On the optimal design of a Financial Stability Fund*

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Abstract

We develop a model of a Financial Stability Fund (the 'Fund' henceforth) for a union of sovereign countries. By design, the contract prevents country defaults, as well as undesired expected losses, which in a union translate into excessive risk mutualizations. A participant country has greater ability to borrow and share risks than using sovereign debt financing. The Fund contract also provides better incentives for the country to reduce endogenous risks. These efficiency gains arise from the ability of the Fund to offer long-term contingent financial contracts, subject to limited enforcement (LE) and moral hazard (MH) constraints. We develop the theory and quantitatively compare the constrained-efficient Fund economy with an incomplete markets economy with default. We calibrate our economy to the euro area 'stressed countries' in the debt crisis (2010–2012). Substantial welfare gains are achieved, particularly in times of crisis. The Fund is, in fact, a risk-sharing, crisis prevention and resolution mechanism, which transforms the participant countries' defaultable sovereign debt into the union's safe assets. In sum, our theory can help to improve current official lending practices and, for example, to eventually design a *European Fiscal Fund*.

Key words: Fiscal unions, Recursive contracts, Debt contracts, Partnerships, Limited enforcement, Moral hazard, Debt restructuring, Debt overhang, Sovereign funds

JEL codes: E43, E44, E47, E63, F34, F36

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1 Introduction

We develop a dynamic model of a *Financial Stability Fund* (Fund) as a long-term partnership between the Fund and participating sovereign countries. Long-term Fund contracts are the instrument of the Fund, and the design, properties and comparison of these contracts with defaultable sovereign debt contracts are the focus of this paper. A fund contract is based on an in-depth analysis of the country's risk profile. It must address four features that are usually seen as most problematic for a lending and risk-sharing institution to be sustainable when the partnership is a union of sovereign countries. First, sovereignty means that countries can always exercise their right to exit the institution (possibly defaulting on their obligations). Second, risk-sharing, as well as crises prevention or resolution transfers should never become undesirably permanent or go beyond the levels of redistribution, or risk mutualization, that are accepted by all partners. To take these two concerns into account, *Fund contracts* are subject to *limited enforcement constraints* (LE) on the borrower and the lender side, respectively. In particular, our specific design assumes that there are no expected losses at any point in time — i.e., the Fund does not provide any redistribution *ex ante* or *ex post* — although it can be easily extended to state-contingent (e.g., solidarity) permanent transfers.[1]

Third, the Fund must take into account moral hazard problems, since governments may be able to reduce future social and economic risks by implementing policy reforms, but often fail to do so whenever these reforms have contemporaneous socio-political costs. Again, sovereignty places constraints here, since the Fund may have limited capacity to fully monitor or enforce policy reform efforts. More importantly, our Fund design respects that national governments have 'ownership' of their policy reforms, while taking into account the potential excessive risk due to the presence of moral hazard. Thus, *Fund contracts* are based on country-specific risk assessments and subject to *moral hazard constraints* (MH). Given that, Fund contracts are 'experience rated.' Therefore, different contracts are offered to countries with different profiles, providing an ex-ante incentive to reduce the risk profile before signing a Fund contract.[2] Moreover, since these contracts incorporate moral hazard constraints, risk-sharing transfers are combined with 'performance-based' long-term rewards (and punishments), which provide incentives for governments to further pursue risk-reduction policy reforms within the contract. Nevertheless, policy reform efforts are not contractable and, accordingly, Fund contracts are not conditional on *ex ante* reforms or austerity packages, which, not surprisingly, are usually perceived as a lender's imposition over the borrowing sovereign country, and often become ineffective (Clancy et al., 2024).

Fourth, risk sharing among *ex ante* equal partners without debt liabilities is relatively easy to design and achieve but, unfortunately, this is not the case in existing unions, for example,

[1]For example, in a more centralized union (e.g., China), or in a politically developed union (e.g., USA or EU), these limited enforcement constraints may be less stringent, reflecting strength of the union (e.g., exit is more unlikely).

[2]In the same way that home-owners may pay for the installation of a proper alarm system before signing a home-insurance contract.

the EMU. In fact, the Euro crisis left a 'debt-overhang problem' which aggravated the euro area divide and the resilience of countries to new crisis, like the pandemic of COVID-19. The Fund contract offers distinct conditions for countries with different growth and risk profiles and can cover large liabilities, provided they are sustainable. Moreover, we show that risky defaultable sovereign debt is more sustainable if it is transformed into safe Fund contracts. With such an operation, the Fund contract is, for the Fund, a safe asset and, for the country, safe sovereign (state-contingent) debt. Therefore, the Fund contract as a safe asset can be backed by the issuance of 'safe bonds'. Finally, the Fund contract can play an important role in resolving existing 'debt-overhang' problems, as well as in creating 'high quality liquid assets', for example, for the euro area.

In sum, the *Financial Stability Fund* is a *constrained-efficient mechanism* which, by integrating the risk-sharing and crisis-resolution functions, becomes a powerful instrument to prevent and confront crises and, as a by-product, also produces safe assets. As stated earlier, one of the main contributions of the paper is the characterization of the constrained efficient Fund contract and the quantitative comparison with the standard instrument used to smooth consumption: sovereign (defaultable) debt financing, which we show is inferior to the Fund.

When characterizing the Fund contract, it should be noted that *limited enforcement* (LE) and *moral hazard* (MH) constraints are *forward-looking* constraints (i.e., the future evolution of the contract is part of the current constraint). Given this, standard dynamic programming techniques cannot be applied to solve the Fund's contracting problem. We use *recursive contracts* (see Marcet and Marimon, 2019) to obtain and characterize the constrained-efficient Fund contract. To our knowledge, this is the first paper using this approach to study optimal lending contracts with LE *and* MH constraints.

The first theoretical contribution is to show that under relatively standard assumptions — in particular, generalizing those on moral hazard of Rogerson (1985) to our dynamic contracting problem — there is a unique solution to the Fund contracting problem. In the characterization of the contract allocation, we show how the LE and MH frictions interact to determine the risk-sharing properties of the Fund contract as well as the maximum sustainable levels of risk sharing and debt. We show how optimal long-term contracts (through state-contingent transfers) can provide sufficient risk-sharing to make sure that borrowers are able to smooth consumption during crisis periods without resorting to default. At the same time, the path of transfers are bounded from above, guaranteeing that the Fund will never accumulate undesired liabilities against the country. We also show that moral hazard considerations lead to transfers that, although deviating from perfect risk-sharing, respond positively to declining government expenditures, which in turn signals high policy effort. Finally, we provide a version of the inverse Euler equation in our environment. In contrast to models that feature only moral hazard (or private information), our model does not feature immiseration, as limited enforcement constraints on the borrower's side prevent it.

The second theoretical contribution of the paper is the proof of the constrained-efficient versions of the Second and First Welfare Theorems. As usual, we prove the SWT by showing that the constrained-efficient allocation can be decentralized with appropriately defined asset prices and holdings. To prove the FWT, however, we have to show that, despite the combination of limited commitment and moral hazard constraints in our setting, the maximization problem of the constrained efficient Fund contract satisfies the interiority and convexity conditions needed for the existence of a saddle-point solution. This, together with the strict concavity assumptions, imply that the Fund allocation is unique and this sets the ground for the proof of our FWT. Note that risk-sharing of an endogenous risk, which can be reduced with effort, involves an external effect, even when effort is contractable (hence, observable): in deciding the level of effort, an impatient sovereign country is likely to neglect the full effect of the effort on the risk-sharing contract. Therefore, with non-contractable effort, the Fund contract must account for the Incentive Compatibility (IC) constraint together with the underlying externality. This can be achieved in different ways. We provide a novel decentralization of the constrained efficient allocation with taxes on Arrow securities which, absent effort, would provide full risk-sharing when limited enforcement constraints are not binding. We show that endogenous borrowing constraints guarantee that the competitive equilibrium is consistent with limited enforcement constraints, while asset-taxes are, in our framework, required to align the private (country-level) incentives for exerting policy effort with the social incentives. In particular, Pigouvian asset-taxes are, in equilibrium, budget neutral and absorb all the asset value variations implied by the moral hazard constraint. In the decentralized economy, the Pigouvian asset-taxes are set by a fiscal authority, which makes the effort and asset choices of the borrower country consistent with the Fund contract.

An alternative decentralization is to require that *all* debt-asset trading satisfies the IC constraint — in particular, the IC constraint is a constraint on the set of feasible borrowing contracts for the country. This is the design pioneered by Prescott and Townsend (1984), but state-contingent contractual constraints may be more difficult to implement than state-contingent asset-taxes, though the latter are not trivial either. In equilibrium, the two decentralizations distort risk-sharing in the same way. As we discuss in Section 3.2, other alternative implementations are also possible.

The third contribution of the paper is a quantitative evaluation of the performance of economies with and without the Fund. In particular, we show how the Euro area 'stressed countries,' would have performed under the Fund during euro crisis (2010–2012). To assess the efficiency of the Fund, we use as a benchmark an incomplete markets model where sovereign countries issue long-term defaultable debt (IMD) to smooth consumption.[3] In order to have a qualitative and quantitative comparison of the two economies, we use the decentralization of the Fund contract described earlier to generate asset holdings and prices that are comparable to those in the IMD economy. Both in the IMD economy and in the Fund economy, interest rates may differ from the risk-free rate. The *positive spreads* in the IMD economy reflect the risk of default. Interestingly, the Fund economy only generates *negative spreads* on sovereign

[3]The only relevant instrument for the 'stressed countries' before the EFSF and ESM programmes were in place.

debt, reflecting the risk that the lender's limited enforcement constraint (i.e., limits for redistribution) is binding. We set the Fund's 'limit for redistribution' to zero, meaning that the Fund never has expected losses.

Our quantitative results are based upon a calibration of the IMD model using data from the Euro area countries that were most affected by the European sovereign debt crisis (Greece, Italy, Portugal, and Spain), with data sample over 1980–2015.[4] The calibrated economy provides a good fit regarding the key variables of interest. In particular, it generates the level of debt and the statistical properties of the spread (mean, volatility and correlation with output) that are in line with the data. We then solve for the constrained-efficient Fund allocation using the same parameters as in the IMD economy to assess quantitatively how the euro area 'stressed countries' would have performed had they had a Fund contract. We compare the IMD and the Fund allocations in a number of ways. We contrast several long run moments of simulated data from both environments, we examine how the two economies respond to severe shocks that resemble the euro-crisis and we evaluate the welfare gains and debt absorbing capacity associated with the Fund. All these comparisons point in the same direction. The Fund is able to provide superior risk sharing (insurance) against shocks through multiple channels. First, it increases the borrowing capacity of the country significantly, smoothing the impact of shocks when they hit through borrowing. This also implies that the Fund can take over very large amount of debt from the borrowing countries without the risk of default episodes. Second, the Fund provides state-contingent payments, generating efficient counter-cyclical primary deficits. Third, while default is costly in the two economies, both because of direct output losses and exclusion from the sovereign debt market, the design of the Fund eliminates default episodes. Fourth, in the absence of default, the borrower does not have to pay any penalties or high spreads on debt whenever borrowing is desirable. Finally, the Fund contract provides incentives for higher risk-reduction effort in normal times, while it allows for lower effort than in the IMD economy in a crisis situation.

Quantitatively, we find that the welfare gains of the Fund are significant: between 7 and 10 percent in consumption-equivalent terms, depending on the state of the economy.[5] The paper then provides a novel decomposition of these welfare gains. We show that the most important sources of welfare gains are the relaxation of the effective borrowing limits, which imply a higher borrowing capacity in the Fund, and the state contingency of payments. In the crisis prone shock states, these two elements constitute at least 95 percent of the total welfare gains, with a somewhat higher contribution by the borrowing capacity.

[4]We do not use the data after 2015, since we wanted to capture the financial and euro crises as shocks, but not to extend the data too long, when euro area governments could internalize the role of the European Stability Mechanism and the ECB in preventing crises.

[5]It is worth to stress that, altough in this paper we assume the Fund absorbs all the sovereign debt, the superior welfare gains can still be achieved when relaxing this exclusivity assumption: following our work, Liu et al. (2023) have shown that similar welfare gains exist if the Fund absorbs an (endogenously determined) minimal part of the debt, with competitive risk-neutral private lenders holding the rest, which becomes safe due to the Fund intervention.

Literature review We are not the first to address how risks could be shared in a monetary union and how to deal with sovereign debt-overhang problems. For example, as an implicit criticism of different proposals to issue some form of joint-liability eurobonds, Tirole (2015) emphasises the asymmetry issue: the optimal (one-period) risk-sharing contract with two symmetric countries is a joint liability debt contract, while the optimal contract between two countries with very different distress probabilities is a debt contract with a cap and no joint liability, where the cap depends on the extent of solidarity that is given by the externality cost of debt default on the lender. With long-term relationships — as they are among sovereign countries that form a union — we show that better contracts can be implemented: the Fund contracts are *constrained-efficient* and they can be implemented as long-term bonds with state-contingent coupons and appropriate taxation of assets.

In terms of optimal long-term contracts, Atkeson (1991) and Thomas and Worrall (1994) study lending contracts in international contexts.[6] Both of these papers consider only limited enforcement from the borrower's side. Similar to our paper, Atkeson (1991) also considers moral hazard, but with respect to consuming or investing the borrowed funds.[7] Similarly to our paper, Atkeson (1991) also proves existence of the constrained efficient allocation. There are two main differences compared to our approach. First, we do not follow the Abreu, Pearce and Stachetti (1986, 1990) methodology as he does, but the Marcet and Marimon (2019) approach, for which we are the first to provide an existence proof with moral hazard. Second, lenders live only for two periods in his economy, and hence they can only offer one period debt contracts. In our environment, instead, the fund offers long term contracts and hence we solve a dynamic saddle-point problem with a 'first-order approach' moral hazard constraint.

Finally, in related and contemporaneous work, Müller et al. (2019) study dynamic sovereign lending contracts with moral hazard with respect to reform policy effort and one-sided limited enforcement. They provide an interesting characterization and decentralization of the constrained efficient allocation in a stylized model (e.g., normal times are an absorbing state) and their mechanism heavily relies on complex *ex post* default procedures. Closer in scope to our work, Dovis (2019) studies a setting with one sided limited enforcement and private information (adverse selection) and he shows that one can capture the state contingency of the optimal contract through partial default and active debt maturity management. The paper

[6] In the context of long term financial contracts, DeMarzo and Fishman (2007) and DeMarzo and Sannikov (2006) study the Fund contract between a risk neutral agent that seeks financing from risk neutral investors under either hidden effort or hidden cash-flows. They show that the Fund contract can be implemented with standard securities (such as long term debt, a line of credit and equity) and with endogenous termination of the contract. In contrast, we consider a risk-averse agent with hidden effort and implement the long term contracts with state-contingent assets, asset taxes and no termination.

[7] Tsyrennikov (2013) studies a quantitative version of Atkeson's environment in which a country borrower seeks investment financing but lenders cannot observe the use of funds by the borrower. The paper shows that moral hazard friction helps the model to replicate some of the key empirical regularities of emerging economies, while limited enforcement frictions have a much limited effect. In contrast, our model features moral hazard because of unobservable effort, and limited enforcement constraints turns out to be more important in our calibration.

rationalizes sovereign default as a decision that is *ex post* inefficient but *ex ante* necessary to sustain the efficient interaction between the contracting parties. In periods of distress, the debt contract is implicitly made state-contingent by allowing for 'excusable' defaults with partial repayments (Grossman and Van Huyck, 1988). Even though such events are rare, they imply losses for the lenders as debt remittance is only partial.

From the perspective of quantitative normative-positive analysis, it is interesting to know the *constrained efficiency* properties of contracts where *ex ante* state-contingencies are replaced by *ex post* active debt management, default episodes or debt renegotiations. This is the focus and contribution of the work of Müller et al. (2019), Dovis (2019) and others. We could have considered other specifications of the incomplete markets with default (IMD) economy, as well as of state-contingent contracts, or arrangements, different from long-term Arrow securities, but our simplifying choices respond to the need of focusing on the design and characterization of the Fund. In fact, our IMD calibration to the euro area four 'stressed countries' during the euro crisis fits remarkably well to the observed level of self-insurance and cyclicality of these countries. Other calibrations with different models can obtain similar fits, but what determines the welfare gains from having a 'constrained-efficient Fund' is the difference between the time-series generated from the fitted IMD model and the economy with the Fund, given the same underlying stochastic process.

Finally, our model of the Fund as a partnership builds on the literature on dynamic optimal contracts with enforcement constraints (e.g., Thomas and Worrall, 1988; Kocherlakota, 1996; Marcet and Marimon, 2019), but, as discussed earlier, we develop the theory further by incorporating moral hazard constraints. There is also a related literature on the decentralization of optimal contracts (e.g., Alvarez and Jermann, 2000; Krueger et al., 2008) and, since we introduce taxes on state-contingent bonds, our paper also relates to the new dynamic public finance literature (e.g., Golosov et al., 2003). Finally, our benchmark incomplete markets economy with long-term debt with default builds on the model of Chatterjee and Eyigungor (2012), who extends the sovereign default models of Eaton and Gersovitz (1981) and Arellano (2008) to long-term debt.

The paper is organized as follows. Section 2 presents the economy with the Fund and with incomplete markets and defaultable long-term sovereign debt. Section 3 shows how to decentralize the Fund contract with state-contingent long-term bonds. Section 4 discusses the calibration. Section 5 quantitatively compares the IMD and Fund regimes, concluding with a welfare comparison and a counterfactual 'euro-crisis' simulation. Section 6 concludes. All proofs and more details of calibration are relegated to Appendix A and C respectively, while Appendix B discusses an alternative implementation of the optimal contract *à la* Prescott and Townsend (1984).

2 The Economy

We consider an infinite-horizon small open economy where the 'benevolent government' acts as a representative agent with preferences for current leisure, $\ell = 1 - n \in [0, 1]$, consumption, $c \geq 0$, and effort, $e \in [0, 1]$, valued by $U(c, n, e) \equiv u(c) + h(1 - n) - v(e)$. We make standard assumptions on preferences: u, h, v are differentiable; $u'(x) > 0, h'(x) > 0$ for $x \geq 0$, $v'(x) > 0$ for $x > 0$, and $v'(0) = 0$; $u''(x) < 0, h''(x) < 0$, $v''(x) > 0$ and $v'''(x) \geq 0$. The government discounts the future at the rate β, satisfying $\beta \leq 1/(1 + r)$, where r is the risk-free world interest rate; in general, we will assume the inequality to be strict.

The country has access to a decreasing-returns labour technology $y = \theta f(n)$, where $f'(n) > 0$, $f''(n) < 0$, and θ is a productivity shock $\theta \in \{\theta_m : m = 1, \ldots, N_\theta\}$, $\theta_m < \theta_{m+1}$, and we assume it is a Markov process with transition probability $\pi^\theta(\theta'|\theta)$. The monotonicity and concavity/convexity properties of the preferences and technology are standard and needed to have a well-behaved optimization problem. The additive separability of preferences is not without loss of generality, but it makes both the analytics and computation more tractable. Finally, the additional requirement on the cost of effort (i.e. $v'''(x) \geq 0$) is needed to guarantee that the Fund contract design problem is concave. For our quantitative results, we will specify functional forms that satisfy these assumptions.

The country also needs to cover its government expenditures, or liabilities, which are represented by g with $g \in \{g_i : i = 1, \ldots, N_g\}$, $g_i > g_{i+1}$. To an extent, g is endogenous, since the current period effort of the government (representative agent) determines the distribution of expenditures next period — i.e. the Markovian transition probability is given by $\pi^g(g'|g, e)$. Costly higher effort results in a distribution of expenditures that *first order stochastically dominates* the distribution with lower effort. We assume that effort is not contractable and, while our theoretical results assume that productivity and government expenditure shocks are independent, in our quantitative results we allow them to be positively correlated. In sum, the exogenous state of the economy is given by $s = (\theta, g) \in S$ with the overall Markovian transition given by $\pi(s'|s, e) = \pi^\theta(\theta'|\theta)\pi^g(g'|g, e)$.

As it is standard in models with private effort, we assume full support: $\pi(s'|s, e) > 0$ for all s', s and $e > 0$. This implies that (i) for interior effort, our model generates an ergodic set of S that includes all possible combinations of shocks with positive probability and (ii) our incentive problem is well-defined, as there are no states of the world where infinite punishments can be used.

Most of our our analysis focuses on a set-up where the country can manage its private and public debt liabilities with the help of a Financial Stability Fund (Fund), which acts as a benevolent risk-neutral principal/planner who has access to the international capital markets at the risk-free rate. Note that, since the country is an open economy, $\theta f(n) - (c + g)$, does not need to be zero period by period, allowing for transfers between the country and the Fund. This framework is compared to an incomplete markets economy with default (IMD), that is, an economy where the government accesses directly the international capital markets by issuing non-contingent defaultable long-term debt. In order to make the economies quantitatively comparable, Section 3.1.3 implements the Fund allocation as a competitive equilibrium.

2.1 The Economy with a *Financial Stability Fund* (Fund)

The Financial Stability Fund (Fund) is modeled as a long-term contract between a Fund (also called lender) and an individual partner (also called country or borrower) who is the government of the small open economy. The Fund contract chooses a state-contingent sequence of consumption, leisure and effort that maximises the life life-time utility of the borrower given some initial level of the borrower's debt. The Fund contract is self-enforcing through the presence of two limited-enforcement constraints. First, we assume that, if the country ever defaults on the Fund contract, it will not be able to sign a new contract with the Fund and will enter the markets for defaultable long-term debt as a defaulter. The Fund contract, however, makes sure that the country never finds it optimal to renege on the contract. Second, the contract also prevents the Fund from accumulating liabilities against the borrower beyond a specific level Z. Note that accumulating liabilities would be equivalent to permanent transfers towards one country and, in a union of countries, this could become institutionally infeasible and the Fund would therefore not commit to this path. Third, since a component of risk (government liabilities) is endogenous, the contract also has an incentive compatibility constraint since effort is non-contractable (i.e., it is private information, or a sovereign's right of the country). Thus, the long-term contract must provide sufficient incentives for the country to implement a (constrained) efficient level of effort. Note that the borrower does not take directly into account that the effort exerted also affects the Fund's payoffs. This externality will be key to understand how the efficient fund allocation distributes consumption, labor and effort across states and over time.

In sum, the Fund contract can provide risk-sharing and consumption smoothing with state-contingent transfers. However, these transfers are constrained by limited enforcement and moral hazard frictions. Note also that the Fund contract is based on a country-specific risk-assessment, as the allocation depends on all the underlying parameters describing preferences, technology and the shock process.

2.1.1 The Long Term Contract

In its extensive form, the *Fund contract* specifies that in state $s^t = (s_0, \ldots, s_t)$, the country consumes $c(s^t)$, uses labour $n(s^t)$ and exercises effort $e(s^t)$, resulting in a transfer to the Fund of $c_l(s^t) = \theta f(n(s^t)) - (c(s^t) + g)$, with $c_l(s^t) < 0$ implying that the country is effectively borrowing. With *two-sided limited enforcement* and *moral hazard*, an optimal Fund contract is a solution to the following problem:

$$\max_{\{c(s^t),n(s^t),e(s^t)\}} \mathbb{E}\left[\mu_{b,0}\sum_{t=0}^{\infty}\beta^t U(c(s^t),n(s^t),e(s^t)) + \mu_{l,0}\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t c_l(s^t)\middle| s_0\right] \tag{1}$$

$$\text{s.t.}\quad \mathbb{E}\left[\sum_{j=t}^{\infty}\beta^{j-t}U(c(s^j),n(s^j),e(s^j))\middle| s^t\right]\geq V^o(s_t), \tag{2}$$

$$v'(e(s^t))=\beta\sum_{s^{t+1}|s^t}\partial_e\pi(s^{t+1}|s^t,e(s^t))V^{bf}(s^{t+1}), \tag{3}$$

$$\mathbb{E}\left[\sum_{j=t}^{\infty}\left(\frac{1}{1+r}\right)^{j-t}c_l(s^j)\middle| s^t\right]\geq Z, \tag{4}$$

$$\text{and}\quad c_l(s^t)=\theta_t f(n(s^t))-c(s^t)-g_t,\quad \forall s^t, t\geq 0, \tag{5}$$

where

$$V^{bf}(s^t)\equiv\mathbb{E}\left[\sum_{j=0}^{\infty}\beta^j U(c(s^{t+j}),n(s^{t+j}),e(s^{t+j}))\middle| s^t\right].$$

In the previous problem, $(\mu_{b,0},\mu_{l,0})$ are the initial Pareto weights, which are key for our interpretation of the Fund contract as a lending contract. In particular, we show in Section 3.1.3 that the initial relative Pareto weight determines uniquely the level of debt that the Fund takes over when the country joins. Note also that the notation is implicit about the fact that expectations are conditional on the implemented effort sequence, as it affects the distribution of the shocks.

Constraints (2) and (4) are the *limited enforcement constraints* for the borrower and the lender, respectively, in state s^t. The outside value for the borrower if it were to break the Fund contract is denoted by $V^o(s_t)$, and the lender can only commit to contracts that deliver *ex post* expected gains in state s that exceed Z. $V^o(s_t)$ will be defined formally in Section 2.2, where we present the economy with incomplete markets and default (IMD), which will serve as the fall-back option for the borrower in the Fund contract. We assume that if a country 'breaks its Fund contract' — i.e. defaults on its Fund liabilities — the cost is the same as defaulting in the IMD economy, although there is no return to the Fund: the Fund commits to never issue a new Fund contract to the defaulting country.[8] The following assumption regarding the IMD economy is necessary for our interiority Assumption 3 on the Fund contract and to define bounds that play a major role in our characterization later. With a finite number of states, this is a fairly mild (unrestrictive) assumption.

Assumption 1 (*The worst and the best options outside the Fund*)**.** *There exist a $\underline{s}\in S$ and a $\overline{s}\in S$ such that $V^o(\underline{s})=\min_s V^o(s)$ and $V^o(\overline{s})=\max_s V^o(s)$; furthermore $V^o(\overline{s})>V^o(\underline{s})$.*[9]

The finite outside option of the lender Z measures the extent of *ex post* redistribution the Fund is willing to tolerate. That is, if $Z<0$ the Fund is allowed to accumulate liabilities up

[8] In contrast, in the IMD economy we assume that, after default with a (low) probability, it is possible to return to the financial market and borrow from private lenders. Private lenders can not collectively commit to a never-lending agreement. Weakening the commitment of the Fund, allowing for (low probability) re-entry to the Fund will translate in lower welfare gains, but as long as there exists a Fund contract, its characterization will be the one presented here. Nevertheless, a Fund with no-reentry commitment we think is a better design.

[9] In our calibrated economies $\underline{s}=(\theta_1,g_1)$ and $\overline{s}=(\theta_{N_\theta},g_{N_g})$.

to level Z, but it cannot commit to sustain any level lower than Z. Clearly, the level of Z has an important impact on the amount of risk sharing in our environment and it can thus be interpreted as solidarity, as in Tirole (2015).[10]

Constraint (3) is the *moral hazard* (i.e., incentive compatibility) constraint with respect to the borrower's effort, which is not contractable and $V^{bf}(s^{t+1})$ represents the value of the Fund contract for the borrower in state s^{t+1}. The interpretation of this constraint is standard: the marginal cost of increasing effort has to be equal to the marginal benefit. The latter is measured as the change in life-time utility due to the change in the distribution of future shocks as a result of the increasing effort.[11] Note that (3) uses the *first-order condition approach*, that is, we replace the agent's full optimization problem with respect to effort by its necessary first-order conditions. Following Rogerson (1985), we now introduce assumptions to guarantee that this condition is also sufficient. To do this, note that if the $\{\theta_t\}$ and $\{g_t\}$ processes are independent we can define:

$$F_j(e,s) = \sum_{i=1}^{j} \pi^g(g' = g_i|s,e).$$

Assumption 2 (*Independence, Differentiability, Monotonicity, and Convexity*)**.** *The $\{\theta_t\}$ and $\{g_t\}$ processes are independent. The cumulative distribution function $F_j(e,s)$ is differentiable in e implying that $\pi^g(g' = g_i|g,e)$ is differentiable for every g and $e > 0$. For every g and $e > 0$, the ratio $\frac{\partial_e \pi^g(g'=g_i|g,e)}{\pi^g(g'=g_i|g,e)}$ is increasing in i; furthermore, $\partial_e^2 F_j(e,s) = \sum_{i=1}^{j} \partial_e^2 \pi^g(g' = g_i|s,e) \geq 0$ and $\partial_e^3 F_j(e,s) \geq 0$.*

The independence assumption is made to have a single shock variable, g, depending on effort, as in the standard moral hazard problem. Except for the last assumption (i.e. $\partial_e^3 F_j(e,s) \geq 0$) these conditions simply generalize the assumptions in Rogerson (1985), so that we can apply his *first-order condition approach* in a simple static Pareto-optimization problem to our dynamic contracting problem with limited enforcement and moral hazard frictions.[12] The last assumption guarantees that, if we replace the equality in (3) with a weak inequality, $\leq$, the corresponding set of feasible efforts, e, is convex, and the Lagrangean of the contract problem is concave in e.

Finally, note that the formulation of the problem implicitly assumes interiority of effort, as we impose the incentive compatibility constraint as equality. In our setting, this is guaranteed,

[10]As will be noted, we can introduce state-dependence of this constraint to allow for 'solidarity permanent transfers' (i.e., $Z(s) < 0$ with positive probability) by properly adapting the few results that rely on its state-independence. Alternatively, we can consider $Z < 0$ large enough that the Fund's limited enforcement constraint (4) is effectively never binding, as in a one-sided limited enforcement environment, but we don't consider such environment a proper description of an economy with a Fund; in particular, in an economy formed by a union of countries, there are limits to permanent transfers across countries — beyond solidarity or agreed redistribution — across countries which, in our Fund contracts, $Z = 0$ deters.

[11]Throughout the paper, we use ∂_x and ∂_x^2 to denote the first and second derivatives of a function with respect to variable x respectively.

[12]More precisely, the monotonicity condition is Rogerson's monotone likelihood-ratio condition (MLR) and $\partial_e^2 F_j(e,s) \geq 0$ is Rogerson's convexity of the distribution condition (CDF).

since full risk sharing is not the optimal allocation and appropriate Inada conditions are imposed on the cost $v(e)$ and the marginal benefit $\partial_e \pi(s'|s,e)$ of effort. Assumption 3 below provides formal conditions to generate interiority, and it guarantees that there are always rents to share — i.e. breaking the contract is not efficient — which also implies that, at any state, at most one of the limited enforcement constraints is binding. This assumption is also necessary for the existence of the saddle-point recursive functional equation (SPFE) provided below, as it guarantees the boundedness of the associated Lagrange multipliers.

Assumption 3 (*Interiority*). *There is an $\epsilon > 0$, such that, for all $s_0 \in S$ there is a program $\{\tilde{c}(s^t), \tilde{n}(s^t), \tilde{e}(s^t)\}_{t=0}^{\infty}$ satisfying constraints (2) and (4) when, on the right-hand side, $V^o(s_t)$ and Z are replaced by $V^o(s_t)+\epsilon$ and $Z+\epsilon$, respectively and, similarly, when in (3) '$v'(e(s^t)) =$' is replaced by '$v'(e(s^t)) + \epsilon \leq$.'*

2.1.2 Recursive Formulation

It is known from Marcet and Marimon (2019) and Mele (2014) that we can rewrite the general fund contract problem as a saddle-point Lagrangian problem:[13]

$$
\begin{aligned}
\text{SP} \min_{\{\gamma_b(s^t),\gamma_l(s^t),\xi(s^t)\}} \max_{\{c(s^t),n(s^t),e(s^t)\}} \mathbb{E}\Bigg[& \sum_{t=0}^{\infty} \beta^t \Big(\mu_{b,t}(s^t) U(c(s^t), n(s^t), e(s^t)) - \xi(s^t) v'(e(s^t)) \\
& \qquad + \gamma_b(s^t) \big[U(c(s^t), n(s^t), e(s^t)) - V^o(s_t) \big] \Big) \\
& + \sum_{t=0}^{\infty} \left(\frac{1}{1+r} \right)^t \Big(\mu_{l,t+1}(s^t) [\theta_t f(n(s^t)) - c(s^t) - g_t] - \gamma_l(s^t) Z \Big) \Bigg| s_0 \Bigg] \qquad (6) \\
\text{s.t. } & \mu_{b,t+1}(s^{t+1}) = \mu_{b,t}(s^t) + \gamma_b(s^t) + \xi(s^t) \frac{\partial_e \pi(s^{t+1}|s^t, e(s^t))}{\pi(s^{t+1}|s^t, e(s^t))}, \\
& \mu_{l,t+1}(s^t) = \mu_{l,t}(s^{t-1}) + \gamma_l(s^t), \text{ with } \mu_{b,0}(s^0) \equiv \mu_{b,0},\ \mu_{l,0}(s^{-1}) \equiv \mu_{l,0} \text{ given,}
\end{aligned}
$$

where $\beta^t \pi(s^t|s_0, \boldsymbol{e}^{t-1}) \gamma_b(s^t)$, $\left(\frac{1}{1+r}\right)^t \pi(s^t|s_0, \boldsymbol{e}^{t-1}) \gamma_l(s^t)$ and $\beta^t \pi(s^t|s_0, \boldsymbol{e}^{t-1}) \xi(s^t)$, with $\boldsymbol{e}^{t-1} \equiv \{e(s^j)\}_{0 \leq j \leq t-1}$, are the Lagrange multipliers of the limited enforcement constraints (2), (4), and incentive compatibility constraint (3), respectively, in state s^t. The above formulation of the problem defines two new co-state variables $\mu_b(s^t)$ and $\mu_l(s^t)$, which represent the temporary Pareto weights of the borrower and the lender respectively. These variables are initialized at the original Pareto weights and they become time-variant because of the limited commitment and moral hazard frictions. In particular, a binding limited enforcement constraint of the borrower (lender) will imply a higher welfare weight of the borrower (lender) so that he does not leave the contract. In addition, the moral hazard friction (whenever $e > 0$ and $\xi > 0$, i.e., whenever the incentive compatibility constraint is binding) implies that the co-state variable of the borrower will increase or decrease depending on the sign of the likelihood ratio

[13] Following Marcet and Marimon (2019), we only consider saddle-point solutions and their corresponding saddle-point multipliers. That is, given $\Phi(a, \lambda)$, (a^*, λ^*) solves SP $\min_\lambda \max_a \Phi(a, \lambda)$ if and only if $\Phi(a, \lambda^*) \leq \Phi(a^*, \lambda^*) \leq \Phi(a^*, \lambda)$, for any feasible action a and Lagrangian multiplier $\lambda \geq 0$.

$\frac{\partial_e \pi^g(g_{t+1}|g_t,e(s^t))}{\pi^g(g_{t+1}|g_t,e(s^t))}$. In particular, a positive likelihood ratio, which occurs with a low government expenditure, provides a good signal about effort and hence the borrower will be rewarded with a higher temporary Pareto weight. Note that the monotonicity assumption guarantees that the likelihood ratio is increasing in g_{t+1}, that is, the Pareto weight of the borrower is rewarded for low realizations of g_{t+1} next period. This leads the borrower to exert higher effort in the current period.

As in Marcet and Marimon (2019) we could write the saddle-point Bellman equation with the co-state vector (μ_b, μ_l) defined in (6), in which case the Fund value function would be homogeneous of degree one in μ, as the above Lagrangian. Therefore, the Fund value function has an Euler representation where μ is the vector of Pareto *positive* weights assigned to the Fund and the contracting partner (the lender and the borrower). Nevertheless, in the above Fund contract, only relative Pareto weights matter for the allocations, and this allows us to reduce the dimensionality of the co-state vector and write the problem recursively by using a convenient normalization. Let $\eta \equiv \beta(1+r) \leq 1$ and define the discounted relative Pareto weight of the borrower as $x_t(s^t) \equiv [\beta(1+r)]^t \mu_{b,t}(s^t)/\mu_{l,t}(s^t)$. We normalize the multipliers as follows:

$$\nu_b(s^t) = \frac{\gamma_b(s^t)}{\mu_{b,t}(s^t)}, \quad \nu_l(s^t) = \frac{\gamma_l(s^t)}{\mu_{l,t}(s^{t-1})}, \quad \varrho(s^t) = \frac{\xi(s^t)}{\mu_{b,t}(s^t)},$$
$$\varphi(s^{t+1}|s^t, e(s^t)) = \varrho(s^t)\frac{\partial_e \pi(s^{t+1}|s^t, e(s^t))}{\pi(s^{t+1}|s^t, e(s^t))}.$$

Note that $\varphi_{t+1}(s^{t+1}|s^t, e(s^t))$ can be positive or negative depending on whether the derivative with respect to effort in the numerator is positive or negative. The law of motion of x can then be defined recursively as:

$$x_{t+1}(s^{t+1}) = \frac{1 + \nu_{b,t}(s^t) + \varphi_{t+1}(s^{t+1}|s^t, e(s_t))}{1 + \nu_{l,t}(s^t)}\eta x_t(s^t), \text{ with } x_0 = \mu_{b,0}/\mu_{l,0}. \tag{7}$$

With this normalization, ν_b and ν_l become the multipliers of the limited enforcement constraints, corresponding to (2) and (4) and ϱ becomes the multiplier of the incentive compatibility constraint corresponding to (3). Moreover, the state vector for the problem (including the new co-state) becomes (x, s). The *Saddle-Point Functional Equation* (**SPFE**) — i.e., the saddle-point version of Bellman's equation — is given by:

$$\begin{aligned} FV(x, s) = \text{SP} \min_{\{\nu_b, \nu_l, \varrho\}} \max_{\{c,n,e\}} \Big\{ & x[(1+\nu_b)U(c,n,e) - \nu_b V^o(s) - \varrho v'(e)] \\ & + [(1+\nu_l)(\theta(s)f(n) - c - g(s)) - \nu_l Z] + \frac{1+\nu_l}{1+r}\mathbb{E}[FV(x', s')\big|s, e] \Big\} \end{aligned} \tag{8}$$

$$\text{s.t. } x'(s') = \frac{1 + \nu_b + \varphi(s'|s,e)}{1+\nu_l}\eta x \quad \text{with} \quad \varphi(s'|s,e) = \varrho\frac{\partial_e \pi(s'|s,e)}{\pi(s'|s,e)} \tag{9}$$

where, as we will see, the Fund's value value functions can be decomposed as follows:

$$FV(x,s) = xV^{bf}(x,s) + V^{lf}(x,s), \text{ with} \tag{10}$$

$$V^{bf}(x,s) = U(c(x,s), n(x,s), e(x,s)) + \beta \mathbb{E}\left[V^{bf}(x'(s';x,s),s')\middle| s, e(x,s)\right], \text{ and} \tag{11}$$

$$V^{lf}(x,s) = c_l(x,s) + \frac{1}{1+r}\mathbb{E}\left[V^{lf}(x'(s',x,s),s')\middle| s, e(x,s)\right], \text{ where} \tag{12}$$

$$c_l(x,s) = \theta(s)f(n(x,s)) - g(s) - c(x,s). \tag{13}$$

The derivation of this recursive SPFE follows the standard procedure of Marcet and Marimon (2019). We study economies where the SPFE equation (8) has a solution for every (x,s). Without moral hazard constraints, which lead to an endogenous g process, a direct application of Marcet and Marimon (2019, Theorem 3 & Corollary) would establish the existence of a unique solution to the SPFE (8) by contraction mapping, and no additional assumptions would be required. We extend their existence and uniqueness results accounting for moral hazard by generalizing the first-order approach of Rogerson (1985).

In what follows, we provide a preliminary characterization of the Fund allocation by looking at the optimality conditions. To simplify notation, we let the policy for the relative Pareto weight be given by $x'_{xs}(s') \equiv x'(s';x,s) \equiv x'(g';x,s)$. The policy functions for consumption of the Fund contract must solve the first-order conditions of the SPFE. In particular, $c(x,s)$ and $n(x,s)$ must satisfy:

$$u'(c(x,s)) = \frac{1+\nu_l(x,s)}{1+\nu_b(x,s)}\frac{1}{x} \tag{14}$$

$$\frac{h'(1-n(x,s))}{u'(c(x,s))} = \theta f'(n(x,s)). \tag{15}$$

These conditions are standard. Given that preferences are separable, the labor supply is undistorted. Moreover, the optimality condition for the borrower's consumption in (14) will play a key role, since it provides a direct link between the optimal consumption allocation $c(x,s)$ and the relative Pareto weight x: if limited enforcement constraints are not binding, the relative weight is the inverse of the marginal utility of consumption and, if they bind, there is a *wedge* given by the corresponding multiplier. The effort policy $e(x,s)$ is determined by the first order condition of the SPFE with respect to e, which can be conveniently expressed as:

$$\begin{aligned} v'(e(x,s)) = {} & \beta \sum_{s'|s} \partial_e \pi(x'_{xs}(s'),s') V^{bf}(x'_{xs}(s'),s') \\ & + \frac{1+\nu_l(x,s)}{1+\nu_b(x,s)}\frac{1}{x}\frac{1}{1+r}\sum_{s'|s} \partial_e \pi(x'_{xs}(s'),s') V^{lf}(x'_{xs}(s'),s') \\ & - \frac{\varrho(x,s)}{1+\nu_b(x,s)}\left[v''(e(x,s)) + \beta \sum_{s'|s} \partial_e^2 \pi(x'_{xs}(s'),s') V^{bf}(x'_{xs}(s'),s')\right]. \end{aligned} \tag{16}$$

Equation (16) balances the marginal cost of effort with the marginal benefits. The first line is the life-time utility benefit of effort to the borrower; the second line is the marginal benefit of effort to the lender, in terms of the borrower's marginal utility, given by (14); the third line accounts for the marginal relaxation/tightening effect of the moral hazard constraint (3) when there is a change in effort. With contractable effort, the Fund problem would not have the *incentive compatibility constraint* (3) and the effort decision would be given by the first two lines, with the second one accounting for the social value of effort. In contrast, with non-contractable effort, as we assume, constraint (3) is present and the first line is equal to zero, namely:

$$v'(e(x,s)) = \beta \sum_{s'|s} \partial_e \pi(s'|s, e(x,s)) V^{bf}(x'_{xs}(s'), s'). \tag{17}$$

In this case, (16) reduces to

$$\begin{aligned} &\frac{1}{1+r} \sum_{s'|s} \partial_e \pi(s'|s, e(x,s)) V^{lf}(x'_{xs}(s'), s') \\ &\qquad = \chi(x,s) \left[v''(e(x,s)) - \beta \sum_{s'|s} \partial_e^2 \pi(s'|s, e(x,s)) V^{bf}(x'_{xs}(s'), s') \right], \end{aligned} \tag{18}$$

where $\chi(x,s) \equiv \frac{x\varrho(x,s)}{1+\nu_l(x,s)}$ can be interpreted as the marginal value of relaxing the ICE constraint in terms of the lender's valuation; that is, (18) accounts for the external effect of effort on the lender's value through its effect on the incentive compatibility constraint. Note that, although incentive compatibility implies that only the borrower's returns affect the effort decision directly, the benefits represented in (18) will affect incentives, as they affect $\varrho(x,s)$ and hence the whole future path of allocations through (9).

Given the policy function $e(x,s)$, we denote by $\{s\}_{e(x,s)}$ the resulting Markov process of $\{\theta, g\}$ shocks. Furthermore, a recursive constrained-efficient Fund allocation also satisfies the following endogenous limited enforcement (constraint qualification) constraints:

$$\nu_b(x,s)\left[V^{bf}(x,s) - V^o(s)\right] = 0 \text{ with } \nu_b(x,s) = 0 \text{ if } V^{bf}(x,s) > V^o(s), \tag{19}$$

$$\nu_l(x,s)\left[V^{lf}(x,s) - Z\right] = 0 \text{ with } \nu_l(x,s) = 0 \text{ if } V^{lf}(x,s) > Z. \tag{20}$$

Note that, (11) and (12) are the first-order conditions with respect to $\nu_b(x,s)$ and $\nu_l(x,s)$, respectively, when these limited enforcement constraints are binding. Similarly, (17) is the first-order condition with respect to $\varrho(x,s)$ when the incentive compatibility constraint is binding, which, as we show, always is. In contrast, while the limited enforcement constraints are in general not binding, given that the solution to (8) is unique, as we show, (11) and (12) are satisfied, even when the limited participation constraints are not binding (see Marcet and Marimon, 2019).

Definition 1 (**Recursive Constrained-Efficient Fund Contract**). *Given an initial rela-*

tive Pareto weight $x(s_0)$ and outside options $\{V^o(s), Z\}$ for the borrower and lender, the policies for the allocations $\{c(x,s), c_l(x,s), n(x,s), e(x,s)\}$, multipliers $\{\nu_b(x,s), \nu_l(x,s), \varrho(x,s)\}$, value functions $\{V^{bf}(x,s), V^{lf}(x,s)\}$, relative Pareto weight $x'_{xs}(s')$, and the laws of motion for $\{\theta, g\}_{e(x,s)}$ are a recursive constrained-efficient Fund contract if they satisfy conditions (9)–(15) and (17)–(20) for all (x,s).

Proposition 1. *Given our assumptions, for any s_0, $x(s_0)$, and outside options $\{V^o(s), Z\}$, there is a unique recursive constrained-efficient Fund contract.*

Proof: See Appendix A.

Two remarks are in order. First, we use the term *recursive constrained-efficient Fund contract* because it is optimal, given the constraints imposed on it, and it has a recursive structure. Nevertheless, thereafter, we will refer to the *unique recursive constrained-efficient Fund contract* simply as the *Fund contract*. Second, on the one-hand, the Fund contract is 'the policy instrument' of the Fund, who in its design takes the constraints of the borrowing country as given and solves for the borrower's recursive policies labour and consumption. On the other hand, with respect to the effort decision, the Fund acts as a Principal, in a Principal-Agent structure, by taking the first-order condition of the borrower as given.

2.1.3 Characterization of the Fund Contract

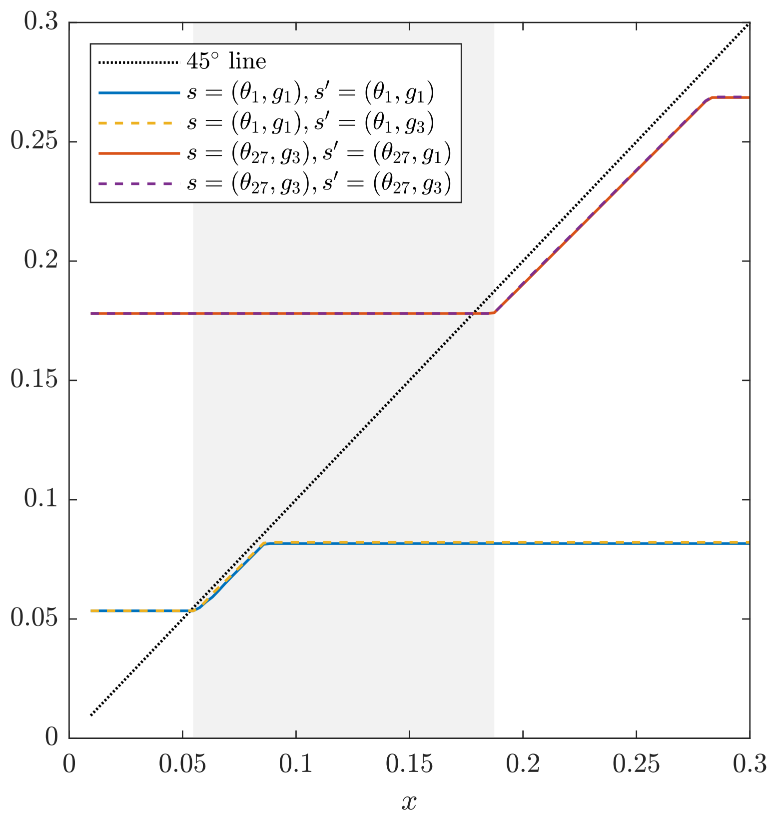
In order to characterize the dynamics of a Fund contract, we define the *threshold x-bounds*: $\underline{x}(s) = \min_x \left\{\nu^b(x,s) = 0\right\}$ and $\overline{x}(s) = \max_x \left\{\nu^l(x,s) = 0\right\}$. Note that they are well defined, since, by Assumption 3, for any s, there is $x(s)$ (and an open set around it) for which both limited enforcement constraints are not binding. Note that, by decreasing $x(s)$, the borrower's LE will eventually be binding and, by increasing it, the lender's LE will be binding. That is, if $x < \underline{x}(s)$ then $\nu_b(x,s) > 0$ and if $x > \overline{x}(s)$ then $\nu_l(x,s) > 0$. The following corollary to Proposition 1 describes the basic dynamic features of Fund contracts.

Corollary 1. *The Fund contract has a steady-state: a long-run stationary allocation determined by a partially endogenous ergodic set of $\{s_t\}$ and an endogenous ergodic set of $\{x(s_t)\}$, with support in $[\underline{x}(\underline{s}),\ \underline{x}(\overline{s})]$. Furthermore, since $\eta < 1$, given any initial condition $x(s_0) > 0$, the Fund contract enters the steady-state in finite time.*

Proof: See Appendix A.

This result follows from the fact that, given any arbitrary initial exogenous state s_0, $x(s_0)$, — possibly $x(s_0) \notin [\underline{x}(\underline{s}),\ \underline{x}(\overline{s})]$ — the minimum and maximum limited enforcement constraints of the borrower are achieved with probability one and, once they have been achieved, x cannot have a smaller or larger value. There is an ergodic set and, therefore, the allocation policies on (x,s), together with the Markov matrix (given by $\pi(s'|s, e(x,s))$ and the law of motion $x'_{xs}(s')$ determine the long-run stationary distribution of allocations (and multipliers). Furthermore, $\eta < 1$ implies that x enters $[\underline{x}(\underline{s}),\ \underline{x}(\overline{s})]$ in finite time. Alternatively, if $\underline{x}(\underline{s}) = \underline{x}(\overline{s})$, Assumption 3 is not satisfied, there are no rents to share and there is no Fund contract. Note that, if $\overline{x}(\underline{s}) \notin [\underline{x}(\underline{s}),\, \underline{x}(\overline{s})]$, there is, effectively, one-sided limited commitment at the steady-state of the Fund contract — i.e. only the borrower's participation constraints may bind at the steady-state — and through the whole implementation of the Fund contract if, in addition, $x(s_o) \leq \underline{x}(\overline{s})$; see Figure 1.

Figure 1: The $x'_{xs}(s')$ policy and the steady-state given by $[\underline{x}(\underline{s}),\, \underline{x}(\overline{s})]$.



Notes: The policy $x'_{xs}(s')$ is evaluated at the extreme values of s, therefore, showing the ergodic set of the Fund contract.

Given the existence of a unique solution to the SPFE and our assumptions, we can establish several important properties. The following ones follow almost directly from the first-order conditions (14) and (15), the resource constraint (13), and the constraint qualification conditions (19) and (20):

Lemma 1. *The Fund contract policy and value functions, solving SPFE (8), satisfy the following properties:*

(a) when the LE constraints do not bind, i.e. at $\underline{x}(s) < x < \overline{x}(s)$*: i)* $x'(s'; x, s)$ *and* $c(x, s)$ *are increasing in* x *and* $c(x, s)$ *does not depend separately on* s*, while, given* (x, s)*,* x' *is increasing in* g'*; ii)* $n(x, s)$ *is decreasing in* x *and increasing in* θ*; iii)* $e(x, s)$ *is decreasing in* x *but not necessarily non-increasing in* g*; iv)* $V^{bf}(x, s)$ *and* $V^{lf}(x, s)$ *are increasing and decreasing in* x*, respectively.*

(b) when the LE constraints bind: i) all policies and value functions are constant, i.e. if $x \leq \underline{x}(s)$, or $x \geq \overline{x}(s)$, then $m(x,s) = m(\underline{x}(s), s)$, or $m(x,s) = m(\overline{x}(s), s)$, respectively, for $m(x,s) = x'(s'; x, s), = c(x,s), = n(x,s), = e(x,s), = V^{bf}(x,s)$ or $= V^{lf}(x,s)$, with $x'(s; \underline{x}(s), s) = \underline{x}(s)$ and $x'(s; \overline{x}(s), s) = \overline{x}(s)$; ii) regarding s, $m(\underline{x}(s), s)$ and $m(\overline{x}(s), s)$ follow the same pattern as in (a) with the exception of consumption, which depends on s iii) the limited enforcement multipliers, $\nu^b(x,s)$, $\nu^l(x,s)$, adapt to keep consumption constant, e.g. $c(x,s) = c(\underline{x}(s), s)$.

Proof: See Appendix A.

This lemma extends the results of the standard two-sided limited commitment model to endogenous labor supply and effort. In the region when neither limited enforcement constraint is binding (a), the allocation is — except for the imperfect risk-sharing — efficient: the consumption of the borrower is increasing with the relative Pareto weight x, while labor supply and effort decrease in x ('wealth effect'); given x, consumption is constant (the imperfect risk-sharing is reflected in x') and labor supply is increasing in θ. Since the borrower is more impatient than the lender, $x' < x$ and, therefore, consumption decreases, as long as x' is not binding. In fact, by (9) and (14), $\mathbb{E}\left[x'(s'; x, s)|s\right]$ and $c(x,s)$ always co-move and, given our Assumption 2, $x'(s'; x, s)$ is non-decreasing in g'. Regarding effort, e, given our convexity assumption on the cost of effort $v(e)$ and the fact that g has a wealth effect on next period expected value (in (3)), one may expect that, given x, effort will be decreasing — or, at least, non-increasing — in g. However, the effort decision (17) is forward looking and maintaining low liabilities (g_i high) may be important if surplus is expected to be tight next period, unless effort is high. Finally, given that x is the relative Pareto weight for the borrower/lender, the monotonicity of the value functions, with respect to x — positive for V^{bf} and negative for V^{lf} — are as expected.

When the limited enforcement constraints bind, Lemma 1 (b) shows that the limited enforcement multipliers do their job of keeping the policies and value functions constant for values of x outside the *threshold x-bounds*. While the policies and value functions follow the patterns of the non-binding limited enforcement constraints in case (a) with respect to s, consumption becomes sensitive to s as equation (14) shows.

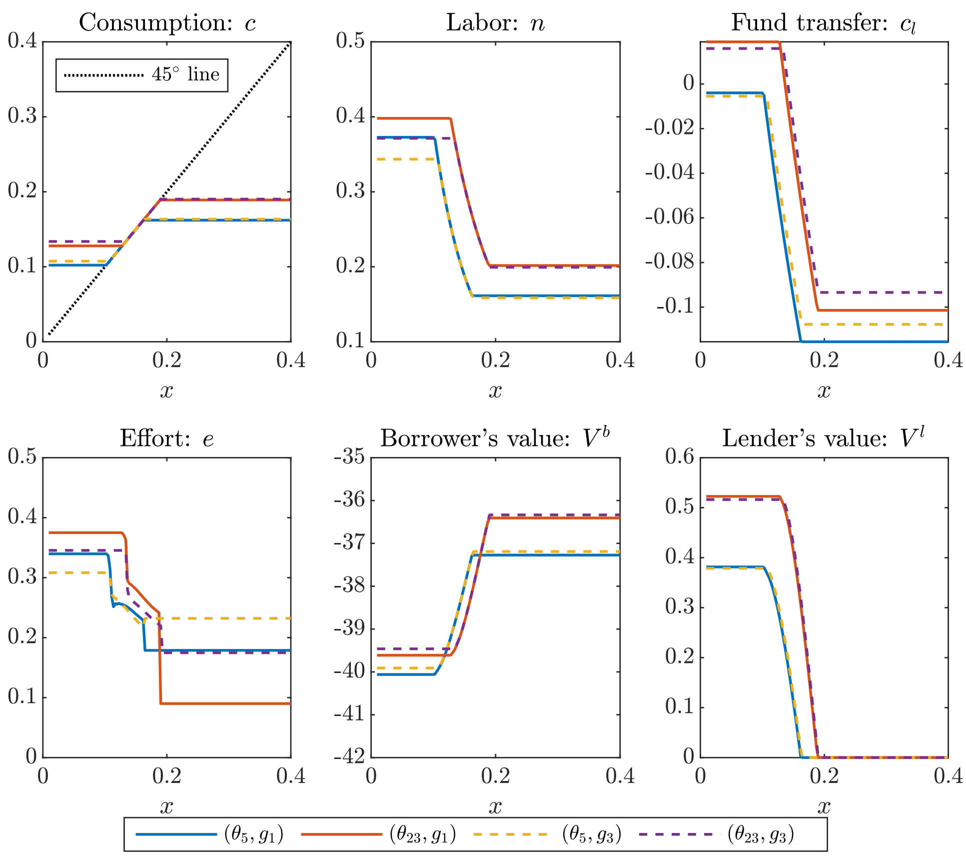
Lemma 1 states an important feature of the Fund contract: when the limited enforcement constraints bind,*iii)* states that $V^{bf}(x,s)$ and $V^{lf}(x,s)$ are increasing and decreasing in x, respectively and *iv)* that they are constant. This implies that the Pareto frontier is downward slopping or flat and, therefore, with a Fund contract there are no *ex post* inefficiencies (in contrast, for example, with Phelan and Townsend, 1991, and Dovis, 2019).[14]

[14] The key difference compared to Phelan and Townsend (1991), who study a similar dynamic moral hazard problem with full commitment, is that they use a utility function that is bounded below and that limits incentive provision substantially for low levels of life-time utility for the agents. To avoid this issue, we use a utility specification that is unbounded below. In addition, Dovis (2019) studies a sovereign debt model with an intermediate import good and shows that the provision of the foreign good becomes inefficiently low when the borrower's lifetime utility is close to autarky. In contrast, we only have labour as a production factor.

Most of the properties described above are reflected and sharpened in Figures 1 and 2, which display the policy functions, starting with the *main engine* of the Fund contract: the graph of the relative Pareto weight of the borrower in Figure 1; in particular, for the economy under study, it shows that both $\underline{x}(s)$ and $\overline{x}(s)$ bounds are active at the steady state, since $\overline{x}(\underline{s}) \in [\underline{x}(\underline{s}), \underline{x}(\overline{s})]$; actually, it suggests that, at the steady-state, there are many *threshold x-bounds*, generating sharp moves followed by periods of decay, with small fluctuations due to imperfect risk-sharing. It also shows that, for extreme values of s, the effect of g, given θ, is very small.[15]

Figure 2 displays the labor, effort and transfer $c_l(x, s) = \theta f(n(x, s)) - c(x, s) - g$ policies, as well as the value functions.[16] All the policies in the figure are displayed for different combinations of the θ and the g states.

Figure 2: Fund policies and value functions.



Note: as a reference, θ_1 and θ_{27} refer to the worst and best productivity shocks respctively, θ_5 and θ_{23} refer to the intermediate bad and good productivity shocks, while g_1 and g_3 refer to the worst and best government consumption shocks respectively.

[15]This effect is, in part, muted by having a positive correlation between θ and g; nevertheless, the effect of g can be larger in less extreme values of s, as in Figure 2.

[16]Section 4 describes the specific stochastic processes and functional forms and provides a more detailed analysis of the simulated economies.

Three observations are worth noting. First, we see how efficiency 'prevails' without binding limited enforcement constraints, $(\nu_b(x,s) = \nu_l(x,s) = 0)$, in the sense that, as Lemma 1 (a) states, with separable preferences, consumption is increasing and labor is decreasing in x (due to 'wealth effects') and that, given x, consumption is independent of s; in fact, with 'log' utility, consumption is equal to the relative Pareto weight, $c = x$ and, given the borrower's relative impatience, $x'_{xs}(s') = \eta x < x$, so that future relative Pareto weights (and consumptions) monotonically decrease over time — through the 'decay line' of slope η — until the borrower's limited enforcement constraint binds. Nevertheless, limited risk-sharing, due to the *always binding* incentive compatibility constraint, makes x', and therefore c' an increasing function of s' — i.e. s' is not fully insured. However, this variability of next-period consumption $c' = x'$, cannot be observed in the graph of the consumption policy function, since it occurs along the 'decay line' (it can be detected in the $x'(s')$ policies of Figure 1 if properly amplified).

Similarly, the lender's limited enforcement constraints deter x from being too high, defining the horizontal lines to the right of the 'decay line' of slope η. This brings us to the second observation: when the limited enforcement constraints bind, changing s has two effects: a potential direct effect and a second effect through the change in $\underline{x}$ or $\overline{x}$, as Lemma 1 *(ii)* states. In particular, a better state (higher θ or lower g) increases both $\underline{x}$ and $\overline{x}$. The way in which effort depends on s shows an interesting pattern that highlights the interaction between the moral hazard and the limited commitment frictions. Through the incentive compatibility constraint, effort depends directly on g, as the probability distribution that effort can affect depends on g due to persistence. However, this effect dependence is constant across x. Hence, it is interesting to notice that $e(x,s)$, as expected, is increasing in g when $x \leq \underline{x}(s)$ — say, from (θ_5, g_1) to (θ_5, g_3) in Figure 2 — but is decreasing in g when $x \geq \overline{x}(s)$ — again, from (θ_5, g_1) to (θ_5, g_3). At the lower bound, given that $\underline{x}(s)$ is increasing, this can be simply due to the wealth effect. However, $\overline{x}(s)$ is increasing as well and we still see the opposite response of effort, i.e. the wealth effect is muted. This is because, whenever the lender's participation constraint is binding (i.e. the borrower country is losing insurance opportunities), maintaining a relatively high effort when g liabilities are low allows the country to lower its deficit — i.e. increase its *fiscal space* — as the graph of the transfer policy $c_l(x,s)$ reveals. Furthermore, this effect is anticipated for relatively high values of unconstrained $x < \overline{x}(s)$, as a form of precautionary deficit reduction when g is low in exchange for a higher deficit when g is high, when x is relatively high and, therefore, the borrower internalizes, to a large extent, the social value of the total surplus; in fact, as can be seen, the plot of the lender's value, V^l is non-decreasing even when x is high.[17] This takes us to the third relevant observation: while the global dynamics are driven by x 'wealth effects', with its 'decay line' and limited enforcement *threshold x-bounds*, the local dynamics are driven by the *always binding* incentive compatibility constraint and its interplay with the lender's limited enforcement constraint (if

[17]Note that a non-monotonic pattern of effort, with moral hazard, also appears in Müller et al. (2019), although, in contrast, in their model it is 'reform effort' and the bind is that high debt levels deter reform.

it binds at the ergodic set).

We now characterize the inverse Euler equation in our setting. First, using the optimality condition with respect to consumption in (14) and the definition of $\varphi(s'|s,e)$, we can express the inverse of the marginal utility of consumption next period as:

$$\frac{1}{u'(c(x'_{xs}(s'),s'))}\frac{1+\nu_l(x'_{xs}(s'),s')}{1+\nu_b(x'_{xs}(s'),s')} = \eta\left[\frac{1}{u'(c(x,s))} + \chi(x,s)\frac{\partial_e\pi(s'|s,e(x,s))}{\pi(s'|s,e(x,s))}\right]. \tag{21}$$

Note that this condition relates consumption in two consecutive periods. If neither limited enforcement is binding next period and there is no moral hazard ($\chi(x,s)=0$) consumption follows a deterministic path determined by η. As we have discussed above, in our environment of relatively impatient borrowers, this implies declining consumption over time. Moral hazard introduces state-contingency in this intertemporal pattern. In particular, consumption is adjusted upwards (downwards) in states that provide a positive (negative) signal about the borrower's effort through the likelihood ratios. Under our monotonicity assumption, this implies that for high (low) g realizations consumption will decrease (increase), ceteris paribus. Finally, if this path reaches a violation of the borrower's (Fund's) limited enforcement constraints, consumption is adjusted upwards as ν_b is positive (downwards as ν_l is positive). Note also that the presence of moral hazard introduces a wedge between the marginal rates of substitution of the borrower and the lender even if the enforcement constraints are not binding. In particular, if $V^{bf}(x'_{xs}(s'),s') > V^o(s')$ and $V^{lf}(x'_{xs}(s'),s') > Z$, equation (21) implies:

$$\beta\frac{u'(c(x'_{xs}(s'),s'))}{u'(c(x,s))}\left[1+\chi(x,s)u'(c(x,s))\frac{\partial_e\pi(s'|s,e(x,s))}{\pi(s'|s,e\,(x,s))}\right] = \frac{1}{1+r}. \tag{22}$$

The following result characterizes the inverse Euler condition in our setting.

Lemma 2. *In a recursive Fund contract the inverse Euler equation takes the following form:*

$$\sum_{s'|s}\pi(s'|s,e)\left[\frac{1}{u'(c(x'_{xs}(s'),s'))}\frac{1+\nu_l(x'_{xs}(s'),s')}{1+\nu_b(x'_{xs}(s'),s')}\right] = \eta\frac{1}{u'(c(x,s))}. \tag{23}$$

Proof: See Appendix A.

In (constrained or unconstrained) dynamic social planning/mechanism design problems, the inverse Euler equation characterizes the intertemporal allocation of consumption. In an unconstrained efficient allocation, this is equivalent to the traditional individual Euler equation that is a result of optimal individual intertemporal consumption choice. Versions of this equation have been derived both for dynamic moral hazard models (see Rogerson, 1985) and for dynamic adverse selection problems (see e.g. Golosov et al., 2003). Our version of this equation, (23) embeds the inverse Euler equations of other problems if limited enforcement constraints are never binding, in which case (23) implies:

$$\mathbb{E}\left[\frac{1}{u'(c(x'_{xs}(s'),s'))}\middle|s\right] \leq \frac{1}{u'(c(x,s))}, \tag{24}$$

with strict inequality if $\eta < 1$. In this latter case, it follows that the inverse of the marginal utility process is a 'positive supermartingale.' Therefore, by the *supermartingale theorem*, consumption converges almost surely to 0 without borrower's limited enforcement constraints, which is the well-known *immiseration* result (see Thomas and Worrall, 1990; Atkeson and Lucas, 1992). Alternatively, if $\eta = 1$, and there is only borrower's one-sided limited commitment (i.e., $\nu_l = 0$), (24) is a 'left bounded positive submartingale' which, without moral hazard, would lead to consumption increase and converge to the level of consumption given by $\underline{x}(s(N))$. In general, limited enforcement constraints of the borrower prevent the immiseration in this environment and put a lower bound on the supermartingale. In sum, in our formulation with $\eta < 1$, two sided-limited commitment and moral-hazard, the (inverse) marginal utility process is characterized by the binding *limited enforcement constraints* recurrently truncating the positive supermartingales processes that are perturbed every period due to moral hazard constraints.

2.2 The Economy with *Incomplete Markets and Default* (IMD)

We now describe the economy with incomplete markets and sovereign debt financing with possible default. This is our second benchmark economy, which plays three roles in our analysis. First, we use this economy as the status quo and we therefore calibrate it to euro area 'stressed countries' — in other words, the *risk assessment* of these countries is done with the IMD model economy. Second, as we have discussed above, the outside option of the borrower in the Fund economy is equivalent to the endogenous outside option of the borrower in the IMD economy. Third, we compare this benchmark economy with the economy with a Fund, to assess the value of introducing this fund in the euro area. The incomplete market model with default is a quantitative version of the seminal model by Eaton and Gersovitz (1981) with endogenous labor supply, policy effort, long-term bonds, and an asymmetric default penalty, to achieve a more complete description of the business cycle dynamics of a small open economy with sovereign debt.[18]

With sovereign debt financing, the borrower can issue or purchase *long-term* bonds, which promise to pay constant cash flows across different states. We model long-term bonds in the same way as Chatterjee and Eyigungor (2012). A unit of long-term bond is parameterized by (δ, κ), where δ is the probability of continuing to pay out the coupon in the current period, and κ is the coupon rate. Alternatively, $1 - \delta$ is the probability of maturing in the current period, and this event is independent over time. The coupon rate κ provides a flexible way to capture the coupon payment, where $\delta\kappa$ equals to the expected coupon payment on each unit of outstanding debt. Note that δ directly captures the maturity of the bond, namely, if $\delta = 0$ and $\kappa = 0$, the bond becomes the standard one-period debt — as in Aguiar and Gopinath (2006) and Arellano (2008) — and, in general, the average maturity of the bond equals to

[18] One can consider alternative benchmark economies where, defaults and/or the bond long-term structure play a larger role as contingencies, as in Müller et al. (2019) or Dovis (2019). Nevertheless, to focus on the Fund mechanism it helps to have a simpler benchmark economy.

$1/(1-\delta)$, which is increasing in δ. By a purchase of one bond we mean, more precisely, the purchase of one unit of a portfolio of a continuum of bonds of infinitesimal size and the same (δ, κ), but with independent realizations within the portfolio. Thus, one unit of bond (δ, κ) repays $(1-\delta) + \delta\kappa$ in any given period (as long as the borrower does not decide to default).

Since the setup of the IMD model is standard, we give only a brief description in what follows. We consider an economy with a continuum of borrowers and of lenders and we describe the decision problems of representative ones. Let b be the size of the long-term bond portfolio held by the borrower at the beginning of a period,[19] and (s, b), $s = (\theta, g)$, be the state. In a given period and state s the sovereign borrower chooses the period consumption, labour and effort (c, n, e) as well as *the value of the debt at maturity* b', taking as given the non-linear price schedule $q(s, b')$.[20] Let $V_n^b(b, s)$ denote the value function of the borrower when the borrower chooses *not* to default. Then it satisfies:

$$V_n^b(b,s) = \max_{c,n,e,b'} U(c,n,e) + \beta \mathbb{E}\left[V^b(b',s')\middle| s,e\right] \quad (25)$$
$$\text{s.t. } c + g + q(s,b')(b' - \delta b) \leq \theta f(n) + (1 - \delta + \delta\kappa) b,$$

where $V^b(b', s')$ denotes the continuation value. When the borrower chooses to default, the value function $V^o(s)$ satisfies

$$V^o(s) = \max_{n,e} \left\{u(\theta^p(\theta) f(n) - g) + h(1-n) - v(e)\right\} + \beta \mathbb{E}\left[(1-\lambda)V^o(s') + \lambda V_n^b(0,s')\middle| s,e\right], \quad (26)$$

where $\theta^p(\theta)$ denotes the productivity net of a penalty, and λ is the probability to come back to the market and be able to borrow again, starting with 0 debt. As is standard in the literature, the outside option is temporary autarky. Moreover, $V^o(s)$ also represents the outside option that the borrower faces in the Fund contract when default is considered.[21] will re-enter the incomplete credit market with probability λ at zero debt.

Finally, the default choice is given by:

$$D(s,b) = 1 \text{ if } V^o(s) > V_n^b(b,s) \text{ and } 0 \text{ otherwise},$$

where $D(s, b) = 1$ denotes default. It follows that the borrower's value function, *prior to* the default decision, is

$$V^b(b,s) = \max\left\{V_n^b(b,s), V^o(s)\right\}, \quad (27)$$

[19] We assume that $b \in [b_{\min}, b_{\max}]$, with $-\infty < b_{\min} < 0 \leq b_{\max} < \infty$., where we will choose $b_{\min}$ and $b_{\max}$ so that in equilibrium the bounds are not binding.

[20] As Ayres et al. (2018) show, for the economy described here, this choice, together with standard assumptions regarding the distribution of θ, guarantees that the equilibrium in the IMD economy is unique and that — as it was already in the cited sovereign debt papers — only fundamental defaults are possible; i.e., there are no 'belief driven' defaults.

[21] This implies, in particular, that after default with the Fund, the borrower stays in autarky for at least one period and, after that, it

When the borrower chooses not to default, the optimality condition with respect to effort takes the following form:

$$v'(e) = \beta \sum_{s'|s} \partial_e \pi(s', s) V^b(b', s'). \tag{28}$$

This equation has a similar form as the incentive compatibility constraint (3) and the same interpretation. Moreover, this condition implies that the optimal effort decision only depends on b through b', hence we can write the policy function as $e(s, b')$. This simplifies considerably the pricing equation of the bond and consequently our computations.

Denoting the expected default rate by $d(s, b') = \mathbb{E}[D(s', b')|s, e(s, b')]$, the equilibrium bond pricing function $q(s, b')$ satisfies the following recursive equation:

$$q(s, b') = \frac{(1-\delta) + \delta\kappa}{1+r}(1 - d(s, b')) + \delta \frac{\mathbb{E}[(1 - D(s', b'))q(s', b''(s', b')|s, e(s, b')]}{1+r} \tag{29}$$

Note that, for a one-period bond ($\delta = 0$), this would reduce to the more familiar expression $q(s, b') = \frac{1-d(s,b')}{1+r}$. Note also that the price of a *riskless* long-term bond (δ, κ) is $q = \frac{(1-\delta)+\delta\kappa}{r+1-\delta}$. Furthermore, the implied interest rate on a risky bond is given by

$$r^i(s, b') = \frac{(1-\delta) + \delta\kappa}{q(s, b')} - (1 - \delta) \tag{30}$$

resulting in a *positive spread* $r^i(s, b') - r \geq 0$, which is strictly positive if $d(s, b') > 0$.

The optimal policies when there is no default $(c(s, b), n(s, b), b'(s, b), e(s, b'(s, b))$ and those when there is default $(n^a(s), e^a(s))$ are standard dynamic programming solutions to (25)–(27), whereas the bond price $q(s, b')$ and implied interest rate $r^i(s, b')$ are a solution to (29) and (30) respectively. Finally, in order to keep track of debt flows and in order to compare with a counterpart for c_l in the Fund contract, it will be useful to define the primary surplus of the borrower, which is also the transfer to the lender, as:

$$c_l^i(s, b) = \theta f(n(s, b)) - (c(s, b) + g) = q(s, b')(b' - \delta b) - (1 - \delta + \delta\kappa)b. \tag{31}$$

In essence, if the country consumes more than it produces, $c_l^i(s, b) < 0$, we say that the country is running a deficit, whereas the country runs a surplus if the opposite. In this sense, we will call $c_l^i(s, b)$ primary surplus (or primary deficit if negative). Here, it is important to note that, in our economy, taxes (and transfers) are implicitly defined by $\theta f(n) - c$. This implies that (31) defines both the primary surplus of the government and the net exports. The two key assumptions behind this equivalence are that only the government has access to any intertemporal borrowing/saving technology and we do not have physical capital accumulation in our model.

3 Implementation of the Fund Contract

In what follows, we show how to implement the Fund contract as a competitive equilibrium with endogenous borrowing constraints and taxes on assets. This will allow us to compare it more directly with the debt contract of the economy with sovereign debt. To do this, we build on the work of Alvarez and Jermann (2000) and Krueger et al. (2008) on limited commitment models, but we consider long-term *state-contingent bonds (assets or securities)* to make them more comparable with the incomplete market model. Our implementation is also related to the new dynamic public finance literature where they show that taxes on capital income or capital holdings are required to provide for efficient incentive provision (see e.g. Golosov et al., 2003). To our knowledge, our paper is the first one that provides such an implementation with both limited commitment and dynamic moral hazard frictions. In Subsection 3.2, we discuss alternative implementations and the differences with ours.

3.1 Implementation with Asset Taxes

3.1.1 Asset Structure

At the beginning of a period, in state s, the borrower holds a portfolio a of securities (δ, κ), where a fraction $1-\delta$ of the portfolio matures in the current period and a fraction δ pays a coupon κ. The borrower can trade in S securities $a'(s')$ with a unit price of $q(s', \boldsymbol{a}'|s)$; and $a'(s')$ pays the *corresponding units of asset* next period only if state s' is realized. The borrower is subject to state contingent taxes $\tau'(s', \boldsymbol{a}'|s)$ on the 'Arrow security' holdings and it receives a lump sum transfer $\overline{\tau}(\boldsymbol{a}, s)$ that make these taxes budget neutral in equilibrium. Note that the price, tax and transfer functions do depend on $\boldsymbol{a}$. However, $\boldsymbol{a}$ in these functions does not represent the individual asset holdings but instead the average asset holdings of borrowers. We will discuss the role of these taxes in the equilibrium in Section 3.1.3. As in the IMD economy, the borrower chooses the amount of net debt issuance $q(s'|\boldsymbol{a}, s)(a'(s')(1+\tau'(s', \boldsymbol{a}'|s)) - \delta a)$. Therefore, the borrower's budget constraint is:

$$c + \sum_{s'|s} q(s', \boldsymbol{a}'|s)(a'(s')(1+\tau'(s', \boldsymbol{a}'|s)) - \delta a) \leq \theta(s)f(n) - g(s) + (1-\delta+\delta\kappa)a + \overline{\tau}(\boldsymbol{a}, s).$$

To make the model as comparable as possible to the IMD economy, we note that the state contingent portfolio can be decomposed into (i) a common 'bond' $\bar{a}'$ that is carried to the next period, is independent of the next period state and is traded at the implicit bond price $q(\boldsymbol{a}, s) = \sum_{s'|s} q(s', \boldsymbol{a}'|s)$, and (ii) an insurance portfolio of S assets $\hat{a}(s')$, with $\hat{a}(s') = a'(s') - \bar{a}'$, $\bar{a}' = \sum_{s'|s} q(s', \boldsymbol{a}'|s)a'(s')/q(\boldsymbol{a}, s)$ and hence $\sum_{s'|s} q(s', \boldsymbol{a}'|s)\hat{a}(s') = 0$.

The budget constraint can then be rewritten as:

$$c + q(\boldsymbol{a}, s)(\bar{a}' - \delta a) + \sum_{s'|s} q(s', \boldsymbol{a}'|s)\hat{a}(s') + \sum_{s'|s} q(s', \boldsymbol{a}'|s)a'(s')\tau'(s', \boldsymbol{a}'|s)$$
$$\leq \theta(s)f(n) - g(s) + (1 - \delta + \delta\kappa)a + \bar{\tau}(\boldsymbol{a}, s). \quad (32)$$

As typical in these type of asset market implementations, alternative forms of asset market structures could potentially deliver the same allocation. However, our main purpose here is to have clear comparison between the two regimes and this asset structure works well for that purpose, since $(a, \bar{a}')$ can be 'identified' with (b, b') in the IMD economy, while $\hat{a}(s')$ corresponds to the additional insurance component provided by the Arrow securities. In addition, we can use the bond price of this equilibrium $q(\boldsymbol{a}, s)$ to compute spreads in this economy, which can be compared with the spreads generated by the IMD economy.

3.1.2 The Recursive Competitive Equilibrium (RCE)

With the above financial structure in hand, we now introduce the equilibrium as a *recursive competitive equilibrium* (in strict sense, a partial equilibrium since the world interest rate is given as the opportunity cost of the lender). In this formulation, the borrower has access to long-term state-contingent assets and solves the following dynamic programming problem:

$$W^b(a, s) = \max_{\{c,n,e,a'(s')\}} U(c, n, e) + \beta\mathbb{E}\left[W^b(a'(s'), s')\big|s, e\right] \quad \text{s.t.} \quad (33)$$

$$c + \sum_{s'|s} q(s', \boldsymbol{a}'|s)(a'(s')(1 + \tau'(s', \boldsymbol{a}'|s) - \delta a) \leq \theta(s)f(n) - g(s) + (1 - \delta + \delta\kappa)a + \bar{\tau}(\boldsymbol{a}, s), \quad (34)$$

$$a'(s') \geq \mathcal{A}_b(s'), \quad (35)$$

where $\mathcal{A}_b(s')$ is the endogenous borrowing constraint which makes the borrower indifferent between fulfilling debt obligations and defaulting (recall (27)); in state s, the limit is defined by the following condition:

$$W^b(\mathcal{A}_b(s), s) = V^o(s). \quad (36)$$

The policies that solve this problem given taxes and transfers $\tau'(s', \boldsymbol{a}'|s)$ and $\bar{\tau}(\boldsymbol{a}, s)$ are denoted by $c(a, s)$, $n(a, s)$, $e(a, s)$, and $a'(s'; a, s)$.[22]

The first-order conditions of (33), with respect to the choice of consumption, labour and effort are given by:

$$u'(c(a, s)) = \lambda(a, s), \quad (37)$$

[22] Note that, since taxes and prices depend on the aggregate state $\boldsymbol{a}$, in principle, all the policies also depend on $\boldsymbol{a}$. To simplify notation, we suppress this dependence in the individual policies, the multipliers and the value functions, unless we want to emphasized.

$$\frac{h'(1-n(a,s))}{u'(c(a,s))} = \theta(s) f'(n(a,s)), \tag{38}$$

$$v'(e(a,s)) = \beta \sum_{s'|s} \partial_e \pi(s'|s,e) W^b(a'(s';a,s),s'), \tag{39}$$

where $\lambda(a,s)$ is the Lagrange multiplier of the intertemporal budget constraint of the borrower. Let $\tilde{\gamma}_b(a,s)$ be Lagrange multiplier on the borrowing constraint,[23] and let

$$A(\boldsymbol{a},s) = (1-\delta+\delta\kappa) + \delta q(\boldsymbol{a},s), \quad \text{with } q(\boldsymbol{a},s) = \sum_{s'|s} q(s',\boldsymbol{a}'|s).$$

Then the first order condition with respect to asset holdings is given by:

$$q(s',\boldsymbol{a}'|s) = \beta\pi(s'|s,e(a,s)) \frac{u'(c(a'(s';a,s),s'))}{u'(c(a,s))(1+\tau'(s',\boldsymbol{a}'|s))} A(\boldsymbol{a}',s') + \frac{\tilde{\gamma}_b(a'(s';a,s),s')}{u'(c(a,s))(1+\tau'(s',\boldsymbol{a}'|s))}, \tag{40}$$

where $\tilde{\gamma}_b(a'(s';a,s)),s') \geq 0$, with $\tilde{\gamma}_b(a'(s';a,s),s') = 0$ if $a'(s';a,s) > \mathcal{A}_b(s')$.

The lender, who has linear preferences for — possibly negative — consumption solves the following problem:

$$W^l(a_l,s) = \max_{\{c_l, a_l'(s')\}} c_l + \frac{1}{1+r} \mathbb{E}\left[W^l(a_l'(s'),s') \middle| s,e\right] \tag{41}$$

$$\text{s.t. } c_l + \sum_{s'|s} q(s',\boldsymbol{a}'|s)(a_l'(s') - \delta a_l) = (1-\delta+\delta\kappa)a_l, \tag{42}$$

$$a_l'(s') \geq \mathcal{A}_l(s'), \tag{43}$$

where $\mathcal{A}_l(s')$ is the endogenous lending constraint, which guarantees that the present value of the debt liabilities is not less than Z; e.g., if $Z = 0$ (43), making sure that there will be no expected losses. In state s, this limit is defined by the following condition:

$$W^l(\mathcal{A}_l(s),s) = Z. \tag{44}$$

The policies that solve this problem are denoted by $c_l(a,s)$ and $a_l'(s';a,s)$.[24] Let $\tilde{\gamma}_l(a,s)$ be the Lagrange multiplier on the borrowing constraint. The optimality condition with respect to asset holdings implies:

$$q(s',\boldsymbol{a}'|s) = \frac{1}{1+r}\pi(s'|s,e(a,s))A(\boldsymbol{a}',s') + \tilde{\gamma}_l(-a_l'(s';a,s),s'), \tag{45}$$

[23] This $\tilde{\gamma}_b(a,s)$ shall be distinguished from the multipliers $\gamma_b(x,s)$ to borrower's limited enforcement constraints used in the recursive formulation of the Fund in Section 2.1.2. The same applies below for $\tilde{\gamma}_l(a,s)$ used for lender's borrowing constraint. In fact, part of the proofs of Propositions 2 and 3 is to show that they are the same or proportional.

[24] To ease the notation, we have used the fact of $a_l = -a$ as required by market clearing condition. This allows us to write the policy function of the lender as a function of a instead of a_l.

with $\tilde{\gamma}_l(-a_l'(s';a,s),s') \geq 0$ and $\tilde{\gamma}_l(-a_l'(s';a,s),s') = 0$ if $a_l'(s';a,s) > \mathcal{A}_l(s')$.

In the competitive equilibrium, the asset market and goods market clearing conditions are given by:

$$a'(s';a,s) + a_l'(s';a,s) = 0, \quad \forall s' \tag{46}$$

$$c(a,s) + c_l(a,s) = \theta(s)f(n(a,s)) - g(s), \tag{47}$$

with the initial asset holdings $a(s_0)$ and $a_l(s_0) = -a(s_0)$ given.

The transfers are determined endogenously such that the government's budget constraint is cleared period by period:

$$\overline{\tau}(\boldsymbol{a},s) = \sum_{s'|s} q(s',\boldsymbol{a}'|s)a'(s';a,s)\tau'(s',\boldsymbol{a}'|s). \tag{48}$$

Finally, given that in our economy there is a 'representative borrower' and a 'representative lender', we have that, in equilibrium, the individual and aggregate asset holdings need to be consistent, that is $a = \boldsymbol{a}$ for all periods and states.

Definition 2 (Recursive Competitive Equilibrium)**.** *Given initial asset holdings $\{a(s_0), a_l(s_0)\}$, borrowing limits $\{\mathcal{A}_b(s'), \mathcal{A}_l(s')\}$, and taxes and transfers $\{\tau'(s',\boldsymbol{a}'|s), \overline{\tau}(\boldsymbol{a},s)\}$, a* Recursive Competitive Equilibrium *(RCE) consists of policy functions for the allocations*

$$\{c(a,s), n(a,s), e(a,s), a'(s';a,s), c_l(a,s), a_l'(s';a,s))\},$$

prices $q(s',\boldsymbol{a}'|s)$, value functions $\{W^b(a,s), W^l(a,s)\}$, and laws of motion for $\{\theta,g\}_{e(a,s)}$ such that: (i) Given the taxes, transfers, outside options and asset prices $q(s',\boldsymbol{a}'|s)$, the policies for the allocations $\{c(a,s), n(a,s), e(a,s), a'(s';a,s)\}$, together with the value function $W^b(a,s)$, solve the borrower's problem (33) given $\mathcal{A}_b(s')$, and the allocations $\{c_l(a,s), a_l'(s';a,s)\}$, together with the value function $W^l(a,s)$, solve the lender's problem (41) given $\mathcal{A}_l(s')$; (ii) the market clearing conditions and government's budget constraint in (46)–(48) are satisfied; and (iii) the aggregate consistency condition is satisfied, namely, $a = \boldsymbol{a}$.

It will be useful to define the Arrow security price in the competitive equilibrium using (40) and (45) as follows:

$$q(s',\boldsymbol{a}'|s) = \pi(s'|s,e(a,s))A(\boldsymbol{a}',s')\max\left\{\beta\frac{u'(c(a',s')}{u'(c(a,s))}\frac{1}{1+\tau'(s';\boldsymbol{a},s)}, \frac{1}{1+r}\right\}, \tag{49}$$

where $A(\boldsymbol{a},s) = (1-\delta+\delta\kappa)+\delta q(\boldsymbol{a},s)$ and $q(\boldsymbol{a},s) = \sum_{s'|s} q(s',\boldsymbol{a}'|s)$. Using the Arrow security prices, we can then define the intertemporal discount factor as:

$$Q(s',\boldsymbol{a}'|s) = \frac{q(s',\boldsymbol{a}'|s)}{A(\boldsymbol{a}',s')}. \tag{50}$$

3.1.3 Implementation of the Fund Contract as a RCE

We now show how that the recursive Fund contract can be implemented as a recursive competitive equilibrium with long-term state contingent assets, state contingent taxes on the assets and endogenous borrowing limits. This allows us to obtain asset prices and holdings supporting the Fund contract allocation, which we can compare to the debt prices and holdings of the incomplete markets economy with defaultable debt. Moreover, we show that, with the right taxes, the recursive competitive equilibrium with endogenous borrowing limits is also a Fund contract.

We first state Proposition 2 below, whose proof is relegated to Appendix A.

Proposition 2. *Given the initial condition $(s_0, x(s_0))$, outside options $\{V^o(s), Z\}$, policy functions for the Fund contract allocations $\{c(x,s), c_l(x,s), n(x,s), e(x,s)\}$, multipliers $\{\nu_l(x,s), \nu_b(x,s), \varrho(x,s)\}$, value functions $\{V^{bf}(x,s), V^{lf}(x,s)\}$, laws of motion for $\{\theta, g\}_{e(x,s)}$ and relative Pareto weights $x'_{xs}(s')$, there are unique value functions $\{W^b(a,s), W^l(a,s)\}$, allocations $\{c(a,s), c_l(a,s), n(a,s), e(a,s)\}$, asset policies $\{a'(s';a,s), a'_l(s';a,s)\}$, with an initial allocation $(a(s_0), a_l(s_0)$, asset prices $q(s', \boldsymbol{a}'|s)$, asset taxes $\{\tau'(s', \boldsymbol{a}'|s), \overline{\tau}(\boldsymbol{a}, s)\}$, borrowing limits $\{\mathcal{A}_b(s'), \mathcal{A}_l(s')\}$ and a law of motion for $\{\theta, g\}_{e(a,s)}$ which constitute a Recursive Competitive Equilibrium (RCE) that implements the Fund allocation.*

Proposition 2 shows that the *recursive Fund contract* with initial Pareto weights (μ_{b0}, μ_{l0}) can be 'implemented' as a *recursive competitive equilibrium* with 'Arrow security' prices and taxes, and endogenous borrowing constraints with specific initial asset holdings $a_0(s_0)$ and $a_0^l(s_0) = -a_0(s_0)$. The first part of the proof is to show that one can derive 'Arrow security' prices and taxes for the Fund contract — which in Section 5 we use to compare the economy with the Fund with the IMD economy — and then map them into RCE prices and taxes with a one-to-one map between (x,s) and (a,s). In particular, the Fund asset price equation, corresponding to (49), is:

$$q(s'|x,s) = \frac{1}{1+r}\pi(s'|s,e(x,s))A(x',s')\max\left\{\frac{1+\nu_l(x',s')}{1+\nu_b(x',s')}\frac{1}{1+\frac{\varphi(s'|s,e(x,s))}{1+\nu_b(x,s)}}\frac{1}{1+\tau(s';x,s)}, 1\right\} \tag{51}$$

$$= \frac{1}{1+r}\pi(s'|s,e(x,s))A(x',s')\max\left\{\frac{1+\nu_l(x',s')}{1+\nu_b(x',s')}, 1\right\}, \tag{52}$$

where $A(x',s') = (1-\delta+\delta\kappa) + \delta q(x,s)$ and $q(x,s) = \sum_{s'|s} q(s'|x,s)$ and the last equality follows from defining Fund Arrow security taxes $\tau(s';x,s)$ as

$$\frac{1}{1+\tau(s';x,s)} = 1 + \chi(x,s)u'(c(x,s))\frac{\partial_e \pi(s'|s,e(x,s))}{\pi(s'|s,e(x,s))}, \tag{53}$$

where $\chi(x,s)$ is given by equation (18), and the fact that the right-hand side of (53) can be written as $1 + \frac{\varphi(s'|s,e)}{1+\nu_b(x,s)}$.

An immediate implication of (52) is that there will be a *negative spread* if and only if the Fund's limited enforcement constraint is binding; i.e., $\nu_l(x', s') > 0$. In other words, asset taxes (transfers) absorb all the price variability needed to internalise how effort affects the lender's value, hence asset prices are decoupled from moral hazard considerations. Let us briefly discuss the role of asset taxes in this implementation. It is clear that taxes need to satisfy (53). Since effort is not contractable in our economies, the Fund imposes the incentive compatibility constraint, which creates a wedge between the intertemporal rate of substitution of the borrower and the lender in the Fund allocation (see Lemma 2). In the equilibrium, the taxes on Arrow securities directly account for this wedge. Note that these taxes also guarantee that, similar to the Fund allocation, the inverse Euler equation (23) holds in equilibrium as well.[25] Nevertheless, private or official lenders do not have legal power to impose state-contingent taxes on sovereign debt. Hence, an appropriate fiscal institution must implement the Pigou taxes described above, consistent with the Fund contract, since the borrower is unlikely to fully endogenize the effect of effort on asset prices (we come back to this in Section 3.2). We consider this implementation as a relevant theoretical contribution of the present paper.

The 'implementation' above allows us to compare prices and asset allocations in the economy with the Fund and in the economy with incomplete markets and default (IMD). To do this, we let $Q(s, x) = \sum_{s'|s} Q(s'|x, s)$, where $Q(s'|x, s)$ is defined as in (50) according to $q(s'|x, s)/A(x', s')$. The implicit interest rate in the decentralized economy can be obtained from the price of the long-term bond:

$$r^f(s, x) = \frac{1}{Q(s, x)} - 1,$$

which results in a possibly *negative spread*, $r^f(s, x) - r \leq 0$, since $Q(s, x) \geq \frac{1}{1+r}$ due to (52).

To understand the negative spread, consider first the case with *no moral hazard*, implying that $\chi(x, s) = \tau(s'; x, s) = 0$ for all s'. Looking at the expression for $q(s'|x, s)$, it is clear that the *negative spread* in this case reflects the fact that the lender's intertemporal limited enforcement constraint is binding for some state tomorrow, that is, $Q(s, x) > \frac{1}{1+r}$ only if $\nu_l(x'(s'; x, s), s') > 0$ for some s'. In that state, the borrower's relative Pareto weight in the Fund contract, $x'(s'; x, s)$, is typically relatively high given s'. Hence, the borrower's liabilities are in risk to become permanent transfers, i.e., the Fund is in danger of making permanent losses. The negative spread then discourages the Fund from making these permanent transfers to the borrower and, in that sense, the negative spread indirectly imposes a constraint on the amount of insurance the borrower can get.

Consider now the case with moral hazard. In this case, whenever the no limited enforcement constraint is binding, the asset taxes make sure that the intertemporal rate of substitu-

[25] Some form of wealth taxes are used often in the dynamic public finance literature (see e.g., Golosov et al., 2003, Kocherlakota, 2004, 2005) to align equilibrium incentives with the inverse Euler equation. Those models exhibit private information instead of private actions and hence taxes take a different form.

tion is equalized across states (see (53) and (21)) and hence equal to $\frac{1}{1+r}$. This implies that exactly the same argument holds as above. In sum, the negative spread, $r^f(s, x) - r < 0$, reflects the wedge that aligns the market price with the lender's unwillingness to provide further insurance or lending to the borrower in some states of the future.

3.1.4 On the (Constrained) First Welfare Theorem

It is well understood that planner's constrained optimization problems can, in general, be decentralized with competitive prices in an akin form to the Second Welfare Theorem of General Equilibrium; our Proposition 2 is an example. But it is also well understood that, in general, this is not the case for the First Welfare Theorem — i.e. the constrained-efficiency of the competitive decentralized allocation — the first reason being the multiplicity of competitive equilibria, with the autarkic solution often being one of them. In our case, there are two additional problems that may prevent a recursive competitive equilibrium from being constrained-efficient. One is that in the Fund contract there is a recursive saddle-point structure where the state is given by (x, s), where x summarizes the past history of multipliers, while the RCE must be able to reproduce the recursive saddle-point structure with the aggregate state $(\boldsymbol{a}, s)$, where $a = \boldsymbol{a}$ summarizes also the individual endogenous state. The second is that, with non-contractable effort, not only there is a moral hazard problem but, linked to it, an externality which, in principle, is not internalized by the borrower. The (constrained) First Welfare Theorem that we state below and prove in Appendix A accounts for these additional issues.

Regarding the first one, the following assumption, which is the *RCE* version of Assumption 3 for the Fund problem, rules out autarkic equilibria and, together with the convexity assumptions, as in Proposition 1, the *RCE* is unique. Basically it says that this is an economy where at any possible state, $(\boldsymbol{a}, s)$, there are gains from trade.

Assumption 4 (*RCE Interiority*)**.** *There is an $\epsilon > 0$, such that, for all states $(\boldsymbol{a}, s)$, the problems of the borrower and the lender — (33) and (41) — have a jointly feasible solution, when the right-hand sides of (35) and (43) are replaced by $\mathcal{A}_b(s')+\epsilon$ and $\mathcal{A}_l(s')+\epsilon$, respectively.*

The core of the proof of the following Proposition 3 is to show how the RCE recovers the recursive saddle-point structure of the Fund problem, which, together with Proposition 2, defines a one-to-one map between the states $(\boldsymbol{a}, s)$ and (x, s). Nevertheless, to account for the moral hazard externality problem, the design of an appropriate system of taxes and transfers for a RCE is not trivial. The following definition lays out what is needed.

Definition 3. *A system of asset taxes and transfers $\{\tau(s', \boldsymbol{a}'|s), \overline{\tau}(\boldsymbol{a}, s)\}$ is constrained-efficient for the allocation of effort, if there is a recursive system of weights $x(\boldsymbol{a}, s)$, and the optimal choice of the borrower $e(a, s)$ in (33) — given the optimal choices, $c(a, s)$, $n(a, s)$,*

$c_l(a,s)$, $a'(s';a,s)$ and $a'_l(s';a,s)$ — is a solution to the following problem:

$$\max_e \Big\{ x(\boldsymbol{a},s)\Big[U(c(a,s),n(a,s),e)+\beta\mathbb{E}\big[W^b(a'(s';a,s),s')\big|s,e\big]\Big]$$

$$+ c_l(a,s)+\frac{1}{1+r}\mathbb{E}\big[W^l(a'_l(s';a,s),s')|s,e\big]\Big\} \tag{54}$$

$$s.t.\quad v'(e)=\beta\sum_{s'|s}\partial_e\pi(s'|s,e)W^b(a'(s';a,s),s'), \tag{55}$$

with the weights $x(\boldsymbol{a},s)$ satisfying the recursion:

$$x'(\boldsymbol{a}',s')=\frac{1+\hat{\gamma}_b(a,s)+\hat{\varphi}(s'|s,e(a,s))}{1+\hat{\gamma}_l(a,s)}\eta x(\boldsymbol{a},s)$$

$$\text{and}\quad \hat{\varphi}(s'|s,e(a,s))=\hat{\varrho}(a,s)\frac{\partial_e\pi(s'|s,e(a,s))}{\pi(s'|s,e(a,s))}, \tag{56}$$

and taxes satisfying

$$\frac{1}{1+\tau'(s',\boldsymbol{a}'|s)}\equiv 1+\frac{\hat{\varphi}(s'|s,e(a,s))}{1+\hat{\gamma}_b(a,s)}, \tag{57}$$

where $\hat{\gamma}_b(a,s)$, $\hat{\gamma}_l(a,s)$, and $\hat{\varrho}(a,s)$ are normalized Lagrange multipliers of (35), (43) and (55), respectively. Furthermore, the tax system satisfies (48).

We can now state the (constrained) FWT, which is proved in Appendix A (along with details about the normalization of the multipliers in the last definition):

Proposition 3. *Given initial asset holdings $\{a(s_0),a_l(s_0)\}$, a recursive competitive equilibrium with policy functions for the allocations*

$$\{c(a,s),n(a,s),e(a,s),a'(s';a,s),c_l(a,s),a'_l(s';a,s)\},$$

laws of motion for $\{\theta,g\}_{e(a,s)}$, prices $q(s',\boldsymbol{a}'|s)$, value functions $\{W^b(a,s),W^l(a,s)\}$, endogenous borrowing limits $\{\mathcal{A}_b(s'),\mathcal{A}_l(s')\}$ satisfying Assumption 4, and a constrained-efficient system of asset taxes and transfers $\{\tau(s';\boldsymbol{a},s),\overline{\tau}(\boldsymbol{a},s)\}$, there exists a Fund contract with initial condition $x(s_0)$, laws of motion $\{\theta,g\}_{e(x,s)}$ and $x'(s';x,s)$, multipliers $\{\nu_l(x,s),\nu_b(x,s),\varrho(x,s)\}$, and allocations $\{c(x,s),c_l(x,s),n(x,s),e(x,s)\}$ with value functions $\{V^{bf}(x,s),V^{lf}(x,s)\}$ that coincide with the competitive equilibrium allocations with $x(s_0)$ being uniquely determined by $a(s_0)$.

A main role of our decentralization is to allow us to compare — with assets, and asset prices — an economy with our benchmark Fund and the IMD economy, as we do in Section 5. Note that $a(s_0)$ determines $x(s_0)$, through the policy $c(a_0,s_0)$. Proposition 2 provides a foundation for this decentralization as a RCE, which preserves the characterization of the Fund contract of Section 2. Proposition 3 shows *what is needed* for a RCE to implement the Fund contract and, hence, achieve constrained-efficiency.

3.2 Alternative Implementations

This section discusses alternative decentralizations of the constrained efficient Fund allocation. As we have seen in the previous sections, the constrained efficient Fund allocation can be decentralized as a competitive equilibrium with asset taxes. One could take the latter assumption a step further and assume that, instead of having the state-contingent debt of the country being subject to taxation and transfers, debt-insurance contracts are subject to the incentive compatibility constraint. That is, following the pioneer work of Prescott and Townsend (1984), who constrain the contract space with the incentive compatibility constraint to implement the optimal Fund contract with moral hazard, we could add to the borrower's problem (33) the IC constraint (55).[26] In Appendix B we show that, provided that the lending and borrowing constraints are also accounted for, there is an equivalence between our tax-and-transfer decentralization and the *extended* — to endogenous borrowing and lending limits — Prescott-Townsend (ext. P&T) decentralization.

Note also that our benchmark economy with the Fund of Section 2 makes a strong assumption about the exclusive role and capacity of the Fund, to the point of fully replacing private sovereign debt lending markets. At the other extreme, in the two decentralizations mentioned above, all the debt of the country is supplied by competitive state-contingent markets with appropriately designed endogenous borrowing limits and either asset taxes or by directly imposed incentive compatibility constraint on permissible choices.[27]

Finally, we would like to emphasize that our decentralization assumes a competitive equilibrium notion with price taking behaviour from both sides, the borrower and the lender, where prices depend on the aggregate borrowing by the country, but not on the effort. In that case, in order to decentralize the moral hazard externality, one needs to impose asset taxes or, as in the other decentralizations we have discussed, directly restrict the contracts that are offered to the borrower to the ones that are incentive compatible. However, the constrained efficient Fund allocation could also be internalized in a setting with non linear prices in which the borrower acts as a monopsonist that controls its own assets and prices, while the (competitive) lenders are price takers. In this case, the borrower would have to take into account that the price of its debt may depend on the amount being borrowed and also the potential effect of the effort on the state-dependent price of borrowing. Effectively implementing the IC constraint in equilibrium.[28] This implies that, if this market power assumption is satisfied, Pigou taxes

[26] While Prescott and Townsend (1984) impose the incentive compatibility constraint directly on the consumption set of the borrowers (see also Kilenthong and Townsend, 2011), Golosov and Tsyvinski (2006), in a different (private information) framework, set up the equilibrium problem with competitive intermediaries offering state-contingent contracts to households. The set of feasible contracts are then restricted, not only by the participation constraints, but also by the incentive compatibility constraint of households. It is easy to see that the two formulations are equivalent, as they both impose the incentive compatibility constraint on the set of permissible allocations.

[27] As we note in the concluding section 6, Liu et al. (2023), following our work (unpublished previous versions), relax the exclusivity assumption, developing an intermediate framework (without endogenous effort), where the Fund and private lenders coexists and implement the constrained-efficient allocation.

[28] We thank a referee for this comment.

are not needed either in a decentralization with non linear prices.[29]Alternatively, one could have a RCE where: given the known preferences of the borrowers, there are non-linear asset prices that make a specific level of effort incentive compatible, and competitive lenders know and understand that with this level of effort and asset prices they can break even (i.e. the expected return is the riskless rate) by offering distorted risk-sharing.[30] Note that this economy is a version of the, already mentioned, Prescott-Townsend's *constrained contract space economy*.

4 Calibration

4.1 Functional Forms and Parameter Values

We calibrate the model parameters so that the IMD economy with defaultable debt is representative of the four 'stressed countries' in the European debt crisis, i.e., Portugal, Italy, Greece, and Spain (henceforth GIPS), over the period 1980–2015.[31] We target key data moments by taking the average across the GIPS countries. The model period is assumed to be one year. The utility of the borrower is additively separable in consumption, leisure and effort. In particular, we assume that $u(c) = \log(c)$, $h(1-n) = \gamma\frac{(1-n)^{1-\sigma}-1}{1-\sigma}$ and $v(e) = \omega e^2$ so that:

$$U(c, n, e) = \log(c) + \gamma\frac{(1-n)^{1-\sigma} - 1}{1-\sigma} - \omega e^2.$$

The preference parameters (σ, γ) are set to $\sigma = 0.34$ and $\gamma = 1.734$. These are used to match the average fraction of working hours, together with the volatility of labor relative to GDP, of the GIPS countries. The effort cost parameter is set to $\omega = 0.0087$ (this choice, with the quadratic functional form, is discussed together with the specification of the government expenditure shock in Section 4.2.2).

We assume that $f(n) = n^\alpha$ with the labor share of the borrower set to $\alpha = 0.566$ to match the average labor share across the GIPS countries.

The risk free interest rate is set to $r = 2.48\%$ to match the average real short-term interest rates of the Euro area. The parameters of the long-term bond (δ, κ) are set to $\delta = 0.814$ and $\kappa = 0.083$ to match the average maturity and the average coupon rate (coupon payment to debt ratio) of long-term debt for the GIPS countries. After a country defaults in the IMD economy, it faces exclusion for a random number of periods, and the probability that it comes back to the market with sovereign debt upon default is set to be $\lambda = 0.264$. Moreover, if a country defaults in the IMD economy, it is subject to an asymmetric default penalty of the

[29]Note that, in the context of a Fund contract, a powerful sovereign could also endogenize that, with effort, the default costs may be reduced to the country's advantage. We do not consider this, more complex, possibility of manipulating the limited enforcement constraints — say, through θ^p — either.

[30]Our interpretation here is that for such economy to emerge, previous experience with Pigou taxation is probably needed.

[31]A description of the data is given in Appendix C.1; in particular, Table C.1.

form (Arellano, 2008):

$$\theta^p = \begin{cases} \bar{\theta}, & \text{if } \theta \geq \bar{\theta} \\ \theta, & \text{if } \theta < \bar{\theta} \end{cases} \quad \text{with } \bar{\theta} = \psi \mathbb{E}\theta,$$

where $\psi = 0.7098$. The latter two parameters (λ, ψ), together with the discount factor $\beta = 0.929$ are chosen to match jointly the spread level, the spread volatility and the average debt to GDP ratio. Note that this implies a different discount factor for the lender of $\frac{1}{1+r} = 0.9758$, as well as a growth rate for the relative Pareto weight of the borrower of $\eta = \beta(1+r) = 0.9684$ in the Fund contract. The fact that the borrower is less patient than the lender implies that the borrower will tend to get indebted in both economies. As it is well known, in the absence of any frictions (limited commitment or moral hazard), consumption of the borrower would converge towards zero in the long run.

The quantitative section focuses on the optimal Fund contract with $Z = 0$, implying that there will be no expected permanent transfers between the borrower and the lender at any time or state. In other words, the Fund is not build on an assumption of solidarity, which would require permanent transfers.[32]

Table 1 summarizes the parameter values, and Appendix C contains additional information regarding data sources.

4.2 Shock Processes

4.2.1 Productivity Shock

We start by constructing the model consistent measure of productivity. The original productivity θ^o_{it} of country i in year t is equal to real output divided by total working hours to the power of α. We further detrend $\{\log \theta^o_{it}\}$ with a country specific linear trend,[33] and then normalize the detrended series to the same mean and volatility across the GIPS countries. The final productivity series $\{\log \theta_{it}\}$ is homogeneous across i, and it can be viewed as repeated samples from the same data generating process, which is representative for the GIPS. Finally, we estimate the following panel Markov regime switching AR(1) model:

$$\log \theta_{it} = (1 - \rho(\varsigma_{it}))\mu(\varsigma_{it}) + \rho(\varsigma_{it}) \log \theta_{it-1} + \sigma(\varsigma_{it})\varepsilon_{it},$$

where $\varsigma_{it} = 1, 2, 3$ denotes the regime of country i at time t, $\mu(\varsigma_{it})$, $\rho(\varsigma_{it})$, and $\sigma(\varsigma_{it})$ are the regime-specific parameters, and $\varepsilon_{it} \overset{\text{iid}}{\sim} N(0, 1)$. Regime ς_{it} follows a Markov chain with a 3×3 transition matrix P. Table 2 displays the estimated parameters of the process. As shown in the table, regime 1 has the lowest conditional mean and the highest conditional volatility.

[32] We also consider a large Z that is never binding, which could potentially have important quantitative consequences for the optimal risk sharing arrangement. It will turn out, however, that the participation constraint of the lender is rarely binding even if $Z = 0$ (around 2.12%), so we focus on this allocation in the quantitative section.

[33] We calculate the productivity trend for each country over 1980–2009, to avoid the influence of the European debt crisis.

Table 1: Parameter values

Parameter	Value	Definition	Target moment
A. Direct measures from data			
α	0.566	labor share	average labor share
r	0.0248	risk-free rate	Euro area short-term risk-free rate
δ	0.814	bond maturity	average bond maturity
κ	0.083	bond coupon rate	average bond coupon rate
B. Based on model solution			
β	0.929	discount factor	average b/y
λ	0.264	return probability	average and volatility of spread
ψ	0.7098	productivity penalty	
σ	0.34	labor elasticity	average n and $\sigma(n)/\sigma(y)$
γ	1.734	labor utility weight	
ϕ	0.975		
ϖ	0.01	g distribution	average, 1 and 99 percentile
w	0.72		of g/y; $\rho(g, y)$, $\sigma(g)/\sigma(y)$,
g_1	0.0385		
g_2	0.0315	g grid points	and $\sigma(ps/y)/\sigma(y)$
g_3	0.0285		
ω	0.0087	effort disutility weight	$\mathbb{E}\zeta(e) = 0.5$
C. By assumption			
Z	0	Fund's outside value	

Notes: Apendix C.1 contains details on data sources; parameters in panel B are calibrated jointly, with groups indicating main sources of identification; and ps denotes primary surplus.

Thus, we can interpret regime 1 as a 'crisis' regime,[34] regime 3 as normal times, while regime 2 as an intermediate more temporary regime that typically precedes a crisis. A Markov regime switching specification allows us to capture the rich dynamics observed in the data, such as the sudden drops of productivity around the financial crisis and the European debt crisis, in a convenient manner.[35] Finally, we discretize the process into a 27-state Markov chain, with 9 grid points in each regime (see Appendix C.2 for more details).

4.2.2 Government Consumption Shock

We first explain the parameterization of π^g, assuming that g is independent of θ, and then extend the baseline specification so that g and θ are correlated, which is the specification we use in the full quantitative model.

[34] The smoothed regime probabilities shown in Appendix C.2 confirm that regime 1 concentrates on the European debt crisis periods.

[35] Bai and Zhang (2010) use a similar regime switching model to calibrate heterogeneous productivity dynamics across countries. Focusing on the euro debt crisis, Kriwoluzky et al. (2019) also choose to capture the rich dynamics in the data by a regime switching structure for the shock process.

Table 2: Parameters of the regime switching productivity process

Regime spec. para.				Regime trans. matrix				Invariant dist.	
	$\mu(\varsigma)$	$\rho(\varsigma)$	$\sigma(\varsigma)$	P	$\varsigma'=1$	$\varsigma'=2$	$\varsigma'=3$		
$\varsigma=1$	-0.1893	0.8680	0.0132	$\varsigma=1$	0.8931	0.0000	0.1069	$\varsigma=1$	0.3289
$\varsigma=2$	0.0118	0.8264	0.0048	$\varsigma=2$	0.1862	0.8138	0.0000	$\varsigma=2$	0.1568
$\varsigma=3$	0.1060	0.9021	0.0129	$\varsigma=3$	0.0116	0.0567	0.9317	$\varsigma=3$	0.5143

Notes: ς denotes the current regime of productivity shock, and ς' denotes that of the next period.

Independent g and θ as a preliminary step In order to parametrize π^g, we adopt the renowned *spanning condition* of Grossman and Hart (1983) by assuming

$$\pi^g(g'|g,e) = \zeta(e)\pi^l(g'|g) + (1-\zeta(e))\pi^h(g'|g), \tag{58}$$

where π^l and π^h are two distributions that are independent of e, while $\pi^h(\cdot|g)$ first-order stochastically dominates $\pi^l(\cdot|g)$ for all g, and the weighting function $\zeta(e) = (1-e)^2 \in (0,1)$ satisfies $\zeta'(e) < 0$ and $\zeta''(e) < 0$. It is straightforward to verify that the *monotonicity* and *convexity* assumptions on π^g are satisfied by (58).[36]

For a parsimonious parameterization, we assume that g takes three values: $g_1 > g_2 > g_2 > 0$. Furthermore, we choose a 2-parameter (ϕ, ϖ) specification for π^h and π^l,

$$\pi^h = \begin{bmatrix} 2\phi-1 & \frac{4}{3}(1-\phi) & \frac{2}{3}(1-\phi) \\ 0 & 2(\phi+2\varpi)-1 & 2(1-\phi-2\varpi) \\ 0 & 0 & 1 \end{bmatrix}, \; \pi^l = \begin{bmatrix} 1 & 0 & 0 \\ 4\varpi & 1-4\varpi & 0 \\ 2\varpi & 2(1-\phi-\varpi) & 2\phi-1 \end{bmatrix}, \tag{59}$$

and calibrate the parameter values so that, when $\bar{\zeta} \equiv \mathbb{E}\zeta(e) = 0.5$ holds in the ergodic mean of the IMD model, the *average* transition matrix $\bar{\pi}^g(g'|g) = \bar{\zeta}\pi^l(g'|g) + (1-\bar{\zeta})\pi^h(g'|g)$ is equal to:[37]

$$\bar{\pi}^g = \begin{bmatrix} \phi & \frac{2}{3}(1-\phi) & \frac{1}{3}(1-\phi) \\ 2\varpi & \phi & 1-\phi-2\varpi \\ \varpi & 1-\phi-\varpi & \phi \end{bmatrix}. \tag{60}$$

The idea underlying such a calibration strategy is indeed reverse engineering: we first use the transition matrix $\bar{\pi}^g$ to fit the process for g in the data, and then split $\bar{\pi}^g$ into π^h and π^l subject to the constraints that $\bar{\pi}^g = 0.5\pi^l + 0.5\pi^h$ and the property that the likelihood ratio of π^h over π^l is monotone in g'. To guarantee the latter condition, we choose the upper

[36] As is well known (e.g., Rogerson, 1985), the functional form for the utility cost of effort is flexible up to a normalization. For the weighting function $\zeta(e)$, the quantitative literature of dynamic moral hazard typically chooses an iso-elasticity specification, with enough curvature to avoid corner solutions (e.g., Tsyrennikov, 2013). For computational convenience, we choose quadratic forms for both $v(e)$ and $\zeta(e)$, so that the optimal choice of e pinned down by the first order approach takes a particularly simple form, which contributes considerably to speeding our numerical solutions for both the IMD and Fund economy.

[37] Note that this specification of the transition matrix is motivated by the one-period-crash Markov chain of Rietz (1988). The numerical values of π^l, π^h, and $\bar{\pi}^g$ are displayed in Appendix C.3.

and lower triangle specification for π^h and π^l respectively. The relevant parameters of the transition matrix are set to $\phi = 0.975$ and $\varpi = 0.01$, while the state space of g is set to $\{0.0385, 0.0315, 0.0285\}$. In addition, we set the effort disutility parameter to $\omega = 0.0087$ so that $\bar{\zeta} = \mathbb{E}\zeta(e) = 0.5$. We discuss the target moments below after introducing correlation between g and θ in the full quantitative model.

Correlated g and θ for the full model In order to capture the fact that there is correlation between g and GDP in the data, as well as the volatility of g and the primary surplus relative to GDP respectively, we assume that the processes for g and θ are correlated in the full quantitative model. To this end, we exploit the Markov regime transition structure in our calibration of θ. In particular, by conditioning the distribution of g' on ς, the regime to which θ belongs, in addition to g and e, it follows that g' and θ' are correlated.[38] Moreover, we introduce one more parameter $w \in [0, 1]$ to control the influence of ς on g': if $w = 0$, g' is independent of ς; if $w = 1$, g' depends only on ς but no longer on g.

As displayed in Table 1, we use 6 parameters related to the distribution of g and the moral hazard setup to match 6 moments in the data: the level and the lower 1 and upper 99 percentiles of g/y, the correlation between g and y, and the relative volatilities of g and the primary surplus with respect to y.

Lastly, to improve the convergence properties of the IMD economy, we follow Chatterjee and Eyigungor (2012) by adding a small iid shock to the g shock described above. In particular, we assume that the shock is uniformly distributed over $[-\bar{m}, \bar{m}] = [-0.001, 0.001]$ and discretize it into 5 equally spaced grid points over the range with equal probability for each point.

4.3 The IMD Model Fit

Table 3 provides an exhaustive account of the model fit with our benchmark calibration. To compute the data moments, we first compute the corresponding moments for each country, and then take the average across the GIPS countries, resulting in a set of moments that are representative of the common features of these countries. Furthermore, for the second moments, we HP filter the data series with a filtering parameter 6.25 to extract the business cycle frequency fluctuations for annual data (Ravn and Uhlig, 2002). To compute the model moments, we compute 50,000 short run simulations of the IMD model with 300 periods each, and we discard the first 100 periods. Similarly to the data moments, we HP filter the simulated data to compute the second moments.[39]

The IMD economy matches most moments remarkably well, with the exception of the average primary surplus to GDP. In particular, the model is able to produce a significant

[38] Appendix C.4 contains the details on the transition probability specification.

[39] Note that there is default in the IMD economy, in which case debt and the primary surplus are zero, by construction, and the spread is not defined. Therefore, all the moments involving the debt to GDP ratio, primary surplus over GDP and the spreads are conditional on borrowing (i.e., not in default).

Table 3: IMD Model Fit and Comparison with Fund

Target Moments				Non-target Moments			
Variables	Data	IMD	Fund	Variables	Data	IMD	Fund
			A. 1^{st} *Moments*				
b'/y (%)	78.33	78.57	191.00	ps/y (%)	-1.00	1.14	4.70
spread (%)	4.15	4.17	-0.003				
g/y %	21.68	21.74	20.97				
1% of g/y	13.38	15.22	14.44				
99% of g/y	32.80	32.14	32.62				
n (%)	36.37	36.56	37.82				
e	n.a.	0.29	0.34				
			B. 2^{nd} *Moments*				
$\sigma(n)/\sigma(y)$	1.00	0.91	0.70	$\sigma(c)/\sigma(y)$	1.51	1.39	0.36
$\sigma(g)/\sigma(y)$	1.02	1.03	0.70	$\rho(c, y)$	0.63	0.64	0.62
$\sigma(ps/y)/\sigma(y)$	1.00	0.97	0.86	$\rho(n, y)$	0.70	0.10	0.94
σ(spread)	1.67	1.74	0.00	$\rho(\text{spread}, y)$	-0.38	-0.06	-0.48
$\rho(g, y)$	0.38	0.38	0.47	$\rho(ps/y, y)$	0.18	0.23	0.93

Notes: all data moments are the averages of country specific moments over GIPS countries; second moments are calculated after removing trends by HP-filter, both in the data and IMD/Fund model solutions; for the Fund solution, debt/output ratio is defined as $\bar{a}'/y$ (cf. (32)); and ps denotes primary surplus.

amount of debt together with a realistic level, volatility and cross-correlation of spreads, but it generates a positive average primary surplus to GDP. Note that, in any stationary model without growth, whenever there is debt in the long run, we need to have primary surplus which allows the country to pay the interest rate on its debt. This is not true in the data, as the countries in the sample were able to run deficits and increase their debt, possibly expecting growth, given that there is (moderate) growth during the sample period. What is more important than the level for our purpose, however, is that we match well the relative volatility of the primary surplus over GDP, the positive correlation of government and technology shocks, and the positive correlation of primary surplus with GDP, even though this last moment is not targeted. Note that this moderate but positive correlation enhances consumption insurance: resources come in whenever the country's output is low.

As the tables reflect, the model also matches well the moments of consumption and labor, although the correlation of labor and GDP is lower than in the data. However, this is not the focus of our analysis and — except for the fact that welfare comparisons are easier with separable preferences — our main results do not depend on our specific choice of preferences.

5 Quantitative Analysis

This section investigates quantitatively whether the Fund improves on the IMD economy using the calibrated parameters described above in the two economies. We do this in three steps. First, we assess how differently the Fund operates in normal times, by comparing the long run properties of the two economies using both a few key statistics and representative long-run simulations in Section 5.1. Then, in Section 5.2, we compute and discuss the welfare implications of these improvements and evaluate the debt absorption capacity of the Fund. We conclude our quantitative analysis in Section 5.3 by contrasting the paths of the two economies under a crisis event that approximates well the onset of the Euro debt crisis for the GIPS countries.

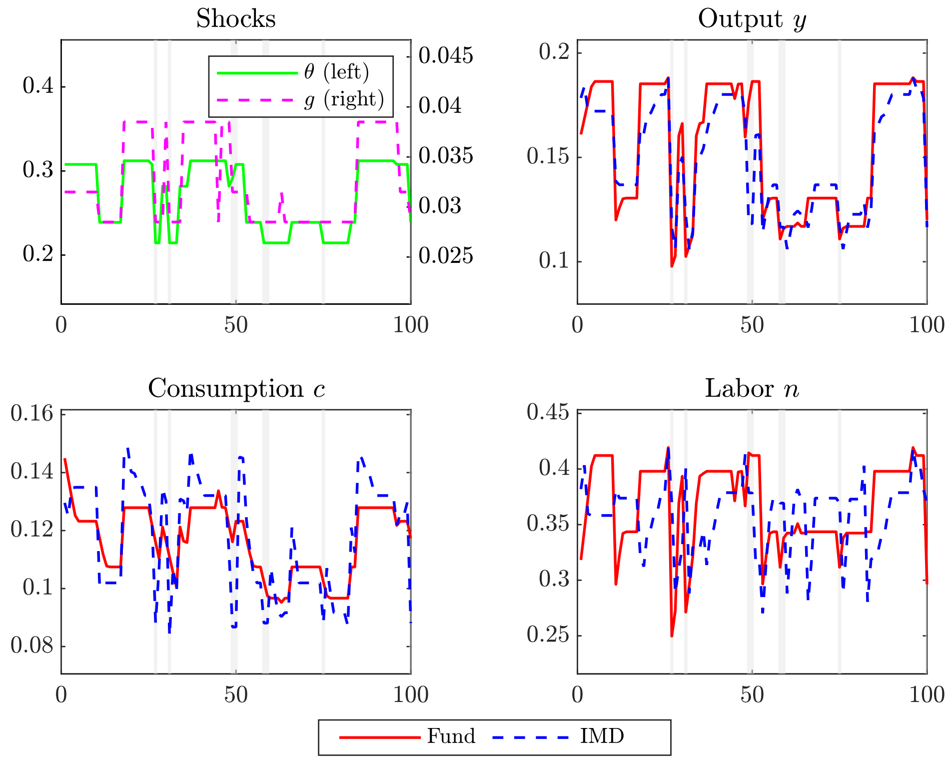
5.1 Comparing the Allocations in 'Normal Times'

We base our description of how the Fund operates during normal times by comparing long-run statistics of the two economies, displayed in Table 3, and by presenting representative simulation paths of both economies on Figures 3 and 4, subject to the same sequence of shocks in the long run stationary distribution. The long-run simulations are initialized at the the ergodic mean of the two economies.

In Figure 3, the upper left panel shows the history of shocks for 100 periods, while the output, consumption and labor allocations in the IMD and Fund regimes are shown in the other panels. In addition, Figure 4 displays the levels of effort, surplus over GDP, debt over GDP and spreads in the two economies. In order to make the two economies comparable, we plot simulations in which they face exactly the same sequence of productivity and government expenditure shocks. The grey periods in the figures correspond to periods of default in the IMD economy. To obtain comparable variables (e.g., debt holdings or spreads) in the two economies, we rely heavily on Section 3.

The differences between the two economies are striking in many dimensions. The first stark difference we would like to emphasise is that the Fund is able to *absorb a much higher level of debt*. The long-run average debt over GDP is 191 percent under the Fund compared with 78.6 percent in the IMD economy. Note that our calibration implies that the borrower is effectively more impatient than the lender (the markets) and hence accumulating debt is desirable. In the IMD economy, a high level of debt cannot be sustained as it increases the probability of future default and consequently the cost of borrowing (the spread). In fact, Figure 3 reflects that defaults are primarily associated with drops in productivity with relatively large levels of initial debt. Moreover, the frequency of default and the long-term nature of debt implies that spreads are relatively high in the IMD economy even in normal times, and they spike just before a default episode, making further borrowing prohibitively costly. Even though the same limited commitment friction (with the same exact outside option, $V^o(s)$) is imposed in the Fund, the Fund allocation provides a much higher utility through improved risk sharing and by avoiding costly default episodes, implying that a much higher debt is sustainable with

Figure 3: Business cycle paths of real variables

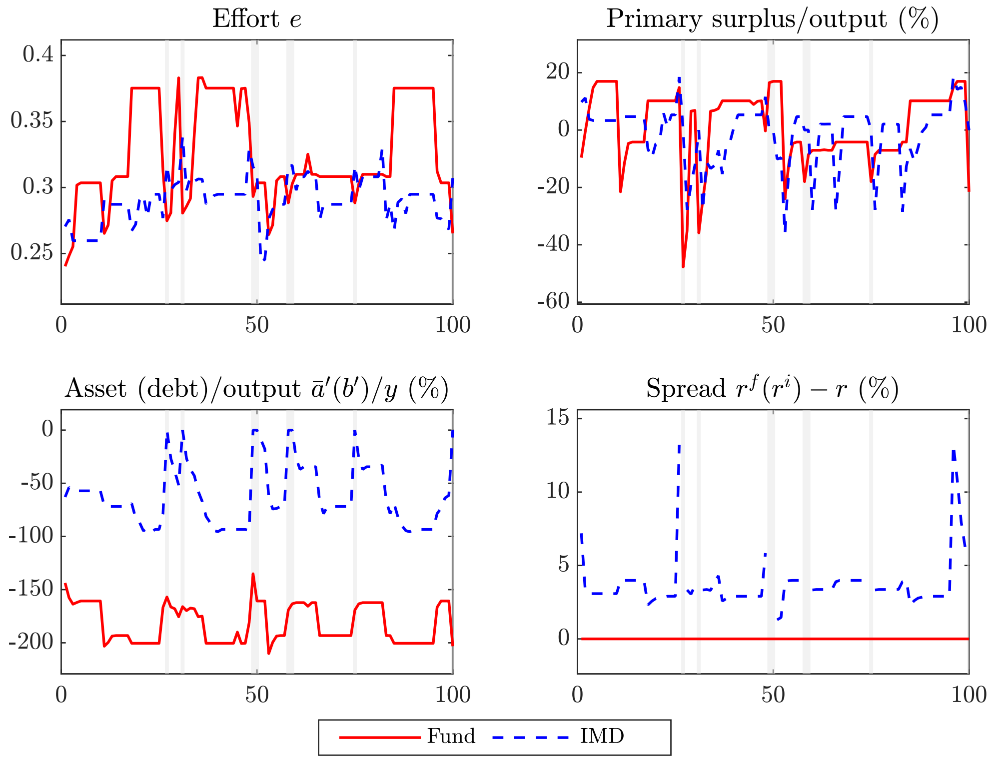


Notes: horizontal axis is for time period; and grey bars indicate default periods in the IMD economy.

the Fund. By eliminating default episodes, even with these much higher levels of debt, the borrower faces no positive spread. As we explained in Section 3.1.3, binding future limited enforcement constraints of the Fund may lead to negative spreads along the equilibrium path. However this does not happen along the particular simulation path displayed in the Figure.

The second key difference between the Fund and IMD allocations is the amount of *consumption insurance*. Table 3 shows that the relative volatility of consumption drops from 139 percent in the IMD economy to 36 percent in the Fund. The smoother path of consumption is also clearly reflected on Figure 3. How does the Fund deliver a smoother consumption path? The key statistic to understand this is the co-movement between output and the primary surplus. Full consumption insurance would imply a very strong co-movement of the primary surplus and output, as constant consumption can be achieved through surpluses (capital outflows or savings) during good times and through deficits (capital inflow or borrowing) during bad times. Under separable preferences, this co-movement is not perfect solely due to the presence of government expenditure shocks. In the IMD economy, the presence of spreads induces an implicity tight borrowing constraint, leading to a mildly procyclical (0.23) primary surplus. At the same time, the Fund allows for much more state-contingency and the procyclicality of the surplus rises to 0.94. Indeed, when we compare the path of θ on Figure 3 with that of the primary surplus on Figure 4, we can visualize the strong co-movement of

Figure 4: Business cycle paths of financial variables



Notes: horizontal axis is for time period; $\bar{a}'$ refers the 'bond' component of asset positions in the Fund economy (cf. (32)); and grey bars indicate default periods in the IMD economy.

the two variables. This is particularly important when the economy is hit by negative shocks, triggering default in the IMD economy. Instead, in most of these cases the borrower enjoys a large primary deficit under the Fund that allows her to keep consumption much smoother. Under incomplete markets, this deficit would imply an immediate increase of outstanding debt. In contrast, Figure 4 shows that the borrower is (partially) insured against this extreme realizations under the Fund and the outstanding debt is actually reduced. Here, consumption smoothing is delivered by the state contingent part of the debt ($\hat{a}(s')$, see (32)), which makes the primary surplus more procyclical.

Finally, the Fund also leads to a *more efficient allocation of labour and effort*. First, as Table 3 reflects, labor supply in the Fund is strongly correlated with output. This is an indication of improved efficiency, as in the unconstrained optimal allocation of this economy, labor supply is solely determined by productivity under our separable utility function specification. Another interesting observation is that the Fund provides on average considerably better incentives for exerting effort, especially in normal times. Table 3 indicates that the long-term mean of effort is 17 percent higher under the fund than under the IMD economy. This is because the Fund provides long run incentives directly, connecting future realizations of the government expenditure shock to future lifetime utility through the law of motion of x

(see (7)). Section 3 shows that these rewards and punishments can be translated to changing of the borrowing terms (the value of the debt-insurance liabilities). In contrast, defaultable debt markets can mostly provide short term incentives through the immediate utility drop and spread rise associated with high realizations of the g shock.

5.2 Welfare Implications

In what follows, we quantify the actual welfare implications of the Fund regime compared to the IMD economy as well as the increased capacity to absorb debt by the Fund. The results are displayed in Table 4. We start the discussion by looking at the debt absorption. The second and third columns of the table display the maximum end of period debt to output ratio in percentage terms that the country has for different values of the shocks.[40] These latter measures are intended to capture the absorbing debt capacity of the borrower. Note that debt capacity is not straightforward to measure in the IMD economy, as there are no explicit debt limits. However, given the impatience of the borrower, the actual debt choices reflect the debt capacity in this case. Hence, we choose the highest equilibrium debt/output ratio for a given state s across all feasible levels of current debt b (or a for the Fund). In the case of complete markets, the borrower has a whole portfolio of debt (and assets) for each future state. However, in Section 3.1.3, we have shown that, from any portfolio of Arrow securities, we can construct a bond component a' and an insurance portfolio, with the bond component being the comparable measure to the debt choice in the incomplete markets economy. Given this, we follow the same logic and we present the maximum of debt/output ratio using the bond component across all values of current debt for a given state.[41]

The difference between the debt capacity in the two economies are striking. As we see, the Fund is able to absorb a much higher debt-to-GDP ratios in all states, while the capacity to absorb debt in the IMD economy is substantially smaller, particularly in bad states. This is because, due to the relatively high persistence of the shocks, a low realization of the shock today implies a high spread on any significant amount of debt, as the country will have only a small chance to pay it back through a better realization of the shocks. Moreover, due to the asymmetric default penalty specification, there is no output penalty for low shock realizations, and default is not particularly costly in this case. Another interesting feature of the IMD economy is that the borrowing limits are relatively loose in normal times (for medium productivity levels). This is due to the fact that, in this case, the countries suffer an output loss upon default and the value of staying in the financial markets is higher in relative terms.

[40] For this exercise, we set the value of the idiosyncratic component of government expenditure to its zero mean.

[41] Note that, for the Fund economy, one can actually compute the maximum borrowing capacity as the bond component of the portfolio that allows for the maximum amount of borrowing across all possible realizations of the future shocks: $\frac{\sum_{s'|s} q(s'|s)\mathcal{A}_b(s')}{\sum_{s'|s} q(s'|s)}$, where $\mathcal{A}_b(s')$ is the state contingent borrowing constraint of the country. However, for comparison with the incomplete markets economy, we choose the alternative measure described in the text, which has values that are necessarily tighter than the maximum borrowing limit.

Table 4: Welfare gains at zero debt and debt capacity

$s = (\theta, g)$	Welfare Gain %	IMD max $\frac{-b'(s,\cdot)}{y(s,\cdot)}$ %	Fund max $\frac{-\bar{a}'(s,\cdot)}{y(s,\cdot)}$ %
(θ_1, g_1)	10.28	1.60	104.08
(θ_{14}, g_1)	8.36	90.89	160.42
(θ_{27}, g_1)	7.27	182.52	273.46
(θ_1, g_3)	9.31	1.87	98.89
(θ_{14}, g_3)	7.73	88.21	169.09
(θ_{27}, g_3)	7.00	183.75	292.82
Average	8.48		

Notes: θ_1 and g_1 denote the worst productivity and government consumption shocks, while θ_{27} and g_3 denote the best shocks; the average welfare gains in the last row equals to $\sum_s \Pr(s)W(s)$, where $\Pr(s)$ denotes the ergodic distribution of shock s, and $W(s)$ denotes the welfare gain for shock s with zero debt; $\bar{a}'$ refers to the 'bond' component of asset holdings in the Fund economy (cf. (32)); and the maximum (end of period) debt capacity is taken over the state space for the current debt, i.e., b for IMD and a for Fund.

Nevertheless, the Fund will be able to support much more borrowing as default becomes less attractive compared to the utility the Fund can deliver. Below, we provide a measure of the welfare impact of this increased debt capacity.

These results also indicate that the Fund can 'take over' very large amounts of debt from potential member countries at the verge of defaulting. For example, if a borrower country has an outstanding debt of around 80 percent of its GDP and it is hit by a crisis (θ_1, g_1), then under the IMD scenario, it will default on its debt, suffering all the consequences of default. In contrast, the Fund will be able to absorb all this debt, enrolling the country in the long-term debt program and able to provide strictly higher utility than under autarky (as the accumulated debt is strictly below the debt absorbing capacity of the Fund), and hence it would not violate the limited enforcement constraint of either parties.

Now, we turn our attention towards the welfare gains. The first column of Table 4 displays the welfare gains of the Fund in (annual) consumption equivalent terms when countries have zero initial debt for different values of the shocks (θ, g) (in the next section, we will measure welfare gains also for an already indebted borrower). The Table reflects that the welfare gains are very substantial under the Fund: the consumption-equivalent steady-state average welfare gain is around 8.5 percent and, even more relevant, the gain is of 10.3 percent in the worst state. As discussed earlier, two of the features of the Fund that lead to welfare gains are the fact that it provides more risk sharing through state contingent assets and the fact that it allows for a much higher debt capacity. Both of these features are particularly important with bad shocks, leading to substantially higher welfare gains. In other words, the welfare gains of the Fund are the highest when the country is in trouble. The gains are still substantial when the country is hit by good shocks. Note that this is partially due to the fact that agents are forward-looking and gain benefits from the future insurance against bad shocks, and partially because at the higher shock levels they still have much higher debt capacity and still benefit from the state contingency of the Fund contract.

Next, we go deeper to inspect how important are these different features of the Fund for the welfare gains. To do this, we propose a novel decomposition of welfare gains that implements a series of counterfactual exercises to evaluate the main channels of welfare improvements. The first important difference between the IMD and Fund economies is that default occurs in equilibrium in the IMD economy but not in the Fund economy. Given this, we first simulate a counterfactual IMD economy where we keep the asset prices, asset holdings and default decisions at the same level, except that *(i) no output penalty* is imposed and *(ii)* no penalty is imposed, in the sense that there is *no market exclusion* after default. By comparing the lifetime values obtained in the first counterfactual economy with the value functions of the IMD economy, and then comparing the values of the first and second economies, we obtain the isolated effect of the output penalty and exclusion, respectively. To evaluate the effect of *(iii)* a *higher debt capacity*, we solve for counterfactual economies with looser constant exogenous debt limits in which default is not allowed. In particular, the debt limits are set at the endogenous borrowing constraints $\mathcal{A}_b(s)$ associated with a given value of the state vector s under the Fund economy. Comparing the value of this counterfactual exercise to case (ii) provides us with the measure of welfare gains due to an increased debt capacity in the Fund (note that all the direct costs of default were already taken care by the previous case). Finally, note that the previous three counterfactuals do not account for the fact that Fund is able to provide *(iv) state-contingent payments* as opposed to the IMD economy (apart from the costly default episodes). This is captured by the (residual) difference between the welfare in the Fund economy and counterfactual (iii).[42] The results of the counterfactuals are displayed in Table 5 below for a selection of initial states.

Table 5: Welfare decomposition at 0 debt/asset for selected shock states

$s = (\theta, g)$	(i) No θ penalty %	(ii) Immediate return to market %	(iii) Greater debt capacity %	(iv) State-contingent insurance %
(θ_1, g_1)	6.58	1.67	63.65	28.10
(θ_1, g_3)	5.31	1.38	51.92	41.39

Notes: see the main text for the explanation on how to decompose the welfare gains into the four components.

The table reflects that, for all values of the shocks, the higher debt capacity and insurance through the state contingent assets provided by the Fund are the two most important factors contributing to the welfare gains. In particular, these two factors account for more than 90% of the welfare gains in both cases presented. We also see that the contribution of not having a penalty upon default is relatively small. The reason is that matching debt levels and spreads simultaneously in the IMD economy requires an asymmetric default penalty that does not impose penalties for low productivity levels, hence the output penalty kicks in only whenever

[42] One caveat is that step (iii) of this decomposition can be computed credibly only for the lowest levels of the shocks because a constant debt limit set at the level of the endogenous borrowing constraint associated with medium shocks under the fund would be not sustainable even under the Fund.

during the default period productivity increases. We also see that the fact that a country is excluded from the financial markets upon default is less important. This is due to the fact that returning to the market with low shocks implies very tight borrowing limits under the IMD economy and hence limited potential welfare gains of market return.

The key result is that both the increase in debt capacity and the state contingency of payments are quantitatively significant in explaining the welfare gains with increased greater debt capacity being the more important component. Whenever low productivity is combined with high government expenditure state-contingency tends to be (relatively) more important because the desire for borrowing is highest in this case and even the relaxed borrowing constraint limits consumption smoothing extensively. To summarize, the Fund leads to substantial welfare gains that arise primarily from the fact that it provides insurance through the state contingent assets as well as a higher debt capacity.[43]

5.3 Comparing the Allocations in Crisis Times

In this section, we investigate how the Fund responds in a crisis situation. In particular, we compute a counterfactual simulation that compares how the representative economy would have done under the IMD and Fund regimes when hit by a crisis that resembles some of the aspects of the Euro debt crisis following the 2008 Financial crisis. To do this, we initialize the economy at a state with low spreads of around 0.8 and a level of debt of around 70% of GDP, which are consistent with the average levels of debt to GDP and spreads in the pre crisis times during 2005–2007. Subsequently, we hit the economy with a negative productivity shock and a bad (high) government shock at period 1 and we compute, under each regime, the average path for 50,000 independent simulations with the same pre crisis initial asset holdings and spreads but different shock realizations from the (partially endogenous) Markov structure of our economy after period zero.

Table 6: Statistics around the onset of European debt crisis

	Periods	Avg. b'/y %	Avg. spread %
Before crisis:	2005–2007	78.31	0.78
Crisis eruption:	2009–2010	99.14	4.04

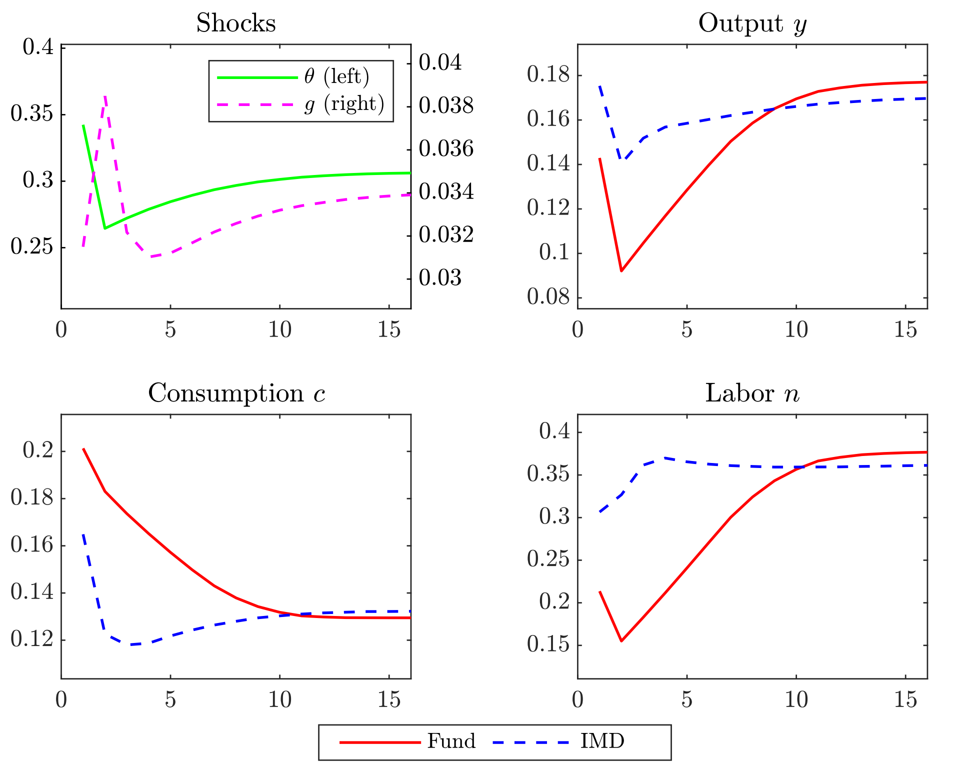
Notes: all moments are the averages over the GIPS countries.

Table 6 displays the pre and post crisis average levels for the debt to GDP, spreads and GDP, while Figures 5 and 6 display the simulated average impulse response paths after the

[43] In a related paper, Evers (2015) studies the amount of risk sharing different fiscal institutions can deliver in a two-country economy, obtaining limited welfare gains. Three important differences between that set-up and ours could explain the differences in welfare gains. First, $\beta(1+r)$ is close to 1 in Evers setting, implying limited gains from being able to borrow extensively. Second, in his two country set-up, insurance can be provided only by the other country and since the two countries are linked through trade and spillovers, insurance provision is limited and costly. Third, in the model of Evers, there are no binding (endogenous) debt constraints, as that model is solved with perturbation methods around the steady state.

crisis for the economy under the IMD and Fund regimes. As the table and figures reflect, the model is able to match relatively well the post crisis increase in debt, the considerable increase in spreads after the crisis, and the decrease in GDP.

Figure 5: Counterfactural simulation of real variables

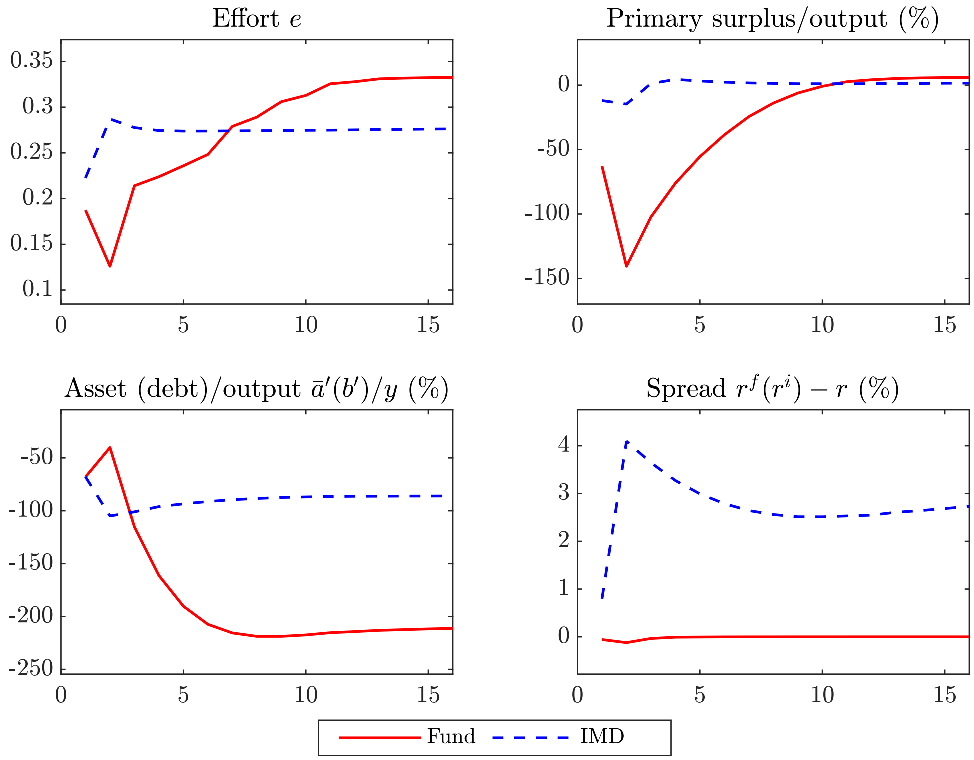


Notes: horizontal axis is for time period; time 1 refers to the pre-crisis period, and time 2 refers to the crisis period; and from time 3 onward, 50,000 IMD/Fund model economies are simulated with stochastic θ and g shocks that are independent in the cross section, and the final results are the average of 50,000 simulations.

The smooth path of all the key variables in the figures reflects the fact that we depict the average path for many independent economies. It is important to note that there are many default episodes in the IMD economy after period 1, generating the positive spreads in Figure 6. For the real variables (shocks, output, consumption, labour, effort, and primary surplus), we take an average over all economies in every period, while for debt over GDP and the spread we only average over for those who are not in default, as these variables are not defined for those who are in default.

Looking at the pictures, the differences between the IMD and the Fund are even more striking than in the long run simulations. The paths for consumption and labor clearly indicate that the Fund is able to stabilize the crisis considerably more in the short run: consumption is higher for the first 10 periods and, due to efficiency considerations, the Fund allows for a reduction in labor supply in the short run. At the same time, for the IMD economy, labor supply needs to increase exactly when productivity is low to limit the consumption drop. In

Figure 6: Counterfactural simulation of financial variables



Notes: horizontal axis is for time period; time 1 refers to the pre-crisis period, and time 2 refers to the crisis period; and from time 3 onward, 50,000 IMD/Fund model economies are simulated with stochastic θ and g shocks that are independent in the cross section; and the final results are the average of 50,000 simulations, but for the last two panels the averages are taken over cross-section units that are not in default at time t.

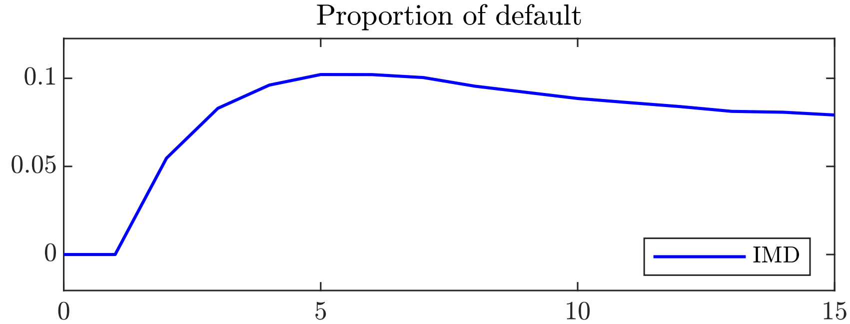
turn, the lower labor supply implies that output drops more under the Fund.

Inspecting Figure 6, we see how consumption smoothing is achieved in the Fund. First of all, under the Fund, the borrower is able to deal with a crisis by running a large deficit during the first periods of the crisis. These deficits are financed by a large reduction of the debt that is in contrast with the sharp rise of debt under the IMD economy during the same period. This debt reduction is due to the state contingent nature of the Fund contract: the country is (partially) insured against severe negative shocks. Also note the dramatic rise of the spread upon the shock in the IMD economy. This shock brings our economy from the normal stage to the intermediate stage of our Markov switching process. That is, the level of productivity is lower but not at the crisis level yet, but the probability of a crisis becomes high. This implies no default in the current period but a high probability of it in future periods, hence a high spread in the IMD economy. This behaviour is confirmed by Figure 7, displaying the proportion of countries defaulting over time. Again, the large positive spreads are eliminated under the Fund economy and replaced by small negative spreads because, in some future states the contingent assets (insurance) requested by the borrower surpass the limits implied by her future primary surplus, risking a permanent loss (transfer) for the Fund.

As a result, the Fund is better off by restraining the borrower.

It is important to note that one has to interpret the paths of debt and the spread under the IMD economy with caution, since we only depict these two variables for the selected set of countries that are not in default. In particular, the fact that debt is increasing under the IMD economy is affecting that the sample of countries depicted for this variable is selected as they experienced (higher) realizations of the shocks not triggering default.

Figure 7: Counterfactural simulation of the default wave



Notes: horizontal axis is for time period; time 1 refers to the pre-crisis period, and time 2 refers to the crisis period; and from time 3 onward, 5,000 IMD model economies are simulated with stochastic θ and g shocks that are independent in the cross section; and the final results are the proportion of cross-section units in default at time t.

In the long run, average consumption in the Fund is slightly lower, while average labor supply is slightly higher. This is is due to the fact that the country accumulates (on average) a much higher stock of debt under the Fund. At first sight, this may imply that the Fund cannot improve welfare compared to the IMD economy, because of lower long-term average consumption and leisure. Note, however, that this is offset by the two other features of the Fund contract. First, as we have seen above, it offers a much smoother consumption. Second, given the fact that the borrower is more impatient than the lender, this front-loading of consumption is increasing *ex ante* welfare. In this counterfactual experiment, the welfare gains achieved by the Fund are around 10.59 percent in consumption equivalent terms. Note that this implies that welfare gains are significantly higher when the Fund is taking over a significant amount of debt (the welfare gains with these initial shocks and zero initial debt would have been be 8.57%).

Finally, the response of effort shows an interesting pattern. While we have seen that the Fund provides better incentives for exerting effort in normal times, the IMD economy imposes more discipline than the Fund in bad times. As we see, effort increases in the IMD economy right after the bad shock, while it decreases in the Fund. In the long run, however, effort is higher in the Fund. In the IMD economy, incentives are provided through prices and through the fact that, when a country is effectively borrowing constrained, higher effort increases the probability of a budget relief (a lower government expenditure). These channels are obviously stronger in crisis times (and under temporary autarky). However, the effort under the Fund indicates that this type of 'austerity' is not part of the efficient allocation.

This counterfactual has demonstrated several important aspects. First, faced with a crisis of similar magnitude to the 2008 financial crisis, a borrower with a similar level of debt as the distressed countries in the EU in the pre-crisis times would have had access to a much larger debt capacity under the Fund, in the sense that the Fund could have absorbed all the preexisting debt and more. Second, the state contingency of the Fund allocation would have lead to considerably more risk sharing through counter-cyclical primary deficits during the crisis times. Third, costly default episodes (both in terms of lack of consumption smoothing and in terms of costly effort) could have been avoided in the Fund. Given this, we find that, even with very limited redistribution, the Fund could have improved efficiency in the EU area considerably and could have led to substantial welfare gains both in normal times and even more in crisis times.

6 Conclusion

By developing and computing a model of a *Financial Stability Fund* with the capacity to implement the *Fund contract* with sovereign countries as a *constrained-efficient mechanism*, we have contributed to the existing literature on risk sharing and sovereign debt, and to the policy debate – for example, in the European Economic Monetary Union (EMU) – on stabilization, debt sustainability, crisis-resolution mechanisms, enhancing fiscal capacity, or making debt a safe asset. The theory lays out what is needed for a Fund contract – the policy instrument of the Fund – to integrate and internalize these different aspects. Its quantification, through calibration, shows that there can be substantial welfare gains, even if we have calibrated the model for euro area 'stressed countries', and we have set a 'tight constraint' on risk-sharing transfers: the Fund cannot commit to any path that would involve positive expected lifetime/permanent transfers towards a borrowing country. We have also shown that accounting for *moral hazard* the Fund provides better incentives to reduce endogenous risks in normal times, without imposing excessive effort in crisis times. The counter-cyclical nature of the Fund contract, and the fact that it transforms defaultable sovereign debt into safe debt, enhances the fiscal space of the sovereign country: debt accumulation is more efficient and sustainable to levels that would have been unsustainable without the Fund contract.

We have made an 'exclusivity assumption' in having the Fund absorbing all the sovereign debt of a participant country but, as already noted, Liu et al. (2023) relaxes this assumption:[44] absorbing only a fraction, the Fund stabilizes *all* the country's sovereign debt. However, the required absorbing capacity for the Fund may be excessive when purely 'belief-driven crises' (not considered here) can also happen. Yet, the required capacity may be achieved if the Fund and the Central Bank jointly intervene (Callegari et al., 2023). All this follow-up work

[44] They maintain an 'exclusivity assumption' on the insurance part of the Fund contract (which is consistent with the fact that large international insurance companies provide private insurance for large catastrophes but they do not provide similar insure to sovereign countries). If there is a moral hazard, monopolizing the insurance market the Fund should be able to internalize the effect of effort as Pigou taxes do in our framework.

is based on the theory of the Fund contract laid out here. Further developments are possible and needed: how to simplify the Fund contract when it accounts for moral hazard; how to make more effective *ex ante* and *ex post* conditionality, or the minimal intervention of the Fund in the sovereign debt market; whether existing fiscal rules can partially implement the Fund contract, etc. But behind these extensions, there always will be the basic idea of the Fund contract introduced here, which transforms unconditional defaultable debt into a state-contingent safe asset making, in turn, the asset side of the Fund's balance sheet safe, therefore allowing it to issue safe and liquid debt (i.e. eurobonds in the EU).

The paper abstracts from capital accumulation, which can affect the outcomes in the presence of limited commitment and moral hazard frictions. As shown by Kehoe and Perri (2004) and Ábrahám and Cárceles-Poveda (2006), combining capital accumulation with limited commitment adds an additional friction due to the fact that capital affects the outside option. In this case, additional instruments (capital income taxes or endogenous limits on capital accumulation) would be needed to address this externality. Nevertheless, we expect that the role of the Fund in enhancing the debt and insurance capacity of a country is likely to remain similarly important in an economy with capital accumulation.

In the aftermath of the euro-crisis, the emphasis was on risk-sharing, solving the debt-overhang problem and producing safe-assets (i.e., safe eurobonds). As noted, our Fund contract, as a *constrained-efficient mechanism*, has the virtue to encompass all these different aspects and to provide a benchmark for other fiscal frameworks. In fact, the EU and EA fiscal framework changed in response to the Covid-19 crisis. Now circa 30% of the euro-area sovereign debt is being held by the Eurosystem and elements of the new European Stability Mechanism (ESM) programmes and of *Next Generation EU* are more in line with the proposed theory. In particular, SURE and RRF debts are designed to avoid transfers;[45] they display more flexible *ex ante* conditionality and, importantly, programmes are financed with eurobonds. Yet, with the need to increase defense spending, war reconstructions, the need to account for climate change and population ageing, in a union with social protection and a world of possibly shrinking global trade, fiscal tightness is, unfortunately, again in the horizon. And so is the design and development of proper fiscal institutions and contracts.[46]

[45] Although, ultimately, are backed by EU Member States' contributions to the EU budget that always must be satisfied.

[46] See, for example, Marimon and Wicht (2021).

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Online Appendix

A Proofs

A.1 Preliminary Lemmas

Lemma 3. $\partial_x FV(x,s) = V^{bf}(x,s) + x\partial_x V^{bf}(x,s) + \partial_x V^{lf}(x,s) = V^{bf}(x,s)$. *This implies the 'efficient risk-sharing' property:* $x\partial_x V^{bf}(x,s) = -\partial_x V^{lf}(x,s)$.

Proof. Given the optimization problem (8), the envelope condition — which is satisfied since the solution is unique (Marimon and Werner, 2021) — implies that $\partial_x FV = V^{bf}(x,s)$. At the same time, the decomposition in (10) implies that $\partial_x FV(x,s) = V^b(x,s) + x\partial_x V^{bf}(x,s) + \partial_x V^{lf}(x,s)$. Combining these two equations delivers the main result. □

Note that we use the 'efficient risk-sharing' property, $\big[x\partial_x V^b(x,s) + \partial_x V^l(x,s)\big] = 0$ when deriving the FOC with respect to e in (16), by letting

$$\partial_e \mathbb{E}\big[FV(x',s')\big|s,e\big] = M(s)\sum_{s'|s}\partial_e^2\pi(x'_{xs}(s'),s')V^b(x'_{xs}(s'),s'), \tag{A.1}$$

where $M(s)$ summarizes the components of $FV(x',s')$ which do not depend on s' or e.

For convenience, we will use the following notation in the next two lemmas: given any $s=(\theta,g)$ we will denote it as $s(i)$ if $g=g_i$, i.e. $s(i)=(\theta,g_i)$. Given our Assumption 2, a statement about the monotonicity of $s(i)$ in i applies to all θ in (θ,g_i); in particular, since effort only affects the distribution of g, $\partial_e\pi(s'(i)|s,e) = \partial_e\pi^g(g'=g_i|g,e)$.

Lemma 4. *i) Given Assumption 2, the law of motion* $x'_{xs}(s'(i))$ *is nondecreasing in* i *and it is constant in* i *if* $\varrho(x,s)=0$. *ii)* $V^b(x'_{xs}(s'(i)),s'(i))$ *and* $FV(x'_{xs}(s'(i)),s'(i))$ *are not decreasing in* i.

Proof. i) Assumption 2 implies that the monotone likelihood-ratio condition: $\frac{\partial_e\pi(s'(i)|s,e)}{\pi(s'(i)|s,e)}$ is nondecreasing in i for every e. Therefore, we only need to recall (9):

$$x'_{xs}(s'(i)) = \frac{1+\nu^b+\varphi(s'(i)|s,e)}{1+\nu_l}\eta x \text{ and } \varphi(s'(i)|s,e) = \varrho\frac{\partial_e\pi(s'(i)|s,e)}{\pi(s'(i)|s,e)}.$$

ii) V^b is either increasing or constant in x (see the proof of Lemma 1) and by Lemma 3 FV is also increasing in x; furthermore, given x, a higher s means a higher surplus and therefore a higher FV and, through risk-sharing, a higher V^b; in sum, both value functions are non-decreasing in i. □

It will be convenient to introduce some additional notation for the following lemmas. Let $\omega^h_{xs}(e) \equiv \sum_{s'|s}\pi(s'|s,e)V^{hf}(x'_{xs}(s'),s')$, for $h=b,l$, and $f\omega_{xs}(e) \equiv x\omega^b_{xs}(e)+\omega^l_{xs}(e)$. Note that, for example $\omega^h_{xs}(e)$, can also be written as:

$$\omega^h_{xs}(e) \equiv \sum_{i=1}^{N_g} \Delta V^h_{xs}(s'(i)) \left[\sum_{j=i}^{N_g} \pi(s'(j)|s,\, e) \right],$$

where

$$\Delta V^h_{xs}(s'(i)) = \begin{cases} \sum_{\theta'|\theta} \pi^\theta(\theta'|\theta,\, e)[V^{hf}(x'_{xs}(s'(i))) - V^{hf}(x'_{xs}(s'(i-1)))], & i > 1, \\ \sum_{\theta'|\theta} \pi^\theta(\theta'|\theta,\, e) V^{hf}(x'_{xs}(s'(1))), & i = 1, \end{cases} \tag{A.2}$$

and we can similarly rewrite $f\omega_{xs}(e)$. Furthermore, let

$$\overline{\omega}^{h\prime}_{xs}(e) \equiv \sum_{i=1}^{N_g} \Delta V^h_{xs}(s'(i)) \left[\sum_{j=i}^{N_g} \frac{\partial \pi(s'(j)|s,\, e)}{\partial e} \right]; \tag{A.3}$$

that is, function $\overline{\omega}^h_{xs}(e)$ is the function $\omega^h_{xs}(e)$ when taking derivatives with respect to e, and only the direct effect of e on the distribution of s is taken into account. We can similarly define $\overline{f\omega}_{xs}(e)$. We use $\overline{\omega}^h_{xs}(e)$ in the derivations that follow, since we are accounting from the fact that the solution to the Fund contract problem satisfies the *efficient risk-sharing* property by Lemma 3, which in this notation is: $x\omega^{b\prime}_{xs}(e) = -\omega^{l\prime}_{xs}(e)$.

Lemma 5. *The functions $\overline{\omega}^b_{xs}(e)$ and $\overline{f\omega}^l_{xs}(e)$ are non decreasing and concave. The saddle-point Lagrangean $\mathcal{L}(x, s)$ (i.e. of the saddle-point Bellman equation) is also concave in e.*

Proof. By Lemma 4 the value of the borrower and the Fund contract are non-decreasing in s and, by Assumption 2, $\partial_e F_n(e, s) \leq 0$, which implies that all the terms within brackets in (A.3) are non-negative — i.e. $\overline{\omega}^{b\prime}_{xs}(e) \geq 0$ and $\overline{f\omega}'_{xs}(e) \geq 0$. Note that, using the latter definition of $\overline{\omega}^h_{xs}(e)$,

$$\overline{\omega}^{h\prime\prime}_{xs}(e) = \sum_{i=1}^{N_g} \Delta V^h_{xs}(s'(i)) \left[\sum_{j=i}^{N_g} \frac{\partial^2 \pi(s'(j)|s,\, e)}{\partial e^2} \right]. \tag{A.4}$$

Similarly, by Assumption 2, $\partial^2_e F_n(e, s) \geq 0$, which implies that all the terms within brackets in (A.4) are non-positive — i.e. $\overline{\omega}^{b\prime\prime}_{xs}(e) \leq 0$ and $\overline{f\omega}''_{xs}(e) \leq 0$.

To see that the above conditions guarantee that, given our assumptions, the Lagrangean $\mathcal{L}(x, s)$ is concave, note that

$$\partial_e \mathcal{L}(x, s) = -\, x(1 + \nu_b)v'(e) - x\varrho v''(e) + \frac{1 + \nu_l}{1 + r}\overline{f\omega}'_{xs}(e) + x\varrho\beta\overline{\omega}^{b\prime\prime}_{xs}(e).$$

Note that $\partial_e \mathcal{L}(x, s) = 0$ is (18) expressed in this more synthetic notation. Therefore,

$$\partial^2_e \mathcal{L}(x, s) = \;-\, x(1 + \nu_b)v''(e) - x\varrho v'''(e) + \frac{1 + \nu_l}{1 + r}\overline{f\omega}''_{xs}(e) + x\varrho\beta\overline{\omega}^{b\prime\prime\prime}_{xs}(e).$$

By assumption, the first two terms are negative and we have just shown that the third is also non-positive; finally, by Assumption 2, $\partial_e^3 F_n(e,s) \geq 0$ and therefore $\varpi_{xs}^{b'''}(e) \leq 0$. □

A.2 Characterization of the Fund Solution

Proof of Lemma 1.

Proof. (a) *i)* It follows from equations (14) — $u'(c(x,s)) = 1/x$ — and (9) — $x'_{xs}(s'(i)) = \varphi(s'(i)|s,e)\eta x$ — and Lemma 4. *ii)* It follows from equation (15) and the monotonicity of $c(x,s)$ on x, i.e., *i)*. *iii)* If $V^{bf}(x,s)$ is increasing and concave in x —— see *iv)* next —, the assumed convexity of v implies, by equation (17), that e is decreasing in x; with respect to $-g$ one would also expect it to be monotone (for example, decreasing if the likelihood-ratio is non-increasing in $-g$ since then $-g$ is a wealth effect); however, if next period there is a positive probability that the limited enforcement constraint of the lender binds, then this monotonicity can be distorted and effort can be higher when $-g$ is lower. *iv)* It follows from the monotonicity of $c(x,s), n(x,s)$ and $e(x,s)$ in x, and our assumptions on $U(c,n,e)$ imply that $V^{bf}(x,s)$ is increasing and strictly concave in x. Then, by Lemma 3, $\partial_x V^{lf}(x,s) < 0$ so that $V^{lf}(x,s)$ is decreasing in x.

(b) *i)* It follows from the fact that the policies, value functions and multipliers are evaluated when the constraints are binding as solutions to the saddle-point problem (SPFE); *ii)* It follows from the fact that s may have a separate effect on the outside values (e.g. it does on $V^o(s)$), and *iii)* It follows from the *constrained qualification* constraints, (19) and (20) and *i)*.

□

A.3 Proof of Proposition 1

The proof parallels and extends the proof of Marcet and Marimon (2019) Theorem 3.

Proof. **Step 1**: *Checking that the necessary assumptions are satisfied.*

Given Assumptions 1, 2 and 3 and our assumptions on preferences, $U(c,n,e)$, and technologies, $f(n), \pi^g(\cdot; g, e)$, our economies satisfy the Marcet and Marimon (2019) assumptions. In particular, A2 on the functions (continuous in (c,n,e) and measurable in s); A3 on the non-empty feasible sets; A5 on convex technologies.[47] Regarding concavity, A6, a clarification is in order. They consider SPFE value functions which are concave in the endogenous state variable and homogeneous of degree one in the co-state Pareto weights. Instead, we merge these conditions in our endogenous co-state x: FV is homogeneous of degree one when we consider both Pareto weights $(x,1)$, and concave in x, given our concavity assumptions, – making V^{bf} strictly concave in x, i.e satisfying A6b – and Lemma 3. The uniform boundedness assumption A4 is also satisfied, since feasible n and effort e are bounded, and consumption

[47]Referring to $\{n(s)|f(n(s)) - g(s) \geq 0\}$, $e \in [0,1]$ and Assumption 3.

c is bounded by the technology and the lender's limited enforcement constraint. Therefore, the rewards ($U(c, n, e)$ and c_l) are bounded as well and, by Assumption 1, the finiteness of $V^o(s)$ and Z imply that the constraint functions are uniformly bounded as well. Finally, our *interiority* assumption is a version of A7b.

Step 2: *'Relaxed Fund Contract problem' and the existence of solutions to this problem.*

We will show first that a solution exists to a '*Relaxed Fund Contract problem*', which is the same as the *Fund Contract problem* except that constraint (3) is replaced by a weak inequality version:

$$\beta \sum_{s^{t+1}|s^t} \frac{\partial \pi(s^{t+1}|s_t,\, e(s^t))}{\partial e(s^t)} V^b(s^{t+1}|s^t) - v'(e(s^t)) \geq 0, \tag{A.5}$$

which can also be written as

$$\beta \overline{\omega}^b_{xs}{}'(e) - v'(e) \geq 0.$$

In particular, given our assumptions,

$$\beta \overline{\omega}^b_{xs}{}'''(e) - v'''(e) \leq 0.$$

Therefore (A.5) defines a convex set of feasible efforts.

Next, we will show that any solution to the 'Relaxed Fund Contract problem' is also a solution of the original problem. Let $\mathfrak{A} = \{(c, n, e) \in \mathbb{R}^3_+ : n \leq 1, e \leq 1\}$. This set is obviously compact and convex. Note that the pay-off of the fund $c_l(s) = \theta f(n) - g - c$ is concave given our concavity assumption on f.

We first decompose the saddle-point recursive contract problem into the choice of actions, $\mathfrak{a} = (c, n, e)$, and multipliers, $\gamma = (\nu_b, \nu_l, \varrho)$, given $FV(x, s)$, as follows:

$$\begin{aligned}
SP^{\mathfrak{a}}(\gamma; x, s) = \Bigg\{ & \mathfrak{a} \in \mathfrak{A} : \text{ for all } \widetilde{\mathfrak{a}} \in \mathfrak{A}, \\
& x\Big[U(\mathfrak{a}) + \beta \sum_{s'|s} \pi(s'|s, e) V^b(x'(s'), s')\Big] \\
& + \Big[c_l(s) + \frac{1}{1+r} \sum_{s'|s} \pi(s'|s, e) V^l(x'(s'), s')\Big] \\
& + x\nu_b \Big[U(\mathfrak{a}) + \beta \sum_{s'|s} \pi(s'|s, e) V^b(x'(s'), s') - V^o(s)\Big] \\
& + \nu_l \Big[c_l(s) + \frac{1}{1+r} \sum_{s'|s} \pi(s'|s, e) V^l(x'(s'), s') - Z\Big] \\
& + x\varrho \Big[\beta \sum_{s'|s} \frac{\partial \pi(s'|s,\, e)}{\partial e} V^b(x'(s'), s') - v'(e)\Big] \\
& \geq x\Big[U(\widetilde{\mathfrak{a}}) + \beta \sum_{s'|s} \pi(s'|s, \widetilde{e}) V^b(\widetilde{x}'(s'), s')\Big]
\end{aligned}$$

$$
\begin{aligned}
&+ \left[\widetilde{c}_l(s) + \frac{1}{1+r} \sum_{s'|s} \pi(s'|s, \widetilde{e}) V^l(\widetilde{x}'(s'), s') \right] \\
&+ x\nu_b \left[U(\widetilde{\mathfrak{a}}) + \beta \sum_{s'|s} \pi(s'|s, \widetilde{e}) V^b(\widetilde{x}'(s'), s') - V^o(s) \right] \\
&+ \nu_l \left[\widetilde{c}_l(s) + \frac{1}{1+r} \sum_{s'|s} \pi(s'|s, \widetilde{e}) V^l(\widetilde{x}'(s'), s') - Z \right] \\
&+ x\varrho \left[\beta \sum_{s'|s} \frac{\partial \pi(s'|s, \, \widetilde{e})}{\partial e} V^b(\widetilde{x}'(s'), s') - v'(\widetilde{e}) \right] \Bigg\},
\end{aligned}
$$

where $\widetilde{x}'(s') = \frac{1+\nu_b}{1+\nu_l} + \varrho \frac{\partial \pi(s'|s,\widetilde{e})/\partial e}{(1+\nu_l)\pi(s'|s,\widetilde{e})}$. Note that our original problem is homogeneous of degree one in $(\mu_{b,0}, \mu_{l,0})$, and that allows us to reformulate the problem using x as a co-state variable. This guarantees, together with our *interiority* assumption (a version of A7b used in Lemma 6A in Marcet and Marimon, 2019), that there exists a positive constant C such that, if γ is the Lagrange multiplier vector, $\|\gamma\| \leq C\|x\|$. But the lender's participation constraint Z sets an upper bound on $\|x\|$ for any feasible contract. Therefore, there exists a $\overline{C}$ such that $\|\gamma\| \leq \overline{C}$, and the set of feasible Lagrange multipliers, $\Gamma = \{\gamma \in \mathbb{R}^3_+ : \|\gamma\| \leq \overline{C}\}$, is also compact and convex. The minimization problem can be written as:

$$
\begin{aligned}
SP^\gamma(\mathfrak{a}; x, s) = \Bigg\{ &\gamma \in \Gamma: \text{ for all } \widehat{\gamma} \in \Gamma, \\
&x \left[U(\mathfrak{a}) + \beta \sum_{s'|s} \pi(s'|s, e) V^b(x'(s'), s') \right] \\
&+ \left[c_l(s) + \frac{1}{1+r} \sum_{s'|s} \pi(s'|s, e) V^l(x'(s'), s') \right] \\
&+ x\nu_b \left[U(\mathfrak{a}) + \beta \sum_{s'|s} \pi(s'|s, e) V^b(x'(s'), s') - V^o(s) \right] \\
&+ \nu_l \left[c_l(s) + \frac{1}{1+r} \sum_{s'|s} \pi(s'|s, e) V^l(x'(s'), s') - Z \right] \\
&+ x\varrho \left[\beta \sum_{s'|s} \frac{\partial \pi(s'|s, \, e)}{\partial e} V^b(x'(s'), s') - v'(e) \right] \\
&\leq x \left[U(\mathfrak{a}) + \beta \sum_{s'|s} \pi(s'|s, e) V^b(\widehat{x}'(s'), s') \right] \\
&+ \left[c_l(s) + \frac{1}{1+r} \sum_{s'|s} \pi(s'|s, e) V^l(\widehat{x}'(s'), s') \right] \\
&+ x\widehat{\nu}_b \left[U(\mathfrak{a}) + \beta \sum_{s'|s} \pi(s'|s, e) V^b(\widehat{x}'(s'), s') - V^o(s) \right]
\end{aligned}
$$

$$
\begin{aligned}
&+\widehat{\nu}_l\left[c_l(s)+\frac{1}{1+r}\sum_{s'|s}\pi(s'|s,e)V^l(\widehat{x}'(s'),s')-Z\right]\\
&+x\widehat{\varrho}\left[\beta\sum_{s'|s}\frac{\partial\pi(s'|s,\,e)}{\partial e}V^b(\widehat{x}'(s'),s')-v'(e)\right]\Bigg\},
\end{aligned}
$$

where $\widehat{x}'(s')=\frac{1+\widehat{\nu}_b}{1+\widehat{\nu}_l}+\widehat{\varrho}\frac{\partial\pi(s'|s,e)/\partial e}{(1+\widehat{\nu}_l)\pi(s'|s,e)}$. Now, if we define the correspondence

$$
SP:\mathfrak{A}\times\Gamma\rightarrow\mathfrak{A}\times\Gamma \text{ by } SP(\mathfrak{a},\gamma;x,s)=\left(SP^{\mathfrak{a}}(\gamma;x,s),SP^{\gamma}(\mathfrak{a};x,s)\right),
$$

one can show — given Lemma 5 — that it is non-empty, convex-valued and upper hemicontinuous, as in Lemma 7A in Marcet and Marimon (2019) (Theorem 3), which applies Kakutani's fixed point theorem to prove the existence of solutions to the saddle-point contracting problem.

In our case, this means that, with the additional (A.5), there is a contract satisfying equations (9)–(10), (13)–(15), (16), (19)–(20) and the following constraint qualification condition:

$$
\varrho\left[\beta\sum_{s'|s}\frac{\partial\pi(s'|s,\,e)}{\partial e}V^b(x'(s'),s')-v'(e)\right]=0, \tag{A.6}
$$

with $\varrho(x,s)=0$ if the term in brackets in (A.6) is non-zero. Now we show that $\varrho(x,s)\neq 0$. Suppose that $\varrho(x,s)=0$. Then (16) reduces to

$$
\begin{aligned}
0=&-v'(e(x,s))+\beta\sum_{s'|s}\frac{\partial\pi(s'|s,e(x,s))}{\partial e}V^b(x'_{xs}(s'),s')\\
&+\frac{1+\nu_l(x,s)}{1+\nu_b(x,s)}\frac{1}{x}\frac{1}{1+r}\sum_{s'|s}\frac{\partial\pi(s'|s,e(x,s))}{\partial e}V^l(x'_{xs}(s'),s').
\end{aligned}
$$

Note that V^l is nondecreasing in i since $\varrho(x,s)\geq 0$. Then we can rewrite

$$
\sum_{s'|s}\frac{\partial\pi(s'|s,e(x,s))}{\partial e}V^l(x'_{xs}(s'),s')
$$

as

$$
\sum_{i=1}^{N}\Delta V^l_{xs}(s'(i))\left[\sum_{j=i}^{\overline{N}}\frac{\partial\pi(s'|s,e(x,s))}{\partial e}\right],
$$

where $\Delta V^l_{xs}(s'(i))$ is defined by (A.2). Note that the first term is equal to zero in this summation. The monotone likelihood ratio implies that all other terms are non-negative, as the terms in the bracket are strictly positive and $\Delta V^h_{xs}(s'(i))\geq 0$ for all $i>1$. The fact that we assume that some risk sharing occurs in this economy (the lender's participation constrain is slack in at least one state realization) implies that some of the terms in the summation will be strictly positive. Given that $\sum_{s'|s} \frac{\partial \pi(s'|s,e(x,s))}{\partial e} V^l(x'_{xs}(s'), s')$ is positive, the first line must be negative, but that would violate (A.5). Hence, we have reached a contradiction and $\varrho(x,s) > 0$ must be true. In sum, the contract satisfies all the conditions of Definition 1.

Step 3: *The relaxed Fund Contract problem and the Fund Contract problem have the same solution.* This is the consequence of $\varrho(x,s) > 0$. Given this, (A.5) implies that the incentive compatibility constraint is satisfied as an equality in the relaxed Fund problem. Hence, the solution is equivalent to the original problem when this constraint was introduced as an equality.

Step 4: *Uniqueness.* FV is monotone in x. Further, it is constant when either of the limited enforcement constraints are binding and concave when both are slack. The same contraction mapping argument used in Theorem 3 of Marcet and Marimon (2019) shows that FV is unique. The strict concavity/convexity assumptions on u, f and v imply that the Recursive Contract allocation is unique and FV is strictly concave in x whenever neither participation constraint is binding, and it is uniquely defined when either is binding. Therefore, the saddle-point solution is unique. □

A.4 Proof of Corollary 1

The proof follows from Stokey et al. (1989) Theorem 12.12. For further details, see Liu et al. (2023) Online Appendix.

A.5 Proofs of Proposition 2 and 3

Proof of Proposition 2. To prove the proposition, we show that we can construct asset prices, asset holdings, policies, multipliers, borrowing limits and value functions, corresponding to the description of the economy of Subsection 3.1.2, such that all the conditions characterizing the recursive competitive equilibrium in Definition 2 are satisfied by the recursive Fund policies and values. The proof is partially based on Alvarez and Jermann (2000), but we account for the presence of a moral hazard incentive compatibility constraint and a risk-neutral lender subject to a limited enforcement constraint. Furthermore, the proof is done in a recursive competitive framework, taking advantage of the Fund's policies and value functions characterization in Lemma 1, and it implements the IC constraint with the introduction of state-contingent assets, where the Arrow security component (paying in units of assets) is subject to Pigou budget-neutral taxation.

Step 1: *Getting $q(s'|\boldsymbol{a}, s)$ from $q(s'|x, s)$ and mapping $(x, s) \to (\boldsymbol{a}, s)$.* As seen in Subsection 3.1.2 we want to obtain Arrow security prices satisfying (49):

$$q(s'|\boldsymbol{a}, s) = \pi(s'|s, e(a,s)) A_{\boldsymbol{a}'}(s') \max\left\{ \beta \frac{u'(c(a', s'))}{u'(c(a,s))} \frac{1}{1+\tau'(s';\boldsymbol{a}, s)}, \frac{1}{1+r} \right\},$$

where $A_{\boldsymbol{a}}(s) = (1 - \delta + \delta\kappa) + \delta q(\boldsymbol{a}, s)$ and $q(\boldsymbol{a}, s) = \sum_{s'|s} q(s'|\boldsymbol{a}, s)$.

By (51) and (52) we get Fund Arrow security prices as:

$$q(s'|x,s) = \frac{1}{1+r}\pi(s'|s,e(x,s))A_{x'}(s')\max\left\{\frac{1+\nu_l(x',s')}{1+\nu_b(x',s')}\frac{1}{1+\frac{\varphi(s'|s,e(x,s))}{1+\nu_b(x,s)}}\frac{1}{1+\tau(s';x,s)},1\right\}$$
$$= \frac{1}{1+r}\pi(s'|s,e(x,s))A_{x'}(s')\max\left\{\frac{1+\nu_l(x',s')}{1+\nu_b(x',s')},1\right\}, \tag{A.7}$$

where, $A_x(s) = (1-\delta+\delta k) + \delta q(x,s)$, with $q(x,s) = \sum_{s'|s} q(s'|x,s)$, and asset security taxes as defined in (53):

$$\frac{1}{1+\tau(s',x,s)} = 1 + \chi(x,s)u'(c(x,s))\frac{\partial_e \pi(s'|s,e(x,s))}{\pi(s'|s,e(x,s))}$$
$$= 1 + \frac{\varphi(s'|s,e)}{1+\nu_b(x,s)}.$$

Therefore, if we map (x,s) into $(\boldsymbol{a},s)$ we obtain $q(s'|\boldsymbol{a},s)$ and $\tau'(s';\boldsymbol{a},s)$. In order to do so, we first write the budget constraint of the borrower in (32), separating its three components, with Fund contract policies, that is:

$$q(x,s)(\bar{a}'(s) - \delta a) = \theta(s)f(n(x,s)) - c(x,s) - g(s) + (1-\delta+\delta\kappa)a, \tag{A.8}$$
$$\sum_{s'|s} q(s'|x,s)\hat{a}(s') = 0 \tag{A.9}$$
$$\text{and} \quad \bar{\tau}(x,s) = \sum_{s'|s} q(s'|x,s)a'(s')\tau'(s';x,s) \tag{A.10}$$

Second, we define

$$Q(s'|x,s) = \frac{q(s'|x,s)}{A_x(s')}, \quad Q(s''|x,s) = \frac{q(s''|x',s')}{A_{x''}(s'')}Q(s'|x,s), \quad \ldots,$$

recursively, for any state and any time in the future, with $Q(s|x,s) = \frac{1}{A_x(s)}$.

Third, by iterating on the budget constraint (A.8) we obtain the initial asset holding allocation $(a(s_0),\, a_l(s_0))$ given by

$$a(s_0) = \sum_{t=0}^{\infty}\sum_{s_t|s_0} Q(s_t|x(s_0),s_0)[c(x_t,s_t) - \theta_t f(n(x_t,s_t)) + g_t], \tag{A.11}$$
$$a_l(s_0) = \sum_{t=0}^{\infty}\sum_{s_t|s_0} Q(s_t|x(s_0),s_0)c_l(x_t,s_t), \tag{A.12}$$

where we have used the fact that the Fund contract allocation is stationary and the value functions are bounded, implying that the transversality conditions are satisfied. Now we have an identification between the initial condition $x(s_0) = \mu_{b,0}/\mu_{l,0}$ and the initial asset holdings $(a(s_0), a_l(s_0))$, where $a_l(s_0) = -a(s_0)$. In order to extend this map to all portfolio of asset holdings, we use the law of motion of x (9), also decomposed as

$$\bar{x}'(s) = \frac{1+\nu_b}{1+\nu_l}\eta x \quad \text{and} \quad \hat{x}'(s') = \frac{\varphi(s'|s,e)}{1+\nu_l}\eta x. \tag{A.13}$$

Now, given any (x, s) — say, $(x(s_0), s_0)$ — we have $a(s)$, by (A.13) and we have $\bar{x}'(s)$ and $\hat{x}'(s')$, which map into $\bar{a}'(s)$ and $\hat{a}(s')$ by (A.8) and (A.9), and we also have $-a_l(s') = a(s') = \bar{a}'(s) + \hat{a}(s')$. Furthermore, we impose the equilibrium condition $\boldsymbol{a} = a$. Finally, as we said, we also have $\tau'(s'; \boldsymbol{a}, s)$ and, by (A.10), we have $\bar{\tau}(a, s)$.

Step 2: *Getting policies, bounds, multipliers and policy functions.* The mapping implies that we can construct the borrower's policies that implement the Fund contract as: $m(a, s) = m(x, s)$, for $m = c, n, e$. Similarly, given the definition of *threshold x-bounds* in Section 2.1.3, we can define the borrowing and lending limits for every s:

$$\mathcal{A}_b(s) = a(\underline{x}(s), s) \text{ and } \mathcal{A}_l(s) = a(\overline{x}(s), s)$$

Note that these limits are history-independent and hence they are functions of only the exogenous state s. Note also that these borrowing constraints imply that $a'(s'; a, s) \geq \mathcal{A}_b(s)$ and $a_l'(s'; a, s) \geq \mathcal{A}_l(s)$ for all s; i.e., the constructed asset holdings satisfy the competitive equilibrium borrowing constraints (35) and (43). We now need to show that the policy functions, as functions of (a, s) and (a_l, s), satisfy all the constraints of borrowers and lenders' competitive equilibrium problems.

Next, we define the multiplier of the borrower's maximization problem as:

$$\lambda(\boldsymbol{a}, s) = \frac{1+\nu_l(x,s)}{1+\nu_b(x,s)}\frac{1}{x}, \tag{A.14}$$

which guarantees that the consumption policy $c(a, s)$ is optimal in the competitive equilibrium as it satisfies (37). Since $c(x, s)$ and $n(x, s)$ satisfy the Fund labor optimality condition in (15), $c(a, s)$ and $n(a, s)$ satisfy the equilibrium labor optimality condition in (38). For the risk-neutral lender $c_l(a_l, s) = c_l(x, s)$ is an optimal consumption policy, as long as the corresponding asset-portfolio is optimal. To see whether the asset policies are optimal competitive policies, we need to show that they bind exactly when the limited enforcement constraints bind in the Fund.

When $a'(s'; a(s), s) > \mathcal{A}_b(s')$, it must be that

$$\begin{aligned} q_x(s'|s) &\equiv \pi(s'|s, e(x,s))A_{x'}(s')\beta\frac{u'(c(a'(s';a,s),s'))}{u'(c(a,s))}\frac{1}{1+\tau_x'(s',s)} \\ &\geq \pi(s'|s, e(x,s))A_{x'}(s')\frac{1}{1+r}. \end{aligned}$$

and, given the taxes we have constructed, also that $\nu_l(x', s') \geq \nu_b(x', s') = 0$; therefore, whenever the weak inequality is an inequality, adding $(1 + \nu_l(x', s')) > 0$ to the right-hand side of the previous equation equates it to (A.7). This changes the inequality into an equality (multiplying the right-hand side).

Similarly, when $a_l'(s'; a(s), s) > \mathcal{A}_l(s')$, it must be that

$$
\begin{aligned}
q(s'|x,s) &\equiv \pi(s'|s, e(x,s)) A_{x'}(s') \frac{1}{1+r} \\
&\geq \pi(s'|s, e(x,s)) A_{x'}(s') \beta \frac{u'(c(a'(s',a,s),s'))}{u'(c(a,s))} \frac{1}{1+\tau_x'(s',s)},
\end{aligned}
$$

and also that $\nu_b(x', s') \geq \nu_l(x', s') = 0$. Then, whenever the weak inequality is an inequality, adding $\frac{\nu_b(x',s')}{u'(c(x,s)(1+\tau_x'(s',s))}$ to the right hand side, equates the inequality. This shows that the asset policies bind when the Fund LE constraints bind. Therefore, setting $\tilde{\gamma}_b(a,s) = \nu_b(x,s)$ and $\tilde{\gamma}_l(a,s) = \nu_l(x,s)$, the asset portfolio policy satisfies the equilibrium optimality conditions with respect to assets prices in (40) and (45).

Finally, provided that the effort policy $e(a,s)$ is also an optimal policy, the identification of the value functions, $W^i(a,s) = V^i(x,s)$ for $i = b, l$, is consistent with their definition: (11) and (12) become (33) and (41). But, by construction, $e(a,s) = e(x,s)$. If $W^b(a,s) = V^b(x,s)$ and $e(x,s)$ satisfies the IC constraint in (17), then $e(a,s)$ satisfies the equilibrium optimality condition for effort (39) as well. □

Proof of Proposition 3. We show that, given a RCE satisfying Definition 3, we can design a Fund contract satisfying equations (9)–(15) and (17)–(20). First, as in the proof of Proposition 2, we obtain a mapping $(\boldsymbol{a}, s) \to (x, s)$, which is also well-defined for the initial state. From (37): $u'(c(a,s)) = \lambda(a,s)$, so we can write (40) as:

$$
\lambda(a,s) q(s'|\boldsymbol{a},s) = \beta \pi(s'|s, e(a,s)) A(\boldsymbol{a}', s') \frac{1+\hat{\gamma}_b(a',s')}{1+\tau'(s';\boldsymbol{a},s)} \lambda(a',s'), \tag{A.15}
$$

where $\hat{\gamma}_b(a', s')$ is the normalized multiplier defined according to

$$
\hat{\gamma}_b(a',s') \equiv \frac{\tilde{\gamma}_b(a',s')}{\beta \pi(s'|s, e(a,s)) A(\boldsymbol{a}', s') \lambda(a',s')}.
$$

Similarly, we can write (45) as:

$$
q(s'|\boldsymbol{a},s) = \frac{1}{1+r} \pi(s'|s, e(a,s)) A(\boldsymbol{a}', s')(1+\hat{\gamma}_l(a',s')), \tag{A.16}
$$

where $\hat{\gamma}_l(a',s') \equiv \tilde{\gamma}_l(a',s') / \left[\frac{1}{1+r} \pi(s'|s, e(a,s)) A(\boldsymbol{a}', s')\right]$. Dividing (A.15) by (A.16) we get:

$$
\lambda(a,s) = \eta \frac{1+\hat{\gamma}_b(a',s')}{1+\hat{\gamma}_l(a',s')} \frac{1}{1+\tau'(s';\boldsymbol{a},s)} \lambda(a',s').
$$

Define $x'(\boldsymbol{a}', s') \equiv \frac{1+\hat{\gamma}_l(a',s')}{1+\hat{\gamma}_b(a',s')} \frac{1}{\lambda(a',s')}$ and, as implied by Definition 3, there exist $\tilde{\varrho}$ and hence $\tilde{\varphi}(s'|s,e)$ at $(\boldsymbol{a}, s)$ such that $\frac{1}{1+\tau'(s';\boldsymbol{a},s)} \equiv 1 + \frac{\tilde{\varphi}(s'|s,e(a,s))}{1+\hat{\gamma}_b(a,s)}$. Then we can obtain the *recursive* *system of weights* $x(\boldsymbol{a}, s)$ in (56):

$$x'(\boldsymbol{a}', s') = \frac{1 + \hat{\gamma}_b(a,s) + \tilde{\varphi}(s'|s, e(a,s))}{1 + \hat{\gamma}_l(a,s)} \eta x(\boldsymbol{a}, s)$$
$$\text{with} \quad \tilde{\varphi}(s'|s, e(a,s)) = \tilde{\varrho}(a,s) \frac{\partial_e \pi(s'|s, e(a,s))}{\pi(s'|s, e(a,s))},$$

and we have the mapping $(\boldsymbol{a}, s) \to (x, s) \equiv (x(\boldsymbol{a}, s), s)$, where the equivalence is notational.

This allows us to identify $m(x,s) = m(a,s)$, for $m = c, n, e$ and c_l, and particularly for the initial state $m(x(s_0), s_0) = m(\boldsymbol{a}(s_0), s_0)$. Furthermore, if we identify $\nu_b(x,s) \equiv \hat{\gamma}_b(a,s)$ and $\nu_l(x,s) \equiv \hat{\gamma}_l(a,s)$, then, we obtain (14):

$$u'(c(x,s)) = \frac{1 + \nu_l(x,s)}{1 + \nu_b(x,s)} \frac{1}{x} = \lambda(a,s) = u'(c(a,s)),$$

and (15). Similarly, we can identify the value functions: $V^{bf}(x,s) = W^b(a,s)$ and $V^{lf}(x,s) = W^l(a,s)$. Then, regarding $e(x,s) = e(a,s)$, note that, by Definition 3, $e(a,s)$ solves (54) and, therefore, satisfies

$$\frac{1}{1+r} \sum_{s'|s} \partial_e \pi(s'|s, e(a,s)) W^l(a', s')$$
$$= \tilde{\chi}(\boldsymbol{a}, s) \left[v''(e(a,s)) - \beta \sum_{s'|s} \partial_e^2 \pi(s'|s, e(a,s)) W^b(a', s') \right],$$

where $\tilde{\chi}(\boldsymbol{a}, s) \equiv \frac{\tilde{\varrho}(a,s)}{1+\hat{\gamma}_l(a,s)} x(\boldsymbol{a}, s)$, which is just a version of (18). Similarly, with our identification of multipliers, the constraint qualification constraints (19) and (20) are satisfied. In sum, equations (9)–(15) and (17)–(20) are satisfied and there is an unique *Fund contract* which implements the RCE. □

Note that taking into account both proofs of Proposition 2 and 3, we have established a *one-to-one* mapping between (x, s) and $(\boldsymbol{a}, s)$.

B On the Prescott-Townsend Implementation

This section discusses an alternative implementation of the optimal Fund allocation following Prescott and Townsend (1984). To maintain comparability to our implementation with asset taxes, we allow the borrower to trade in long run state contingent assets. However, in the absence of asset taxes, the externality of effort and the fact that there is two sided limited commitment in the optimal contract are captured by two additional constraints which are imposed directly in the borrower's problem: the incentive compatibility constraint and a state-contingent upper bound on the borrower's asset holdings. The problem of the borrower can be written as follows:

$$
\begin{aligned}
W^b(a,s) = \max_{\{c,n,e,a'(s')\}} & U(c,n,e) + \beta \mathbb{E}\left[W^b(a'(s'),s')\middle|s,e\right] \quad \text{s.t.} \\
c + \sum_{s'|s} q(s'|\boldsymbol{a},s)a'(s')A(\boldsymbol{a}',s') & \leq \theta(s)f(n) - g(s) + aA(\boldsymbol{a},s), \\
a'(s')A(\boldsymbol{a}',s') & \geq \mathcal{A}_b(s'), \\
v'(e) & = \beta \sum_{s'|s} \frac{\partial \pi(s'|s,e)}{\partial e} W^b(a'(s'),s'), \\
-a'(s')A(\boldsymbol{a}',s') & \geq \mathcal{A}_l(s'),
\end{aligned} \tag{B.1}
$$

with $A(\boldsymbol{a},s) = 1-\delta+\delta\kappa+\delta q(\boldsymbol{a},s)$ and the borrowing limits $\mathcal{A}_b(s')$ and $\mathcal{A}_l(s')$ are endogenous as in the implementation with asset taxes. Note first that, in contrast with the decentralization of Section 3, constraint (B.1) is a constraint in the borrower's problem, not in the lender's problem; however, regarding the RCE, both formulations are equivalent, although they have different interpretations. Second, we are expressing the borrowing (and saving) constraints in terms of the per period payoffs of the borrower's long term asset, as this simplifies the algebra and it makes it more compatible with the original Fund design problem.

As in the equilibrium with asset taxes, competitive risk neutral international lenders participate in this market as well. They do not face any constraints beyond a no-Ponzi condition and this implies that they price the long term securities as follows:

$$
q(\boldsymbol{a},s'|s) = \frac{1}{1+r}\pi(s'|s,e)A(\boldsymbol{a}',s'), \tag{B.2}
$$

Using the previous equation, we can replace the price in the budget constraint of the borrower and obtain the following condition:

$$
c + \frac{1}{1+r}\sum_{s'|s}\pi(s'|s,e)a'(s')A(\boldsymbol{a}',s') \leq \theta\left(s\right)f(n) - g(s) + aA(\boldsymbol{a},s) \tag{B.3}
$$

This condition, together with incentive compatibility, guarantees that all feasible allocations are incentive compatible and acceptable by international lenders. It is easy to show that this equilibrium satisfies the optimality conditions of the optimal Fund allocation and can therefore implement it. To see this, note that the Lagrangean of the borrower's problem is:

$$
\begin{aligned}
\max_{\{c,n,e,a'(s')\}} & U(c,n,e) + \beta\sum_{s'|s}\pi(s'|s,e)W^b(a'(s'),s') \\
& + \lambda_b(a,s)\left[\theta\left(s\right)f(n) - g(s) + aA(\boldsymbol{a},s) - c - \frac{1}{1+r}\sum_{s'|s}\pi(s'|s,e)a'(s')A(\boldsymbol{a}',s')\right] \\
& + \gamma_b(a'(s'),s')\pi(s'|s,e)[a'(s')A(\boldsymbol{a}',s') - \mathcal{A}_b(s')]
\end{aligned}
$$

$$
\begin{aligned}
&+\psi(a,s)\left[\beta\sum_{s'|s}\partial_e\pi(s'|s,e)W^b(a'(s'),s')-v'(e)\right]\\
&+\gamma_l(a'(s'),s')\pi(s'|s,e)[-a'(s')A(\boldsymbol{a}',s')-\mathcal{A}_l(s')].
\end{aligned}
$$

First, the labor optimality condition of the Fund contract is satisfied. Second, the optimality condition of consumption and the envelope condition imply:

$$
u'(c)=\lambda_b\left(a,s\right)
$$

$$
\frac{\partial W^b(a,s)}{\partial a}=\lambda_b(a,s)A\left(\boldsymbol{a},s\right)=u'(c)A(\boldsymbol{a},s).
$$

Moroever, the optimality condition with respect to $a'(s')$ is:

$$
\begin{aligned}
0=\beta&\frac{\pi(s'|s,e)\partial W^b(a'(s'),s')}{\partial a'(s')}-\lambda_b(a,s)q(s'|\boldsymbol{a},s)A(\boldsymbol{a}',s')\\
&+\psi(a,s)\beta\partial_e\pi(s'|s,e)\frac{\partial W^b(a'(s'),s')}{\partial a'(s')}\\
&+\gamma_b(a'(s'),s')\pi(s'|s,e)A(\boldsymbol{a}',s')-\gamma_l(a'(s'),s')\pi(s'|s,e)A(\boldsymbol{a}',s').
\end{aligned}
$$

After substituting for the equilibrium price, the previous equation can be rewritten as:

$$
\begin{aligned}
&\left[\frac{1}{\lambda_b(a,s)}+\frac{\psi\left(a,s\right)}{\lambda_b(a,s)}\frac{\partial_e\pi(s'|s,e)}{\pi(s'|s,e)}\right]\beta\frac{\partial W^b(a'(s'),s')}{\partial a'(s')}\\
&\qquad=\frac{A(\boldsymbol{a}',s')}{1+r}-\frac{\gamma_b(a'(s'),s')-\gamma_l(a'\left(s'\right),s')}{\lambda_b(a,s)}A(\boldsymbol{a}',s').
\end{aligned}
$$

Substituting the envelope condition and the optimality condition for consumption (which allows to eliminate $A(\boldsymbol{a}',s')$), we obtain:

$$
\begin{aligned}
&\left[\frac{1}{u'(c)}+\frac{\psi(a,s)}{\lambda_b(a,s)}\frac{\partial_e\pi(s'|s,e)}{\pi(s'|s,e)}\right]\beta(1+r)\\
&\qquad=\frac{1}{u'(c(a'(s'),s'))}\left[1-\frac{(1+r)(\gamma_b(a'(s'),s')-\gamma_l(a'(s'),s'))}{\lambda_b(a,s)}\right].
\end{aligned}
$$

Note that $\beta(1+r)=\eta$ and set $\chi(x,s)=\frac{\psi(a,s)}{\lambda_b(a,s)}$. Whenever $\gamma_l(a'(s'),s')=0$, we set $\nu_l(x',s')=0$ and $\frac{1}{1+\nu_b(x',s')}=1-\frac{(1+r)\gamma_b(a'(s'),s')}{\lambda_b(a,s)}$. Moreover, whenever $\gamma_b(a'(s'),s')=0$, we set $\nu_b(x',s')=0$ and $1+\nu_l(x',s')=1+\frac{(1+r)\gamma_l(a'(s'),s')}{\lambda_b(a,s)}$. Hence this equilibrium also delivers constrained efficient consumption allocations.

We now turn to the optimality condition with respect to effort, which is given by:

$$
0=\,-v'(e)+\beta\partial_e\pi(s'|s,e)W^b(a'(s'),s')+\psi(a,s)\left[\beta\sum_{s'|s}\partial_e^2\pi(s'|s,e)W^b(a'(s'),s')-v''(e)\right]
$$

$$- \lambda_b(a,s)\frac{1}{1+r}\sum_{s'|s}\partial_e\pi(s'|s,e)a'(s')A(\boldsymbol{a'},s')$$

The incentive compatibility constraint simplifies the above equation to:

$$\frac{1}{1+r}\sum_{s'|s}\partial_e\pi(s'|s,e)a'(s')A(\boldsymbol{a'},s') = \frac{\psi(a,s)}{\lambda_b(a,s)}\left[v''(e) - \beta\sum_{s'|s}\partial_e^2\pi(s'|s,e)W^b(a'(s'),s')\right]$$

implying that effort is also constrained efficient.

C More Details on the Calibration

C.1 Data Sources and Model Consistent Measures

The main data sources and relevant definitions of data variables are listed in Table C.1. To map the data to the model, we construct model consistent data measures as below.

Table C.1: Data Sources and Definitions

Series	Time	Sources[a]	Unit
Output	1980–2015	AMECO (OVGD)	1 billion 2010 constant euro
Government consump.	1980–2015	AMECO (OCTG)	1 billion 2010 constant euro
Total working hours	1980–2015	AMECO (NLHT)[b]	1 million hours
Employment	1980–2015	AMECO (NETD)	1000 persons
Government debt	1980–2015	AMECO EDP[c]	end-of-year percentage of GDP
Debt service	1980–2015	AMECO (UYIGE)[d]	end-of-year percentage of GDP
Primary surplus	1980–2015	AMECO (UBLGIE)[e]	end-of-year percentage of GDP
Bond yields	1980–2015	AMECO (ILN)[f]	percentage, nominal
Debt maturity	1990–2010	OECD, EuroStat, ESM[g]	years
Labor share	1980–2015	AMECO[h]	percentage

[a] Strings in parentheses indicate AMECO labels of data series.
[b] PWT 8.1 values for Greece in 1980–1982.
[c] General government consolidated gross debt; ESA 2010 and former definition, linked series.
[d] AMECO for 1995–2015; European Commission *General Government Data* (GDD 2002) for 1980–1995.
[e] AMECO linked series for 1995–2015; European Commission *General Government Data* (GDD 2002) for 1980–1995.
[f] A few missing values for Greece and Portugal replaced by EuroStat long-term government bond yields.
[g] Average across different data sources; sporadic time coverage over countries, see text below; ESM data are obtained from private correspondence.
[h] Compensation of employees (UWCD) plus gross operating surplus (UOGD) minus gross operating surplus adjusted for imputed compensation of self-employed (UQGD), then divided by nominal GDP (UVGD).

Labor input For the aggregate labor input n_{it}, we use two series from AMECO, the aggregate working hours H_{it} and the total employment E_{it} of each country over the period 1980–2015. We calculate the normalized labor input as $n_{it} = H_{it}/(E_{it} \times 5200)$, assuming 100 hours of disposable time per worker per week. However, for most of the data moment computations, we use H_{it} directly, since the per worker annual working hours do not show a significant cyclical pattern and both the level and the trend do not affect the computation of the moments.

Fiscal position and private consumption We hold the premise of fitting the *observed* fiscal behavior across the GIPS countries, so that we use directly the *data measures* of government consumption and primary surplus to calibrate the model. However, the cost of such a strategy is on the model consistent measure of private consumption. Note that in the model, primary surplus equals to $y - g - c$, therefore private consumption equals to y minus the sum of g and primary surplus. This is the model consistent measure of private consumption we use in our calibration. Nevertheless, due to small magnitudes in primary surplus relative to GDP, the model consistent measure of private consumption tracks closely the dynamics of the alternative data measure of consumption,[48] and the correlation between the two measure is well beyond 0.97.

Government debt, spread, and maturity Since one of the major purposes of this paper is to provide a quantitative assessment of the Euro Area 'stressed' countries, we choose to capture the overall debt burden of those countries by calibrating the general government consolidated gross debt. Indeed, Bocola et al. (2019) argue that matching the overall public debt allows a quantitative sovereign default model to better fit crisis dynamics.

We use the nominal long-term bond yields in AMECO to measure the nominal borrowing costs of the Euro Area 'stressed' countries. For the nominal risk free rate, we use the annualized short-term (3M) interest rates in the Euro money market (obtained from EuroStat with label `irt_st_q`) for 1999–2015, and the annualized short-term (3M) bond return of Germany (obtained from EuroStat with label `irt_h_mr3_q`) for 1980–1998, before the start of Euro. To convert the nominal risk-free rate into real rate, we subtract GDP deflator of Germany from the former series. To arrive at a meaningful measure of the *real* spread, i.e., a spread unaffected by expected inflation hence rightly reflecting the 'stressed' countries' credit risk, we split the sample into two parts. After the introduction of Euro, we can directly use the spread between the 'stressed' countries' long-term nominal bond yields and the nominal risk-free rate, since all rates are denominated in euro and thus subject to the same inflation expectation. The question is much trickier for the period before Euro. Motivated by Du and Schreger (2016), we use spot and forward exchange rates (retrieved from Datastream) to convert the German nominal risk free rate into each stressed country's local currency, hence deriving a synthetic local currency risk free rate, and then take the difference between the local nominal long-term bond yield with the synthetic risk free rate. Since the synthetic risk free rate is denominated in the local currency as well, it is subject to the same inflation expectations as the long-term bond yield, and consequently, the difference is equivalent to the real spread.

[48]Indeed, the alternative measure is private absorption defined as the sum of private consumption and investment as measured in the data, since there is no capital in our model.

The information on the maturity structure of the government debt for the GIPS countries is not comprehensive. The overall time coverage is unequal across countries: 1998–2010 and 2014–2015 for Ireland, 1998–2015 for Greece, 1991–2015 for Spain, 1990–2015 for Italy, and 1995–2015 for Portugal.

C.2 More Details on the Productivity Shock Estimation

We implement the panel Markov regime switching AR(1) estimation of the productivity process following the expectation maximization approach outlined in Hamilton (1990). To overcome the local maximum problem, we randomize the initialization by 50,000 times. Apart from the parameter estimation results reported in the main text, Figure C.1 shows the smoothed probability for each regime across the GIPS countries. Evidently, regime 3 concentrates around the global financial crisis and the European debt crisis. As a last remark, we discretize the regime switching AR(1) process with 9 grid points for each regime using the method detailed in Liu (2017).

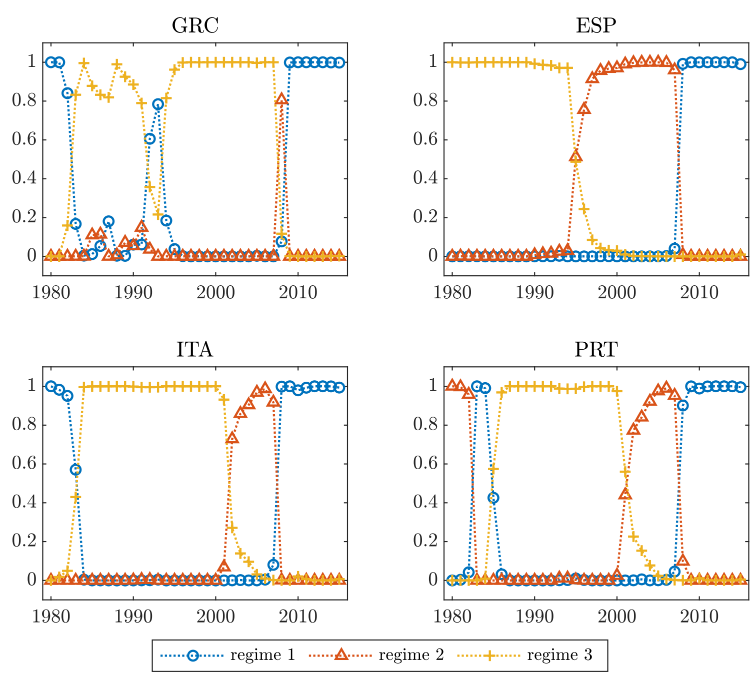

Figure C.1: Smoothed probability for each regime

C.3 The Numerical Values of the Transition Matrix for g

The parameter values imply the following transition matrices:

$$\bar{\pi}^g = \begin{bmatrix} 0.9750 & 0.0167 & 0.0083 \\ 0.0200 & 0.9750 & 0.0050 \\ 0.0100 & 0.0150 & 0.9750 \end{bmatrix}, \quad \pi^h = \begin{bmatrix} 0.95 & 0.0333 & 0.0167 \\ 0 & 0.99 & 0.01 \\ 0 & 0 & 1 \end{bmatrix}, \quad \pi^l = \begin{bmatrix} 1 & 0 & 0 \\ 0.04 & 0.96 & 0 \\ 0.02 & 0.03 & 0.95 \end{bmatrix}.$$

Note that the *average* distribution $\bar{\pi}^g$ of g constrains the possible effect of effort. Even in this 'extreme' case, the effect of effort is limited. For example by moving effort from 0 to 1 the borrower can increase the chance of reducing government expenditure from 0 to only 7% if the current expenditure is very high.

C.4 Transition Probabilities for Correlated g and θ

Based on the convenient fact that the number of regimes for θ and the number of values g can take both equal to 3, we extend the baseline conditional distribution (58) for $\pi^g(g'|g,e)$ to $\tilde{\pi}^g(g'|\varsigma, g, e)$ as follows:

$$\begin{aligned} &\tilde{\pi}^g(g' = g_j | \varsigma = i, g = g_k, e) \\ = &\, w[\zeta(e)\pi^l(j|4-i) + (1-\zeta(e))\pi^h(j|4-i)] \\ &+ (1-w)[\zeta(e)\pi^l(j|k) + (1-\zeta(e))\pi^h(j|k)], \quad i,j = 1,2,3, \end{aligned} \tag{C.1}$$

where i denotes the regime of θ, j denotes the value of future g', and k denotes the value of current g. The additional parameter $w \in [0,1]$ controls for the influence on the distribution of g' coming from regime ς of θ: if $w = 1$, then g' only depends on ς but not on g; in contrast, if $w = 0$, then g' does not depend on ς, and the transition probability is identical to the baseline specification in (58). Moreover, the index $4 - i$ in the second line suggests that when the current regime for θ is high, i.e., i is larger, then not only future θ' is high since ς is persistent, but g' also tends to be higher by the persistence inherent to the structure of π^h and π^l in (59). This feature induces positive correlation between g and θ for any w. Given $\tilde{\pi}^g$ so defined, it is straightforward to construct the overall transition matrix $\pi(s'|s,e)$ accordingly.