\Big] \\
& + x\varrho\Big[\beta \sum_{s'|s} \frac{\partial \pi(s'|s,\, e)}{\partial e} V^b(x'(s'), s') - v'(e)\Big] \\
& \geq x\Big[U(\widetilde{\mathfrak{a}}) + \beta \sum_{s'|s} \pi(s'|s, \widetilde{e}) V^b(\widetilde{x}'(s'), s')\Big] \\
& + \Big[\widetilde{c}_l(s) + \frac{1}{1+r} \sum_{s'|s} \pi(s'|s, \widetilde{e}) V^l(\widetilde{x}'(s'), s')\Big] \\
& + x\nu_b\Big[U(\widetilde{\mathfrak{a}}) + \beta \sum_{s'|s} \pi(s'|s, \widetilde{e}) V^b(\widetilde{x}'(s'), s') - V^o(s)\Big] \\
& + \nu_l\Big[\widetilde{c}_l(s) + \frac{1}{1+r} \sum_{s'|s} \pi(s'|s, \widetilde{e}) V^l(\widetilde{x}'(s'), s') - Z\Big] \\
& + x\varrho\Big[\beta \sum_{s'|s} \frac{\partial \pi(s'|s,\, \widetilde{e})}{\partial e} V^b(\widetilde{x}'(s'), s') - v'(\widetilde{e})\Big] \Bigg\},
\end{aligned}
$$

where $\widetilde{x}'(s') = \frac{1+\nu_b}{1+\nu_l} + \varrho \frac{\partial \pi(s'|s,\widetilde{e})/\partial e}{(1+\nu_l)\pi(s'|s,\widetilde{e})}$. Note that our original problem is homogeneous of degree one in $(\mu_{b,0}, \mu_{l,0})$, and that allows us to reformulate the problem using $x$ as a co-state variable. This guarantees, together with our *interiority* assumption (a version of A7b used in Lemma 6A in Marcet and Marimon, 2019), that there exists a positive constant $C$ such that, if $\gamma$ is the Lagrange multiplier vector, $\|\gamma\| \leq C\|x\|$. But the lender's participation constraint $Z$ sets an upper bound on $\|x\|$ for any feasible contract. Therefore, there exists a $\overline{C}$ such that $\|\gamma\| \leq \overline{C}$, and the set of feasible Lagrange multipliers, $\Gamma = \{\gamma \in \mathbb{R}^3_+ : \|\gamma\| \leq \overline{C}\}$, is also compact and convex. The minimization problem can be written as:

$$
SP^{\gamma}(\mathfrak{a}; x, s) = \Bigg\{ \gamma \in \Gamma : \text{ for all } \widehat{\gamma} \in \Gamma,
$$

$$
\begin{aligned}
& x\left[U(\mathfrak{a})+\beta\sum_{s'|s}\pi(s'|s,e)V^b(x'(s'),s')\right]\\
&+\left[c_l(s)+\frac{1}{1+r}\sum_{s'|s}\pi(s'|s,e)V^l(x'(s'),s')\right]\\
&+x\nu_b\left[U(\mathfrak{a})+\beta\sum_{s'|s}\pi(s'|s,e)V^b(x'(s'),s')-V^o(s)\right]\\
&+\nu_l\left[c_l(s)+\frac{1}{1+r}\sum_{s'|s}\pi(s'|s,e)V^l(x'(s'),s')-Z\right]\\
&+x\varrho\left[\beta\sum_{s'|s}\frac{\partial\pi(s'|s,\,e)}{\partial e}V^b(x'(s'),s')-v'(e)\right]\\
&\leq x\left[U(\mathfrak{a})+\beta\sum_{s'|s}\pi(s'|s,e)V^b(\widehat{x}'(s'),s')\right]\\
&+\left[c_l(s)+\frac{1}{1+r}\sum_{s'|s}\pi(s'|s,e)V^l(\widehat{x}'(s'),s')\right]\\
&+x\widehat{\nu}_b\left[U(\mathfrak{a})+\beta\sum_{s'|s}\pi(s'|s,e)V^b(\widehat{x}'(s'),s')-V^o(s)\right]\\
&+\widehat{\nu}_l\left[c_l(s)+\frac{1}{1+r}\sum_{s'|s}\pi(s'|s,e)V^l(\widehat{x}'(s'),s')-Z\right]\\
&+x\widehat{\varrho}\left[\beta\sum_{s'|s}\frac{\partial\pi(s'|s,\,e)}{\partial e}V^b(\widehat{x}'(s'),s')-v'(e)\right]\Bigg\},
\end{aligned}
$$

where $\widehat{x}'(s')=\frac{1+\widehat{\nu}_b}{1+\widehat{\nu}_l}+\widehat{\varrho}\frac{\partial\pi(s'|s,e)/\partial e}{(1+\widehat{\nu}_l)\pi(s'|s,e)}$. Now, if we define the correspondence

$$SP:\mathfrak{A}\times\Gamma\to\mathfrak{A}\times\Gamma \text{ by } SP(\mathfrak{a},\gamma;x,s)=\left(SP^{\mathfrak{a}}(\gamma;x,s),SP^{\gamma}(\mathfrak{a};x,s)\right),$$

one can show — given Lemma 5 — that it is non-empty, convex-valued and upper hemicontinuous, as in Lemma 7A in Marcet and Marimon (2019) (Theorem 3), which applies Kakutani's fixed point theorem to prove the existence of solutions to the saddle-point contracting problem.

In our case, this means that, with the additional (A.5), there is a contract satisfying equations (9)–(10), (13)–(15), (16), (19)–(20) and the following constraint qualification condition:

$$\varrho\left[\beta\sum_{s'|s}\frac{\partial\pi(s'|s,\,e)}{\partial e}V^b(x'(s'),s')-v'(e)\right]=0, \tag{A.6}$$

with $\varrho(x,s)=0$ if the term in brackets in (A.6) is non-zero. Now we show that $\varrho(x,s)\neq 0$.

Suppose that $\varrho(x,s)=0$. Then (16) reduces to

$$\begin{aligned}0 = & -v'(e(x,s)) + \beta \sum_{s'|s} \frac{\partial \pi(s'|s,e(x,s))}{\partial e} V^b(x'_{xs}(s'),s') \\ & + \frac{1+\nu_l(x,s)}{1+\nu_b(x,s)} \frac{1}{x} \frac{1}{1+r} \sum_{s'|s} \frac{\partial \pi(s'|s,e(x,s))}{\partial e} V^l(x'_{xs}(s'),s').\end{aligned}$$

Note that $V^l$ is nondecreasing in $i$ since $\varrho(x,s)\geq 0$. Then we can rewrite

$$\sum_{s'|s} \frac{\partial \pi(s'|s,e(x,s))}{\partial e} V^l(x'_{xs}(s'),s')$$

as

$$\sum_{i=1}^{N} \Delta V^l_{xs}(s'(i)) \left[ \sum_{j=i}^{\overline{N}} \frac{\partial \pi(s'|s,e(x,s))}{\partial e} \right],$$

where $\Delta V^l_{xs}(s'(i))$ is defined by (A.2). Note that the first term is equal to zero in this summation. The monotone likelihood ratio implies that all other terms are non-negative, as the terms in the bracket are strictly positive and $\Delta V^h_{xs}(s'(i)) \geq 0$ for all $i>1$. The fact that we assume that some risk sharing occurs in this economy (the lender's participation constrain is slack in at least one state realization) implies that some of the terms in the summation will be strictly positive. Given that $\sum_{s'|s} \frac{\partial \pi(s'|s,e(x,s))}{\partial e} V^l(x'_{xs}(s'),s')$ is positive, the first line must be negative, but that would violate (A.5). Hence, we have reached a contradiction and $\varrho(x,s)>0$ must be true. In sum, the contract satisfies all the conditions of Definition 1.

**Step 3**: *The relaxed Fund Contract problem and the Fund Contract problem have the same solution.* This is the consequence of $\varrho(x,s)>0$. Given this, (A.5) implies that the incentive compatibility constraint is satisfied as an equality in the relaxed Fund problem. Hence, the solution is equivalent to the original problem when this constraint was introduced as an equality.

**Step 4**: *Uniqueness.* $FV$ is monotone in $x$. Further, it is constant when either of the limited enforcement constraints are binding and concave when both are slack. The same contraction mapping argument used in Theorem 3 of Marcet and Marimon (2019) shows that $FV$ is unique. The strict concavity/convexity assumptions on $u$, $f$ and $v$ imply that the Recursive Contract allocation is unique and $FV$ is strictly concave in $x$ whenever neither participation constraint is binding, and it is uniquely defined when either is binding. Therefore, the saddle-point solution is unique. □

### A.4 Proof of Corollary 1

The proof follows from Stokey et al. (1989) Theorem 12.12. For further details, see Liu et al. (2023) Online Appendix.

### A.5 Proofs of Proposition 2 and 3

**Proof of Proposition 2.** To prove the proposition, we show that we can construct asset prices, asset holdings, policies, multipliers, borrowing limits and value functions, corresponding to the description of the economy of Subsection 3.1.2, such that all the conditions characterizing the recursive competitive equilibrium in Definition 2 are satisfied by the recursive Fund policies and values. The proof is partially based on Alvarez and Jermann (2000), but we account for the presence of a moral hazard incentive compatibility constraint and a risk-neutral lender subject to a limited enforcement constraint. Furthermore, the proof is done in a recursive competitive framework, taking advantage of the Fund's policies and value functions characterization in Lemma 1, and it implements the IC constraint with the introduction of state-contingent assets, where the Arrow security component (paying in units of assets) is subject to Pigou budget-neutral taxation.

**Step 1**: *Getting $q(s'|\boldsymbol{a}, s)$ from $q(s'|x, s)$ and mapping $(x, s) \to (\boldsymbol{a}, s)$.* As seen in Subsection 3.1.2 we want to obtain Arrow security prices satisfying (49):

$$q(s'|\boldsymbol{a}, s) = \pi(s'|s, e(a, s))A_{\boldsymbol{a}'}(s') \max\left\{\beta \frac{u'(c(a', s'))}{u'(c(a, s))} \frac{1}{1 + \tau'(s'; \boldsymbol{a}, s)}, \frac{1}{1 + r}\right\},$$

where $A_{\boldsymbol{a}}(s) = (1 - \delta + \delta\kappa) + \delta q(\boldsymbol{a}, s)$ and $q(\boldsymbol{a}, s) = \sum_{s'|s} q(s'|\boldsymbol{a}, s)$.

By (51) and (52) we get Fund Arrow security prices as:

$$\begin{aligned} q(s'|x, s) &= \frac{1}{1 + r}\pi(s'|s, e(x, s))A_{x'}(s') \max\left\{\frac{1 + \nu_l(x', s')}{1 + \nu_b(x', s')} \frac{1}{1 + \frac{\varphi(s'|s, e(x,s))}{1 + \nu_b(x,s)}} \frac{1}{1 + \tau(s'; x, s)}, 1\right\} \\ &= \frac{1}{1 + r}\pi(s'|s, e(x, s))A_{x'}(s') \max\left\{\frac{1 + \nu_l(x', s')}{1 + \nu_b(x', s')}, 1\right\}, \end{aligned} \tag{A.7}$$

where, $A_x(s) = (1 - \delta + \delta k) + \delta q(x, s)$, with $q(x, s) = \sum_{s'|s} q(s'|x, s)$, and asset security taxes as defined in (53):

$$\begin{aligned} \frac{1}{1 + \tau(s', x, s)} &= 1 + \chi(x, s)u'(c(x, s)) \frac{\partial_e \pi(s'|s, e(x, s))}{\pi(s'|s, e(x, s))} \\ &= 1 + \frac{\varphi(s'|s, e)}{1 + \nu_b(x, s)}. \end{aligned}$$

Therefore, if we map $(x, s)$ into $(\boldsymbol{a}, s)$ we obtain $q(s'|\boldsymbol{a}, s)$ and $\tau'(s'; \boldsymbol{a}, s)$. In order to do so, we first write the budget constraint of the borrower in (32), separating its three components, with Fund contract policies, that is:

$$q(x, s)(\bar{a}'(s) - \delta a) = \theta(s)f(n(x, s)) - c(x, s) - g(s) + (1 - \delta + \delta\kappa)a, \tag{A.8}$$

$$\sum_{s'|s} q(s'|x, s)\hat{a}(s') = 0, \tag{A.9}$$

$$\text{and}\quad \bar{\tau}(x,s) = \sum_{s'|s} q(s'|x,s)a'(s')\tau'(s';x,s) \tag{A.10}$$

Second, we define

$$Q(s'|x,s) = \frac{q(s'|x,s)}{A_x(s')}, \quad Q(s''|x,s) = \frac{q(s''|x',s')}{A_{x''}(s'')}Q(s'|x,s), \quad \ldots,$$

recursively, for any state and any time in the future, with $Q(s|x,s) = \frac{1}{A_x(s)}$.

Third, by iterating on the budget constraint (A.8) we obtain the initial asset holding allocation $(a(s_0),\, a_l(s_0))$ given by

$$a(s_0) = \sum_{t=0}^{\infty}\sum_{s_t|s_0} Q(s_t|x(s_0),s_0)[c(x_t,s_t) - \theta_t f(n(x_t,s_t)) + g_t], \tag{A.11}$$

$$a_l(s_0) = \sum_{t=0}^{\infty}\sum_{s_t|s_0} Q(s_t|x(s_0),s_0)c_l(x_t,s_t), \tag{A.12}$$

where we have used the fact that the Fund contract allocation is stationary and the value functions are bounded, implying that the transversality conditions are satisfied. Now we have an identification between the initial condition $x(s_0) = \mu_{b,0}/\mu_{l,0}$ and the initial asset holdings $(a(s_0), a_l(s_0))$, where $a_l(s_0) = -a(s_0)$. In order to extend this map to all portfolio of asset holdings, we use the law of motion of $x$ (9), also decomposed as

$$\bar{x}'(s) = \frac{1+\nu_b}{1+\nu_l}\eta x \quad \text{and} \quad \hat{x}'(s') = \frac{\varphi(s'|s,e)}{1+\nu_l}\eta x. \tag{A.13}$$

Now, given any $(x,s)$ — say, $(x(s_0),s_0)$ — we have $a(s)$, by (A.13) and we have $\bar{x}'(s)$ and $\hat{x}'(s')$, which map into $\bar{a}'(s)$ and $\hat{a}(s')$ by (A.8) and (A.9), and we also have $-a_l(s') = a(s') = \bar{a}'(s) + \hat{a}(s')$. Furthermore, we impose the equilibrium condition $\boldsymbol{a} = a$. Finally, as we said, we also have $\tau'(s';\boldsymbol{a},s)$ and, by (A.10), we have $\bar{\tau}(a,s)$.

**Step 2**: *Getting policies, bounds, multipliers and policy functions.* The mapping implies that we can construct the borrower's policies that implement the Fund contract as: $m(a,s) = m(x,s)$, for $m = c,n,e$. Similarly, given the definition of *threshold $x$-bounds* in Section 2.1.3, we can define the borrowing and lending limits for every $s$:

$$\mathcal{A}_b(s) = a(\underline{x}(s),s) \text{ and } \mathcal{A}_l(s) = a(\overline{x}(s),s)$$

Note that these limits are history-independent and hence they are functions of only the exogenous state $s$. Note also that these borrowing constraints imply that $a'(s';a,s) \geq \mathcal{A}_b(s)$ and $a_l'(s';a,s) \geq \mathcal{A}_l(s)$ for all $s$; i.e., the constructed asset holdings satisfy the competitive equilibrium borrowing constraints (35) and (43). We now need to show that the policy functions, as functions of $(a,s)$ and $(a_l,s)$, satisfy all the constraints of borrowers and lenders' competitive equilibrium problems.

Next, we define the multiplier of the borrower's maximization problem as:

$$\lambda(\boldsymbol{a}, s) = \frac{1 + \nu_l(x, s)}{1 + \nu_b(x, s)} \frac{1}{x}, \tag{A.14}$$

which guarantees that the consumption policy $c(a, s)$ is optimal in the competitive equilibrium as it satisfies (37). Since $c(x, s)$ and $n(x, s)$ satisfy the Fund labor optimality condition in (15), $c(a, s)$ and $n(a, s)$ satisfy the equilibrium labor optimality condition in (38). For the risk-neutral lender $c_l(a_l, s) = c_l(x, s)$ is an optimal consumption policy, as long as the corresponding asset-portfolio is optimal. To see whether the asset policies are optimal competitive policies, we need to show that they bind exactly when the limited enforcement constraints bind in the Fund.

When $a'(s'; a(s), s) > \mathcal{A}_b(s')$, it must be that

$$\begin{aligned} q_x(s'|s) &\equiv \pi(s'|s, e(x, s))A_{x'}(s')\beta\frac{u'(c(a'(s'; a, s), s'))}{u'(c(a, s))}\frac{1}{1 + \tau'_x(s', s)} \\ &\geq \pi(s'|s, e(x, s))A_{x'}(s')\frac{1}{1 + r}. \end{aligned}$$

and, given the taxes we have constructed, also that $\nu_l(x', s') \geq \nu_b(x', s') = 0$; therefore, whenever the weak inequality is an inequality, adding $(1 + \nu_l(x', s')) > 0$ to the right-hand side of the previous equation equates it to (A.7). This changes the inequality into an equality (multiplying the right-hand side).

Similarly, when $a'_l(s'; a\,(s)\,, s) > \mathcal{A}_l(s')$, it must be that

$$\begin{aligned} q(s'|x, s) &\equiv \pi(s'|s, e(x, s))A_{x'}(s')\frac{1}{1 + r} \\ &\geq \pi(s'|s, e(x, s))A_{x'}(s')\beta\frac{u'(c(a'(s', a, s), s'))}{u'(c(a, s))}\frac{1}{1 + \tau'_x(s', s)}, \end{aligned}$$

and also that $\nu_b(x', s') \geq \nu_l(x', s') = 0$. Then, whenever the weak inequality is an inequality, adding $\frac{\nu_b(x', s')}{u'(c(x, s)(1 + \tau'_x(s', s))}$ to the right hand side, equates the inequality. This shows that the asset policies bind when the Fund LE constraints bind. Therefore, setting $\tilde{\gamma}_b(a, s) = \nu_b(x, s)$ and $\tilde{\gamma}_l(a, s) = \nu_l(x, s)$, the asset portfolio policy satisfies the equilibrium optimality conditions with respect to assets prices in (40) and (45).

Finally, provided that the effort policy $e(a, s)$ is also an optimal policy, the identification of the value functions, $W^i(a, s) = V^i(x, s)$ for $i = b, l$, is consistent with their definition: (11) and (12) become (33) and (41). But, by construction, $e(a, s) = e(x, s)$. If $W^b(a, s) = V^b(x, s)$ and $e(x, s)$ satisfies the IC constraint in (17), then $e(a, s)$ satisfies the equilibrium optimality condition for effort (39) as well. □

**Proof of Proposition 3.** We show that, given a RCE satisfying Definition 3, we can design a Fund contract satisfying equations (9)–(15) and (17)–(20). First, as in the proof of

Proposition 2, we obtain a mapping $(\boldsymbol{a}, s) \to (x, s)$, which is also well-defined for the initial state. From (37): $u'(c(a, s)) = \lambda(a, s)$, so we can write (40) as:

$$\lambda(a,s)q(s'|\boldsymbol{a},s) = \beta\pi(s'|s,e(a,s))A(\boldsymbol{a}',s')\frac{1+\hat{\gamma}_b(a',s')}{1+\tau'(s';\boldsymbol{a},s)}\lambda(a',s'), \tag{A.15}$$

where $\hat{\gamma}_b(a', s')$ is the normalized multiplier defined according to

$$\hat{\gamma}_b(a',s') \equiv \frac{\tilde{\gamma}_b(a',s')}{\beta\pi(s'|s,e(a,s))A(\boldsymbol{a}',s')\lambda(a',s')}.$$

Similarly, we can write (45) as:

$$q(s'|\boldsymbol{a},s) = \frac{1}{1+r}\pi(s'|s,e(a,s))A(\boldsymbol{a}',s')(1+\hat{\gamma}_l(a',s')), \tag{A.16}$$

where $\hat{\gamma}_l(a', s') \equiv \tilde{\gamma}_l(a', s')/\left[\frac{1}{1+r}\pi(s'|s, e(a, s))A(\boldsymbol{a}', s')\right]$. Dividing (A.15) by (A.16) we get:

$$\lambda(a,s) = \eta\frac{1+\hat{\gamma}_b(a',s')}{1+\hat{\gamma}_l(a',s')}\frac{1}{1+\tau'(s';\boldsymbol{a},s)}\lambda(a',s').$$

Define $x'(\boldsymbol{a}', s') \equiv \frac{1+\hat{\gamma}_l(a',s')}{1+\hat{\gamma}_b(a',s')}\frac{1}{\lambda(a',s')}$ and, as implied by Definition 3, there exist $\tilde{\varrho}$ and hence $\tilde{\varphi}(s'|s, e)$ at $(\boldsymbol{a}, s)$ such that $\frac{1}{1+\tau'(s';\boldsymbol{a},s)} \equiv 1 + \frac{\tilde{\varphi}(s'|s,e(a,s))}{1+\hat{\gamma}_b(a,s)}$. Then we can obtain the *recursive system of weights* $x(\boldsymbol{a}, s)$ in (56):

$$x'(\boldsymbol{a}',s') = \frac{1+\hat{\gamma}_b(a,s)+\tilde{\varphi}(s'|s,e(a,s))}{1+\hat{\gamma}_l(a,s)}\eta x(\boldsymbol{a},s)$$
$$\text{with}\quad \tilde{\varphi}(s'|s,e(a,s)) = \tilde{\varrho}(a,s)\frac{\partial_e\pi(s'|s,e(a,s))}{\pi(s'|s,e(a,s))},$$

and we have the mapping $(\boldsymbol{a}, s) \to (x, s) \equiv (x(\boldsymbol{a}, s), s)$, where the equivalence is notational.

This allows us to identify $m(x, s) = m(a, s)$, for $m = c, n, e$ and $c_l$, and particularly for the initial state $m(x(s_0), s_0) = m(\boldsymbol{a}(s_0), s_0)$. Furthermore, if we identify $\nu_b(x, s) \equiv \hat{\gamma}_b(a, s)$ and $\nu_l(x, s) \equiv \hat{\gamma}_l(a, s)$, then, we obtain (14):

$$u'(c(x,s)) = \frac{1+\nu_l(x,s)}{1+\nu_b(x,s)}\frac{1}{x} = \lambda(a,s) = u'(c(a,s)),$$

and (15). Similarly, we can identify the value functions: $V^{bf}(x, s) = W^b(a, s)$ and $V^{lf}(x, s) = W^l(a, s)$. Then, regarding $e(x, s) = e(a, s)$, note that, by Definition 3, $e(a, s)$ solves (54) and, therefore, satisfies

$$\frac{1}{1+r}\sum_{s'|s}\partial_e\pi(s'|s,e(a,s))W^l(a',s')$$

$$= \tilde{\chi}(\boldsymbol{a}, s) \left[ v''(e(a,s)) - \beta \sum_{s'|s} \partial_e^2 \pi(s'|s, e(a,s)) W^b(a', s') \right],$$

where $\tilde{\chi}(\boldsymbol{a}, s) \equiv \frac{\tilde{\varrho}(a,s)}{1+\tilde{\gamma}_l(a,s)} x(\boldsymbol{a}, s)$, which is just a version of (18). Similarly, with our identification of multipliers, the constraint qualification constraints (19) and (20) are satisfied. In sum, equations (9)–(15) and (17)–(20) are satisfied and there is an unique *Fund contract* which implements the RCE. □

Note that taking into account both proofs of Proposition 2 and 3, we have established a *one-to-one* mapping between $(x, s)$ and $(\boldsymbol{a}, s)$.

## B On the Prescott-Townsend Implementation

This section discusses an alternative implementation of the optimal Fund allocation following Prescott and Townsend (1984). To maintain comparability to our implementation with asset taxes, we allow the borrower to trade in long run state contingent assets. However, in the absence of asset taxes, the externality of effort and the fact that there is two sided limited commitment in the optimal contract are captured by two additional constraints which are imposed directly in the borrower's problem: the incentive compatibility constraint and a state-contingent upper bound on the borrower's asset holdings. The problem of the borrower can be written as follows:

$$\begin{aligned} W^b(a, s) = \max_{\{c,n,e,a'(s')\}} & U(c, n, e) + \beta \mathbb{E}\left[W^b(a'(s'), s') \middle| s, e\right] \quad \text{s.t.} \\ c + \sum_{s'|s} q(s'|\boldsymbol{a}, s) a'(s') A(\boldsymbol{a}', s') &\leq \theta(s) f(n) - g(s) + aA(\boldsymbol{a}, s), \\ a'(s') A(\boldsymbol{a}', s') &\geq \mathcal{A}_b(s'), \\ v'(e) &= \beta \sum_{s'|s} \frac{\partial \pi(s'|s, e)}{\partial e} W^b(a'(s'), s'), \\ -a'(s') A(\boldsymbol{a}', s') &\geq \mathcal{A}_l(s'), \end{aligned} \tag{B.1}$$

with $A(\boldsymbol{a}, s) = 1 - \delta + \delta\kappa + \delta q(\boldsymbol{a}, s)$ and the borrowing limits $\mathcal{A}_b(s')$ and $\mathcal{A}_l(s')$ are endogenous as in the implementation with asset taxes. Note first that, in contrast with the decentralization of Section 3, constraint (B.1) is a constraint in the borrower's problem, not in the lender's problem; however, regarding the RCE, both formulations are equivalent, although they have different interpretations. Second, we are expressing the borrowing (and saving) constraints in terms of the per period payoffs of the borrower's long term asset, as this simplifies the algebra and it makes it more compatible with the original Fund design problem.

As in the equilibrium with asset taxes, competitive risk neutral international lenders participate in this market as well. They do not face any constraints beyond a no-Ponzi condition and this implies that they price the long term securities as follows:

$$q(\boldsymbol{a}, s'|s) = \frac{1}{1+r}\pi(s'|s,e)A(\boldsymbol{a}', s'), \tag{B.2}$$

Using the previous equation, we can replace the price in the budget constraint of the borrower and obtain the following condition:

$$c + \frac{1}{1+r}\sum_{s'|s}\pi(s'|s,e)a'(s')A(\boldsymbol{a}', s') \leq \theta(s) f(n) - g(s) + aA(\boldsymbol{a}, s) \tag{B.3}$$

This condition, together with incentive compatibility, guarantees that all feasible allocations are incentive compatible and acceptable by international lenders. It is easy to show that this equilibrium satisfies the optimality conditions of the optimal Fund allocation and can therefore implement it. To see this, note that the Lagrangean of the borrower's problem is:

$$\begin{aligned}
&\max_{\{c,n,e,a'(s')\}} U(c,n,e) + \beta\sum_{s'|s}\pi(s'|s,e)W^b(a'(s'), s') \\
&\quad + \lambda_b(a,s)\left[\theta(s) f(n) - g(s) + aA(\boldsymbol{a}, s) - c - \frac{1}{1+r}\sum_{s'|s}\pi(s'|s,e)a'(s')A(\boldsymbol{a}', s')\right] \\
&\quad + \gamma_b(a'(s'), s')\pi(s'|s,e)[a'(s')A(\boldsymbol{a}', s') - \mathcal{A}_b(s')] \\
&\quad + \psi(a,s)\left[\beta\sum_{s'|s}\partial_e\pi(s'|s,e)W^b(a'(s'), s') - v'(e)\right] \\
&\quad + \gamma_l(a'(s'), s')\pi(s'|s,e)[-a'(s')A(\boldsymbol{a}', s') - \mathcal{A}_l(s')].
\end{aligned}$$

First, the labor optimality condition of the Fund contract is satisfied. Second, the optimality condition of consumption and the envelope condition imply:

$$u'(c) = \lambda_b(a,s)$$

$$\frac{\partial W^b(a,s)}{\partial a} = \lambda_b(a,s)A(\boldsymbol{a}, s) = u'(c)A(\boldsymbol{a}, s).$$

Moroever, the optimality condition with respect to $a'(s')$ is:

$$\begin{aligned}
0 = {} & \beta\frac{\pi(s'|s,e)\partial W^b(a'(s'), s')}{\partial a'(s')} - \lambda_b(a,s)q(s'|\boldsymbol{a}, s)A(\boldsymbol{a}', s') \\
& + \psi(a,s)\beta\partial_e\pi(s'|s,e)\frac{\partial W^b(a'(s'), s')}{\partial a'(s')} \\
& + \gamma_b(a'(s'), s')\pi(s'|s,e)A(\boldsymbol{a}', s') - \gamma_l(a'(s'), s')\pi(s'|s,e)A(\boldsymbol{a}', s').
\end{aligned}$$

After substituting for the equilibrium price, the previous equation can be rewritten as:

$$\left[\frac{1}{\lambda_b(a,s)}+\frac{\psi\,(a,s)}{\lambda_b(a,s)}\frac{\partial_e\pi(s'|s,e)}{\pi(s'|s,e)}\right]\beta\frac{\partial W^b(a'(s'),s')}{\partial a'(s')}$$
$$=\frac{A(\boldsymbol{a}',s')}{1+r}-\frac{\gamma_b(a'(s'),s')-\gamma_l(a'\,(s')\,,s')}{\lambda_b(a,s)}A(\boldsymbol{a}',s').$$

Substituting the envelope condition and the optimality condition for consumption (which allows to eliminate $A(\boldsymbol{a}', s')$), we obtain:

$$\left[\frac{1}{u'(c)}+\frac{\psi(a,s)}{\lambda_b(a,s)}\frac{\partial_e\pi(s'|s,e)}{\pi(s'|s,e)}\right]\beta(1+r)$$
$$=\frac{1}{u'(c(a'(s'),s'))}\left[1-\frac{(1+r)(\gamma_b(a'(s'),s')-\gamma_l(a'(s'),s'))}{\lambda_b(a,s)}\right].$$

Note that $\beta(1+r)=\eta$ and set $\chi(x,s)=\frac{\psi(a,s)}{\lambda_b(a,s)}$. Whenever $\gamma_l(a'(s'),s')=0$, we set $\nu_l(x',s')=0$ and $\frac{1}{1+\nu_b(x',s')}=1-\frac{(1+r)\gamma_b(a'(s'),s')}{\lambda_b(a,s)}$. Moreover, whenever $\gamma_b(a'(s'),s')=0$, we set $\nu_b(x',s')=0$ and $1+\nu_l(x',s')=1+\frac{(1+r)\gamma_l(a'(s'),s')}{\lambda_b(a,s)}$. Hence this equilibrium also delivers constrained efficient consumption allocations.

We now turn to the optimality condition with respect to effort, which is given by:

$$0=-v'(e)+\beta\partial_e\pi(s'|s,e)W^b(a'(s'),s')+\psi(a,s)\left[\beta\sum_{s'|s}\partial_e^2\pi(s'|s,e)W^b(a'(s'),s')-v''(e)\right]$$
$$-\lambda_b(a,s)\frac{1}{1+r}\sum_{s'|s}\partial_e\pi(s'|s,e)a'(s')A(\boldsymbol{a}',s')$$

The incentive compatibility constraint simplifies the above equation to:

$$\frac{1}{1+r}\sum_{s'|s}\partial_e\pi(s'|s,e)a'(s')A(\boldsymbol{a}',s')=\frac{\psi(a,s)}{\lambda_b(a,s)}\left[v''(e)-\beta\sum_{s'|s}\partial_e^2\pi(s'|s,e)W^b(a'(s'),s')\right]$$

implying that effort is also constrained efficient.

## C More Details on the Calibration

### C.1 Data Sources and Model Consistent Measures

The main data sources and relevant definitions of data variables are listed in Table C.1. To map the data to the model, we construct model consistent data measures as below.

**Labor input** For the aggregate labor input $n_{it}$, we use two series from AMECO, the aggregate working hours $H_{it}$ and the total employment $E_{it}$ of each country over the period 1980–2015. We calculate the normalized labor input as $n_{it}=H_{it}/(E_{it}\times 5200)$, assuming 100 hours of allocatable time per worker per week. However, for most of the data moment computations, we use $H_{it}$ directly, since the per worker annual working hours do not show a

Table C.1: Data Sources and Definitions

| Series | Time | Sources[a] | Unit |
|---|---|---|---|
| Output | 1980–2015 | AMECO (OVGD) | 1 billion 2010 constant euro |
| Government consump. | 1980–2015 | AMECO (OCTG) | 1 billion 2010 constant euro |
| Total working hours | 1980–2015 | AMECO (NLHT)[b] | 1 million hours |
| Employment | 1980–2015 | AMECO (NETD) | 1000 persons |
| Government debt | 1980–2015 | AMECO EDP[c] | end-of-year percentage of GDP |
| Debt service | 1980–2015 | AMECO (UYIGE)[d] | end-of-year percentage of GDP |
| Primary surplus | 1980–2015 | AMECO (UBLGIE)[e] | end-of-year percentage of GDP |
| Bond yields | 1980–2015 | AMECO (ILN)[f] | percentage, nominal |
| Debt maturity | 1990–2010 | OECD, EuroStat, ESM[g] | years |
| Labor share | 1980–2015 | AMECO[h] | percentage |

[a] Strings in parentheses indicate AMECO labels of data series.
[b] PWT 8.1 values for Greece in 1980–1982.
[c] General government consolidated gross debt; ESA 2010 and former definition, linked series.
[d] AMECO for 1995–2015; European Commission *General Government Data* (GDD 2002) for 1980–1995.
[e] AMECO linked series for 1995–2015; European Commission *General Government Data* (GDD 2002) for 1980–1995.
[f] A few missing values for Greece and Portugal replaced by EuroStat long-term government bond yields.
[g] Average across different data sources; sporadic time coverage over countries, see text below; ESM data are obtained from private correspondence.
[h] Compensation of employees (UWCD) plus gross operating surplus (UOGD) minus gross operating surplus adjusted for imputed compensation of self-employed (UQGD), then divided by nominal GDP (UVGD).

significant cyclical pattern and both the level and the trend do not affect the computation of the moments.

**Fiscal position and private consumption** We hold the premise of fitting the *observed* fiscal behavior across the GIPS countries, so that we use directly the *data measures* of government consumption and primary surplus to calibrate the model. However, the cost of such a strategy is on the model consistent measure of private consumption. Note that in the model, primary surplus equals to $y - g - c$, therefore private consumption equals to $y$ minus the sum of $g$ and primary surplus. This is the model consistent measure of private consumption we use in our calibration. Nevertheless, due to small magnitudes in primary surplus relative to GDP, the model consistent measure of private consumption tracks closely the dynamics of the alternative data measure of consumption,[2] and the correlation between the two measure is well beyond 0.97.

**Government debt, spread, and maturity** Since one of the major purposes of this paper is to provide a quantitative assessment of the Euro Area 'stressed' countries, we choose to capture the overall debt burden of those countries by calibrating the general government consolidated gross debt. Indeed, Bocola et al. (2019) argue that matching the overall public debt allows a quantitative sovereign default model to better fit crisis dynamics.

[2] Indeed, the alternative measure is private absorption defined as the sum of private consumption and investment as measured in the data, since there is no capital in our model.

We use the nominal long-term bond yields in AMECO to measure the nominal borrowing costs of the Euro Area 'stressed' countries. For the nominal risk free rate, we use the annualized short-term (3M) interest rates in the Euro money market (obtained from EuroStat with label `irt_st_q`) for 1999–2015, and the annualized short-term (3M) bond return of Germany (obtained from EuroStat with label `irt_h_mr3_q`) for 1980–1998, before the start of Euro. To convert the nominal risk-free rate into real rate, we subtract GDP deflator of Germany from the former series. To arrive at a meaningful measure of the *real* spread, i.e., a spread unaffected by expected inflation hence rightly reflecting the 'stressed' countries' credit risk, we split the sample into two parts. After the introduction of Euro, we can directly use the spread between the 'stressed' countries' long-term nominal bond yields and the nominal risk-free rate, since all rates are denominated in euro and thus subject to the same inflation expectation. The question is much trickier for the period before Euro. Motivated by Du and Schreger (2016), we use spot and forward exchange rates (retrieved from Datastream) to convert the German nominal risk free rate into each stressed country's local currency, hence deriving a synthetic local currency risk free rate, and then take the difference between the local nominal long-term bond yield with the synthetic risk free rate. Since the synthetic risk free rate is denominated in the local currency as well, it is subject to the same inflation expectations as the long-term bond yield, and consequently, the difference is equivalent to the real spread.

The information on the maturity structure of the government debt for the GIPS countries is not comprehensive. The overall time coverage is unequal across countries: 1998–2010 and 2014–2015 for Ireland, 1998–2015 for Greece, 1991–2015 for Spain, 1990–2015 for Italy, and 1995–2015 for Portugal.

### C.2 More Details on the Productivity Shock Estimation

We implement the panel Markov regime switching AR(1) estimation of the productivity process following the expectation maximization approach outlined in Hamilton (1990). To overcome the local maximum problem, we randomize the initialization by 50,000 times. Apart from the parameter estimation results reported in the main text, Figure C.1 shows the smoothed probability for each regime across the GIPS countries. Evidently, regime 3 concentrates around the global financial crisis and the European debt crisis. As a last remark, we discretize the regime switching AR(1) process with 9 grid points for each regime using the method detailed in Liu (2017).

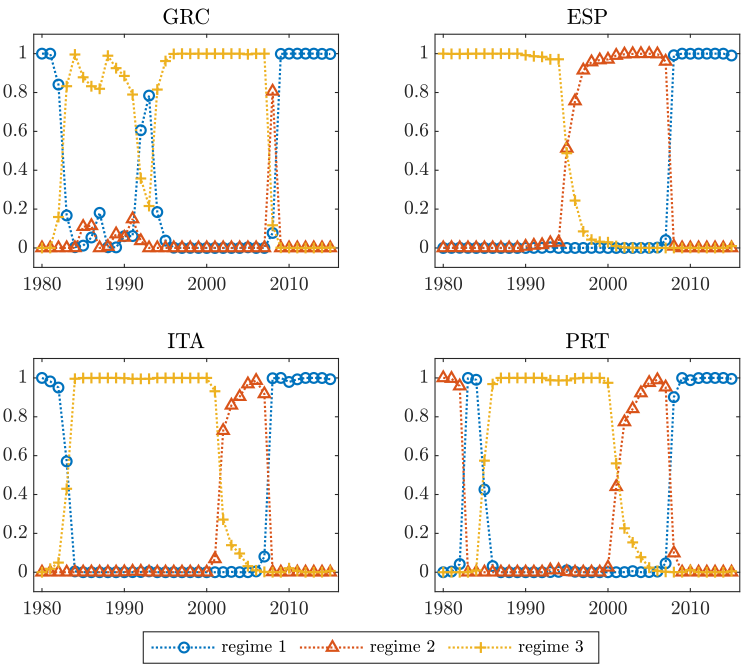


Figure C.1: Smoothed probability for each regime

### C.3 The Numerical Values of the Transition Matrix for $g$

The parameter values imply the following transition matrices:

$$\bar{\pi}^g = \begin{bmatrix} 0.9750 & 0.0167 & 0.0083 \\ 0.0200 & 0.9750 & 0.0050 \\ 0.0100 & 0.0150 & 0.9750 \end{bmatrix}, \ \pi^h = \begin{bmatrix} 0.95 & 0.0333 & 0.0167 \\ 0 & 0.99 & 0.01 \\ 0 & 0 & 1 \end{bmatrix}, \ \pi^l = \begin{bmatrix} 1 & 0 & 0 \\ 0.04 & 0.96 & 0 \\ 0.02 & 0.03 & 0.95 \end{bmatrix}.$$

Note that the *average* distribution $\bar{\pi}^g$ of $g$ constrains the possible effect of effort. Even in this 'extreme' case, the effect of effort is limited. For example by moving effort from 0 to 1 the borrower can increase the chance of reducing government expenditure from 0 to only 7% if the current expenditure is very high.

### C.4 Transition Probabilities for Correlated $g$ and $\theta$

Based on the convenient fact that the number of regimes for $\theta$ and the number of values $g$ can take both equal to 3, we extend the baseline conditional distribution (58) for $\pi^g(g'|g,e)$ to $\tilde{\pi}^g(g'|\varsigma,g,e)$ as follows:

$$\begin{aligned} &\tilde{\pi}^g(g'=g_j|\varsigma=i,g=g_k,e) \\ =\ & w[\zeta(e)\pi^l(j|4-i)+(1-\zeta(e))\pi^h(j|4-i)] \\ &+(1-w)[\zeta(e)\pi^l(j|k)+(1-\zeta(e))\pi^h(j|k)], \quad i,j=1,2,3, \end{aligned} \tag{C.1}$$

where $i$ denotes the regime of $\theta$, $j$ denotes the value of future $g'$, and $k$ denotes the value of current $g$. The additional parameter $w \in [0, 1]$ controls for the influence on the distribution of $g'$ coming from regime $\varsigma$ of $\theta$: if $w = 1$, then $g'$ only depends on $\varsigma$ but not on $g$; in contrast, if $w = 0$, then $g'$ does not depend on $\varsigma$, and the transition probability is identical to the baseline specification in (58). Moreover, the index $4 - i$ in the second line suggests that when the current regime for $\theta$ is high, i.e., $i$ is larger, then not only future $\theta'$ is high since $\varsigma$ is persistent, but $g'$ also tends to be higher by the persitency inherent to the structure of $\pi^h$ and $\pi^l$ in (59). This feature induces positive correlation between $g$ and $\theta$ for any $w$. Given $\tilde{\pi}^g$ so defined, it is straightforward to construct the overall transition matrix $\pi(s'|s, e)$ accordingly.

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