Stability

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Outline

- Expectations in macroeconomics and sources of business cycles
  - The role of expectations;
  - Two examples;
  - Competing theories on business cycles.
- Basic tools: difference equations and stability of nonlinear system
  - Concepts;
  - First-order difference equations;
  - Higher-order difference equations;
  - Simultaneous difference equations.
- Uhlig Toolkit
  - Solving for the recursive law of motion using the method of undetermined coefficients;
  - Solve with Toolkit 4.1.
- RBC models with sunspot equilibria
- References

Expectations in macroeconomics

The role of expectations

- Central difference between economics and natural sciences: forward-looking decisions made by economic agents;
- Expectations play a key role;
  - Examples: consumption theory; investment decisions; asset prices, etc.
- The role of expectations: they influence the time path of the economy, and the time path of the economy influences expectations.
  - Rational expectation (RE): mathematical conditional expectation of the relevant variables;
  - The expectations are conditioned on all of the information available to the decision makers

Expectations in macroeconomics

Two examples

- Example 1. Cobweb model

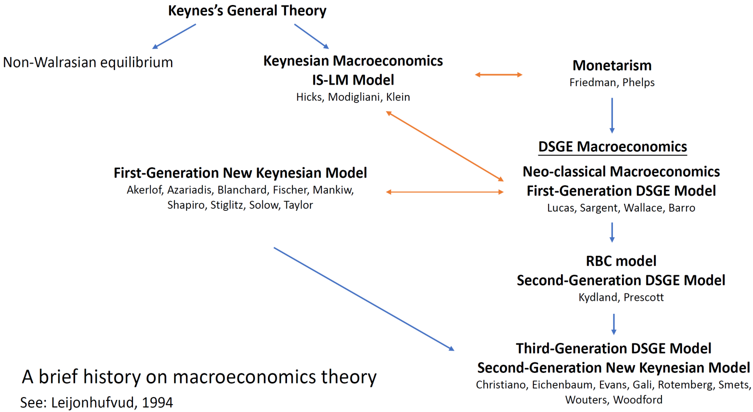
$$\begin{aligned} d_t &= m_I - m_p p_t + v_{1t}, \\ s_t &= r_I + r_p p_t^e + v_{2t}, \\ s_t &= d_t, \end{aligned}$$

  where m_I, m_p, r_I and r_p are all positive constant.
- Example 2. Cagan model

$$m_t - p_t = -\psi \left(p_{t+1}^e - p_t \right), \; \psi > 0.$$

Sources of business cycles

Some competing theories



A brief history on macroeconomics theory

See: Leijonhufvud, 1994

Sources of business cycles

News view of business cycles

- Business cycles are mainly the result of agents having incentives to continuously anticipate the economy's future demands.
  - If an agent can properly anticipate a future need...
  - If many agents adopt similar behavior...
  - However, errors are possible...
- Trace back to Pigou (1927)

The very source of fluctuations is the "*wave-like swings in the mind of the business world between errors of optimism and errors of pessimism*."

- Keynes' 1936 notion of animal spirits.
- Then what are optimism and pessimism in business cycles?
  - an entirely psychological phenomenon?
  - self-fulfilling fluctuations? The macroeconomy is inherently unstable
  - news view?

Concepts on difference equations

Definition

Discrete time: time is taken to be a discrete variable (integer number, like 1,2,3...)

Definition

First-order difference is

$$\Delta y_t = y_{t+1} - y_t,$$

where y_t is the value of y in the t^{th} period. Second-order difference is

$$\Delta^2 y_t = \Delta\left(\Delta y_t\right) = y_{t+2} - 2y_{t+1} + y_t.$$

Note 1: "period" - rather than point - of time.
Note 2: y has a unique value in each period of time.

Concepts

Definition

Difference Equation:

$$\Delta y_t = y_{t+1} - y_t = c,$$

or

$$y_{t+1} - y_t = ay_t + b.$$

Note: the choice of time subscripts is arbitrary, i.e. it does not make any difference if we write it as $y_{t+1} - y_t = c$ or as $y_{t+2} - y_{t+1} = c$.

Solving a first-order difference equation

- Iterative method
- General method

$$y_{t+1} + ay_t = c,$$

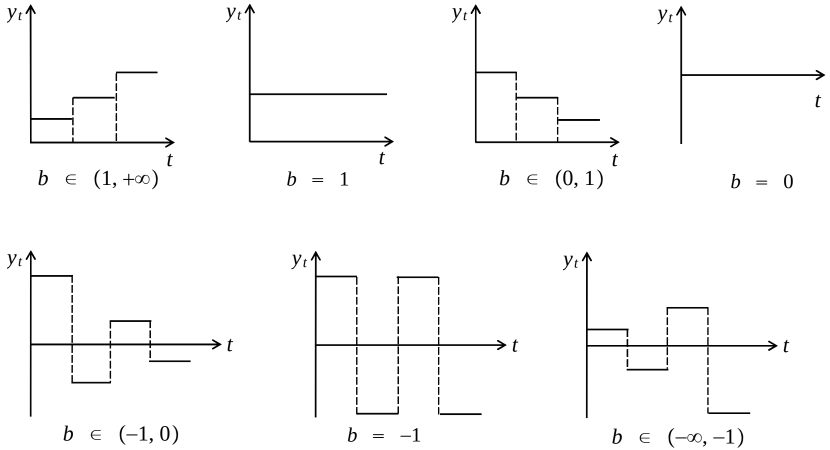
  The general solution y_t consists of y_p (particular solution) and y_c (complementary solution)
- Complementary solution: Try $y_t = Ab^t$ (A is arbitrary) and get $b = -a$.
- Particular solution: 1. if $a \neq -1$, solve $y_p = \frac{c}{1+a}$; 2. if $a = -1$, solve $y_p = ct$.
- Get y_t

$$y_t = y_c + y_p = \begin{cases} A\left(-a\right)^t + \frac{c}{1+a}, a \neq -1 \\ A + ct, a = -1 \end{cases}.$$

- How about A? - Determined by y_0.

Dynamic stability of equilibrium

- The significance of b:



$$y_t = b^t$$

- Nonoscillatory (oscillatory) if $b > (<) 0$;

Dynamic stability of equilibrium

- The role of A
  - Scale effect: magnitude
  - Mirror effect: sign
- Example: A market model with inventory

$$Q_{dt} = \alpha - \beta P_t,$$

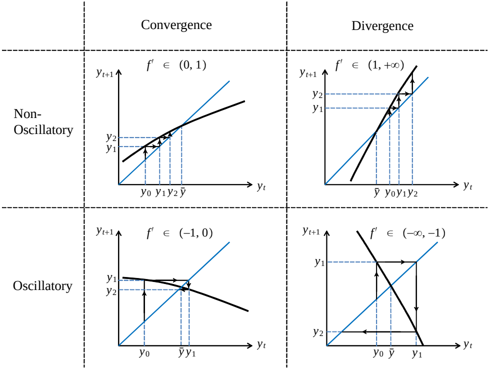
$$Q_{st} = -\gamma + \delta P_t,$$

$$P_{t+1} = P_t - \sigma \left(Q_{st} - Q_{dt} \right),$$

where $\alpha, \beta, \gamma, \delta, \sigma > 0$. Solve the time path of P_t.

Non-linear difference equations: qualitative-graphic approach

$$y_{t+1} = f\left(y_t\right).$$



Stability of Non-Linear System

Higher-order difference equations

- Second-order linear difference equations with constant coefficients and constant term

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = c.$$

- Particular solution y_p: the intertemporal equilibrium level of y.
  - If $a_1 + a_2 \neq -1$, try $y_p = k$ and get $k = \frac{c}{1+a_1+a_2}$;
  - If $a_1 + a_2 = -1$ and $a_1 \neq -2$, try $y_p = kt$ and get $k = \frac{c}{2+a_1}$;
  - If $a_1 + a_2 = -1$ and $a_1 = -2$, try $y_p = kt^2$ and get $k = \frac{c}{2}$.
- Complementary solution: the deviation from the equilibrium for every time period

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = 0.$$

Try $y_t = Ab^t$ and get the characteristic equation

$$b^2 + a_1 b + a_2 = 0.$$

Higher-order difference equations

- Case 1: two distinct real roots: $a_1^2 > 4a_2$

$$y_c = A_1 b_1^t + A_2 b_2^t.$$

- Case 2: repeated real roots: $a_1^2 = 4a_2$ and thus $b = b_1 = b_2 = -\frac{a_1}{2}$

$$y_c = A_3 b^t + A_4 t b^t.$$

- Case 3: complex roots: $a_1^2 < 4a_2$

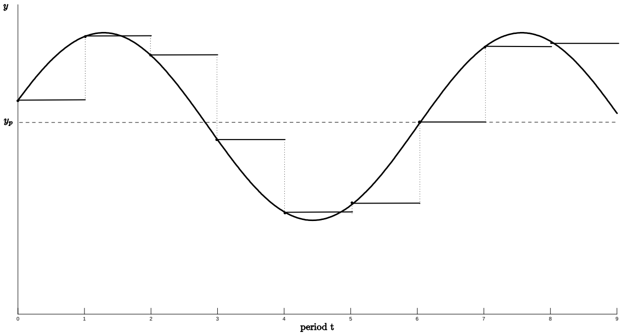
$$b_{1,2} = h \pm vi, \text{ where } h = -\frac{a_1}{2} \text{ and } v = \frac{\sqrt{4a_2 - a_1^2}}{2}.$$

$$y_c = A_1 b_1^t + A_2 b_2^t = A_1 (h + vi)^t + A_2 (h - vi)^t.$$

Higher-order difference equations

- Because $(h \pm vi)^t = R^t(\cos\theta t \pm i \sin\theta t)$ where $R = \sqrt{h^2 + v^2} = \sqrt{a_2}$, $\cos\theta = \frac{h}{R} = -\frac{a_1}{2\sqrt{a_2}}$ and $\sin\theta = \frac{v}{R} = \sqrt{1 - \frac{a_1^2}{4a_2}}$, we have

$$y_c = R^t (A_5 \cos\theta t + A_6 \sin\theta t).$$



Convergence of the time path

- Distinct roots
  - Dominant root: the root with the higher absolute value
  - A time path will be convergent iff the dominant root is less than 1 in absolute value
- Repeated roots: if $|b| < 1$, we have convergence
- Complex roots:
  - if $R < 1$, i.e. $|b| < 1$, we have damped stepped fluctuation
  - if $R > 1$, i.e. $|b| > 1$, we have explosive stepped fluctuation

Simultaneous difference equations

- Relation between higher-order difference equation and simultaneous difference equations

$$y_{t+2} + a_1 y_{t+1} + a_2 y_t = c.$$

Define $x_t = y_{t+1}$. We will have

$$\begin{aligned} x_{t+1} + a_1 x_t + a_2 y_t &= c, \\ y_{t+1} &= x_t. \end{aligned}$$

In matrix form, it is

$$\begin{pmatrix} x_{t+1} \\ y_{t+1} \end{pmatrix} = \begin{pmatrix} -a_1 & -a_2 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_t \\ y_t \end{pmatrix} + \begin{pmatrix} c \\ 0 \end{pmatrix}.$$

- Simultaneous difference equations

$$X_{t+1} = AX_t + b,$$

where $X_t = (x_{1t}, x_{2t}, ...x_{nt})'$, A is a constant matrix with coefficients a_{ij}, $i, j = 1...n$, and $b = (b_1, b_2, ...b_n)'$.

Simultaneous difference equations

- Particular solution: $x_{t+1} = x_t = x$ and $y_{t+1} = y_t = y$;
- Complementary solution: substituting $x_t = mb^t$ and $y_t = nb^t$, we have the characteristic equation and the characteristic roots b_1 and b_2.
- Characteristic equation

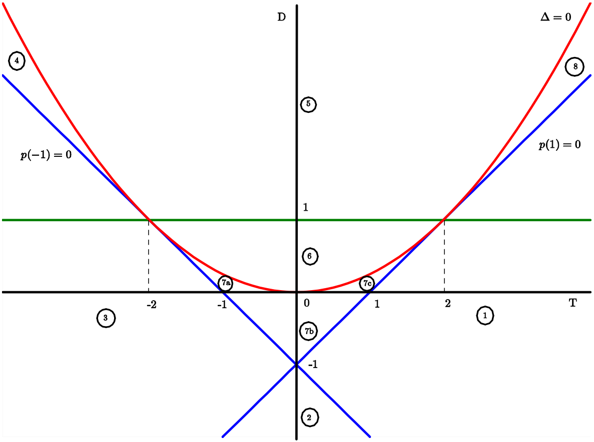
$$p(b) = |A - bI| = b^2 - \mathcal{T}b + \mathcal{D} = 0,$$

where $\mathcal{T}$ and $\mathcal{D}$ are the trace and determinant of matrix A.
- Moreover, we know that

$$\mathcal{T} = b_1 + b_2, \;\; \mathcal{D} = b_1 b_2,$$

$$p(b) = (b - b_1)(b - b_2).$$

Stability property



Stability Triangle

Stability property

- Red line: $\Delta = \mathcal{T}^2 - 4\mathcal{D} = 0$
  - Region above the red line: $\Delta < 0 \Rightarrow$ complex roots;
  - Region below the red line: $\Delta > 0 \Rightarrow$ real roots;
- Blue lines: $p(1) = 1 - \mathcal{T} + \mathcal{D} = 0$ and $p(-1) = 1 + \mathcal{T} + \mathcal{D} = 0$
  - Region above (below) the right blue line: $p(1) > (<) 0$;
  - Region above (below) the left blue line: $p(-1) > (<) 0$;
- Green line: $D = 1$
  - Region above the green line: $|b| > 1$;
  - Region below the green line: $|b| < 1$.

Stability property

Region	$p(b_i)$	b_i	Stability
1	$p(1) < 0,\ p(-1) > 0$	$\lvert b_1 \rvert < 1,\ b_2 > 1$	saddle
2	$p(1) < 0,\ p(-1) < 0$	$b_1 b_2 < 0,\ \lvert b_i \rvert > 1$	explosive
3	$p(1) > 0,\ p(-1) < 0$	$\lvert b_1 \rvert < 1,\ b_2 < -1$	saddle
4	$p(1) > 0,\ p(-1) > 0$ $\mathcal{D} > 1,\ \mathcal{T} < -2$	$b_i < -1$	explosive
5	$\Delta < 0,\ \mathcal{D} > 1$	$\lvert b_i \rvert > 1$	explosive
6	$\Delta < 0,\ \mathcal{D} < 1$	$\lvert b_i \rvert < 1$	stable
7	$p(1) > 0,\ p(-1) > 0$ $\mathcal{D} < 1$	$\lvert b_i \rvert < 1$	stable
8	$p(1) > 0,\ p(-1) > 0$ $\mathcal{D} > 1,\ \mathcal{T} > 2$	$b_i > 1$	explosive

Stability property

- Conditions for a saddle: $p\left(1\right) < 0$, $p\left(-1\right) > 0$ or $p\left(1\right) > 0$, $p\left(-1\right) < 0$

$$\begin{aligned} 1 - \mathcal{T} + \mathcal{D} &< 0 \text{ and } 1 + \mathcal{T} + \mathcal{D} > 0; \text{ or} \\ 1 - \mathcal{T} + \mathcal{D} &> 0 \text{ and } 1 + \mathcal{T} + \mathcal{D} < 0. \end{aligned}$$

  Or equivalent as

$$\left|\mathcal{T}\right| > \left|1 + \mathcal{D}\right|.$$

- Conditions for two stable roots: $\Delta < 0$, $\mathcal{D} < 1$ or $p\left(1\right) > 0$, $p\left(-1\right) > 0$, $\mathcal{D} < 1$

$$\mathcal{D} < 1,\ 1 - \mathcal{T} + \mathcal{D} > 0 \text{ and } 1 + \mathcal{T} + \mathcal{D} > 0,$$

  i.e.

$$\mathcal{D} < 1 \text{ and } \left|\mathcal{T}\right| < 1 + \mathcal{D}.$$

Stability property

Local Uniqueness/Multiplicity

Definition

Predetermined variable: the variable whose initial value is given, as k, h, and b;

Definition

Jump variable: the variable whose initial value is not given, as c, l, and p (sometimes).

Theorem

Conditions for local uniqueness/multiplicity:
1. If the number of stable roots = the number of predetermined variables $\Rightarrow$ Saddle path (Determinacy);
2. If the number of stable roots $<$ the number of predetermined variables $\Rightarrow$ Source (Explosive);
3. If the number of stable roots $>$ the number of predetermined variables $\Rightarrow$ Sink (Indeterminacy).

Solve for the recursive law of motion with method of undetermined coefficients.

State variables: $\hat{k}_{t-1}$, $\hat{z}_t$

The dynamics of the model should be described by **recursive laws of motion** in terms of the state variables,

$$\begin{aligned}\hat{k}_t &= v_{kk}\hat{k}_{t-1} + v_{kz}\hat{z}_t,\\ \hat{c}_t &= v_{ck}\hat{k}_{t-1} + v_{cz}\hat{z}_t.\end{aligned}$$

We need to solve for v_{kk}, v_{kz}, v_{ck} and v_{cz}, the "undetermined" coefficients.

Coefficient interpretation: *elasticities*.

Recall: the log-linearized system consists of

$$\hat{k}_t = \frac{1}{\beta}\hat{k}_{t-1} - \frac{\overline{C}}{\bar{K}}\hat{c}_t + \frac{\widetilde{\delta}}{\alpha\beta}\hat{z}_t,$$

$$\sigma \mathrm{E}_t \hat{c}_{t+1} + \widetilde{\delta}\left(1-\alpha\right)\hat{k}_t - \widetilde{\delta}\mathrm{E}_t\hat{z}_{t+1} = \sigma\hat{c}_t,$$

$$\hat{z}_t = \psi\hat{z}_{t-1} + \varepsilon_t.$$

Recursivity

- Substitute the postulated linear recursive law of motion into the dynamic equations until only $\hat{k}_{t-1}$ and $\hat{z}_t$ remain. E.g.

$$\mathrm{E}_t \hat{z}_{t+1} = \psi \hat{z}_t,$$

$$\begin{aligned} \mathrm{E}_t \hat{c}_{t+1} &= \mathrm{E}_t \left(v_{ck} \hat{k}_t + v_{cz} \hat{z}_{t+1} \right) \\ &= v_{ck} \left(v_{kk} \hat{k}_{t-1} + v_{kz} \hat{z}_t \right) + v_{cz} \psi \hat{z}_t \\ &= v_{ck} v_{kk} \hat{k}_{t-1} + \left(v_{ck} v_{kz} + v_{cz} \psi \right) \hat{z}_t . \end{aligned}$$

- Compare coefficients.

Recursivity

For the first equation (budget constraint)

$$\begin{aligned}
\hat{k}_t &= \frac{1}{\beta}\hat{k}_{t-1} - \frac{\bar{C}}{\bar{K}}\hat{c}_t + \frac{\widetilde{\delta}}{\alpha\beta}\hat{z}_t, \\
v_{kk}\hat{k}_{t-1} + v_{kz}\hat{z}_t &= \frac{1}{\beta}\hat{k}_{t-1} - \frac{\bar{C}}{\bar{K}}\left(v_{ck}\hat{k}_{t-1} + v_{cz}\hat{z}_t\right) + \frac{\widetilde{\delta}}{\alpha\beta}\hat{z}_t,
\end{aligned}$$

$$\left(\frac{1}{\beta} - \frac{\bar{C}}{\bar{K}}v_{ck} - v_{kk}\right)\hat{k}_{t-1} + \left(\frac{\widetilde{\delta}}{\alpha\beta} - \frac{\bar{C}}{\bar{K}}v_{cz} - v_{kz}\right)\hat{z}_t = 0.$$

Comparing coefficients: since the equation has to be satisfied for any value of $\hat{k}_{t-1}$ and $\hat{z}_t$, we have

$$\begin{aligned}
\text{for } \hat{k}_{t-1} &: \qquad v_{kk} = \frac{1}{\beta} - \frac{\bar{C}}{\bar{K}}v_{ck} \\
\text{for } \hat{z}_t &: \qquad v_{kz} = \frac{\widetilde{\delta}}{\alpha\beta} - \frac{\bar{C}}{\bar{K}}v_{cz}
\end{aligned}$$

Recursivity

For the second equation (Euler equation/asset pricing)

$$\sigma \mathrm{E}_t \hat{c}_{t+1} + \widetilde{\delta} \left(1 - \alpha\right) \mathrm{E}_t \hat{k}_t - \widetilde{\delta} \mathrm{E}_t \hat{z}_{t+1} = \sigma \hat{c}_t,$$

$$\begin{aligned} & \sigma \left[v_{ck} v_{kk} \hat{k}_{t-1} + \left(v_{ck} v_{kz} + v_{cz} \psi \right) \hat{z}_t \right] + \widetilde{\delta} \left(1 - \alpha\right) \left(v_{kk} \hat{k}_{t-1} + v_{kz} \hat{z}_t \right) - \\ = \; & \sigma \left(v_{ck} \hat{k}_{t-1} + v_{cz} \hat{z}_t \right), \end{aligned}$$

$$\begin{aligned} & \left[\sigma v_{ck} v_{kk} + \widetilde{\delta} \left(1 - \alpha\right) v_{kk} - \sigma v_{ck} \right] \hat{k}_{t-1} + \\ & \left[\sigma \left(v_{ck} v_{kz} + v_{cz} \psi \right) + \widetilde{\delta} \left(1 - \alpha\right) v_{kz} - \widetilde{\delta} \psi - \sigma v_{cz} \right] \hat{z}_t \\ = \; & 0. \end{aligned}$$

Comparing coefficients, we have

$$\begin{aligned} \text{for } \hat{k}_{t-1} \quad &: \quad \sigma v_{ck} \left(1 - v_{kk}\right) = \widetilde{\delta} \left(1 - \alpha\right) v_{kk}, \\ \text{for } \hat{z}_t \quad &: \quad \sigma v_{cz} \left(1 - \psi\right) = \left[\sigma v_{ck} + \widetilde{\delta} \left(1 - \alpha\right) \right] v_{kz} - \widetilde{\delta} \psi. \end{aligned}$$

Comparing coefficients

Collecting the results, and comparing coefficients on $\hat{k}_{t-1}$,

$$\sigma v_{ck}\left(1-v_{kk}\right)=\widetilde{\delta}\left(1-\alpha\right)v_{kk}, \tag{1}$$

$$v_{kk}=\frac{1}{\beta}-\frac{\bar{C}}{\bar{K}}v_{ck}. \tag{2}$$

To solve v_{kk}, we substitute v_{ck} and obtain a quadratic equation

$$v_{kk}^{2}-\left[1+\frac{1}{\beta}-\frac{\widetilde{\delta}\left(1-\alpha\right)}{\sigma}\frac{\bar{C}}{\bar{K}}\right]v_{kk}+\frac{1}{\beta}=0.$$

Solving the quadratic equation

$$0 = v_{kk}^2 - \gamma v_{kk} + \frac{1}{\beta},$$

where

$$\gamma = 1 + \frac{1}{\beta} + \frac{\widetilde{\delta}\,(1-\alpha)}{\sigma}\frac{1 - [1 - (1-\alpha)\,\delta]\,\beta}{\alpha\beta}.$$

The solution is a high school math problem:

$$v_{kk} = \frac{\gamma}{2} - \sqrt{\left(\frac{\gamma}{2}\right)^2 - \frac{1}{\beta}}.$$

Why we delete the root

$$\frac{\gamma}{2} + \sqrt{\left(\frac{\gamma}{2}\right)^2 - \frac{1}{\beta}}.\ ?$$

Solving the quadratic equation

$$\begin{aligned} (v_{kk} - \lambda_1)(v_{kk} - \lambda_2) &= 0, \\ v_{kk}^2 - (\lambda_1 + \lambda_2)v_{kk} + \lambda_1\lambda_2 &= 0, \end{aligned}$$

$$\begin{aligned} \lambda_1\lambda_2 &= \frac{1}{\beta} > 1, \\ \lambda_1 + \lambda_2 &= \gamma > 1 + \frac{1}{\beta}, \\ \text{hence, } \lambda_1 + \lambda_2 &> 1 + \lambda_1\lambda_2, \\ (1 - \lambda_1)(\lambda_2 - 1) &> 0. \end{aligned}$$

so λ_1 and λ_2 both positive, one root > 1, and the other root < 1.
We need to delete the explosive solution!

Solving the quadratic equation

Once v_{kk} is solved, the others can be solved easily.
Plugging it into equation (2),

$$v_{ck} = \left(\frac{\widetilde{\delta}}{\alpha\beta} - \delta \right) \left(\frac{1}{\beta} - v_{kk} \right),$$

we get v_{ck}.

Solving the quadratic equation

Then for coefficients on $\hat{z}_t$

$$v_{kz} = \left(\delta - \frac{\widetilde{\delta}}{\alpha\beta}\right) v_{cz} + \frac{\widetilde{\delta}}{\alpha\beta}, \tag{3}$$

$$\sigma v_{cz}\left(1-\psi\right) = \left[\sigma v_{ck} + \widetilde{\delta}\left(1-\alpha\right)\right] v_{kz} - \widetilde{\delta}\psi, \tag{4}$$

i.e.

$$v_{cz} = \frac{\sigma v_{ck} + \widetilde{\delta}\left(1-\alpha\right) - \alpha\beta\psi}{\sigma\left(1-\psi\right) - \left[\sigma v_{ck} + \widetilde{\delta}\left(1-\alpha\right)\right]\left(\delta - \frac{\widetilde{\delta}}{\alpha\beta}\right)} \frac{\widetilde{\delta}}{\alpha\beta},$$

where v_{ck} is known. Substituting v_{cz} into (3), we solve v_{kz}.

Solving the quadratic equation

We could obtain

$$
\begin{aligned}
\hat{k}_t &= v_{kk}\hat{k}_{t-1} + v_{kz}\hat{z}_t, \\
\hat{c}_t &= v_{ck}\hat{k}_{t-1} + v_{cz}\hat{z}_t, \\
\hat{z}_t &= \psi\hat{z}_{t-1} + \varepsilon_t,
\end{aligned}
$$

where

$$
v_{kk} = \frac{\gamma}{2} - \sqrt{\left(\frac{\gamma}{2}\right)^2 - \frac{1}{\beta}}, \; v_{ck} = \left(\frac{\widetilde{\delta}}{\alpha\beta} - \delta\right)\left(\frac{1}{\beta} - v_{kk}\right),
$$

$$
v_{cz} = \frac{\sigma v_{ck} + \widetilde{\delta}(1-\alpha) - \alpha\beta\psi}{\sigma(1-\psi) - \left[\sigma v_{ck} + \widetilde{\delta}(1-\alpha)\right]\left(\delta - \frac{\widetilde{\delta}}{\alpha\beta}\right)}\frac{\widetilde{\delta}}{\alpha\beta},
$$

$$
v_{kz} = \left(\delta - \frac{\widetilde{\delta}}{\alpha\beta}\right)v_{cz} + \frac{\widetilde{\delta}}{\alpha\beta}.
$$

Calibration and simulation

Assuming quarterly data with

$$\begin{aligned} \beta &= 0.99 & \alpha &= 0.36 \\ \sigma &= 1.0 & \delta &= 0.025 \\ \bar{Z} &= 1 & \psi &= 0.95 \end{aligned}$$

Then we get...

Calibration and simulation

- *Impulse response* analysis: trace out all variables for $\varepsilon_1 = 1$, $\varepsilon_t = 0$ for $t > 1$, when starting from the steady state.
- Because

$$\hat{z}_t = \psi \hat{z}_{t-1} + \varepsilon_t,$$

we have

$$\hat{z}_1 = \psi \hat{z}_0 + \varepsilon_1 = \varepsilon_1,$$

$$\hat{z}_2 = \psi \hat{z}_1 + \varepsilon_2 = \psi \varepsilon_1,$$

$$\hat{z}_j = \psi^{j-1} \varepsilon_1.$$

and

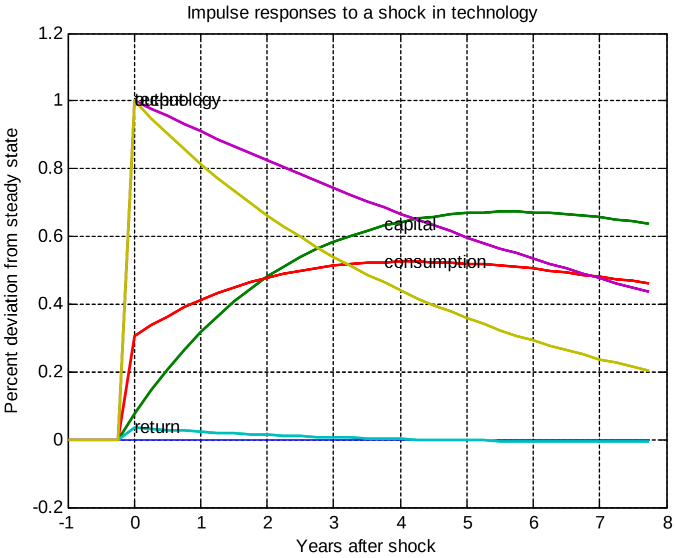
$$\hat{k}_1 = v_{kk} \hat{k}_0 + v_{kz} \hat{z}_1 = v_{kz} \varepsilon_1,$$

$$\hat{k}_2 = v_{kk} \hat{k}_1 + v_{kz} \hat{z}_2 = v_{kk} v_{kz} \varepsilon_1 + v_{kz} \psi \varepsilon_1 = \left(v_{kk} + \psi \right) v_{kz} \varepsilon_1,$$

$$\hat{k}_j = v_{kk} \hat{k}_{j-1} + v_{kz} \psi^{j-1} \varepsilon_1 = \sum_{i=0}^{j-1} \left(v_{kk}^i \psi^{j-i-1} \right) v_{kz} \varepsilon_1.$$

Calibration and simulation

Impulse response functions



Calibration and simulation

Baseline model vs.

$$\sigma = 100$$

Does not change steady states.
1) $v_{kk} \uparrow$ due to risk aversion, consumption smoothing, lower intertemporal elasticity of substitution
2) $v_{kz} \downarrow$ less sensitive to technology shock to better smooth consumption

Calibration and simulation

Baseline model vs.

$$\delta = 0.1$$

Reduce steady state size of the economy dramatically due to higher depreciation rate.

	$\bar{K}$	$\bar{Y}$	$\bar{C}$
$\delta = 0.025$	38	3.7	2.75
$\delta = 0.1$	6.4	1.95	1.3

1) $v_{kk} \downarrow$ due to higher depreciation rate
2) $v_{kz} \uparrow$ due to less stock of capital and higher MPK. So return and output both respond more proportionally.

Solve with Toolkit 4.1

The structure of the problem.

There is an $m \times 1$ endogenous state vector x_t, an $n \times 1$ vector of other endogenous variables y_t, and a $k \times 1$ vector of exogenous stochastic processes z_t. The equilibrium relationships between these variables are fully characterized by the list of equations we just collected after log-linearization. We can cast these equations into three blocks:

$$\begin{aligned} 0 &= Ax_t + Bx_{t-1} + Cy_t + Dz_t \\ 0 &= E_t[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t] \\ z_t &= Nz_{t-1} + \epsilon_{t+1}; \quad E_t[\epsilon_{t+1}] = 0 \end{aligned}$$

where C is of size $l \times n$, $l \geq n$ and of rank n, K is of size $(m + n - l) \times n$, and N has only stable eigenvalues. In total there are $m + n + k$ equations.

Solve with Toolkit 4.1

Recursive law of motion

$$
\begin{aligned}
x_t &= Px_{t-1} + Qz_t \\
y_t &= Rx_{t-1} + Sz_t \\
z_t &= Nz_{t-1} + \epsilon_t
\end{aligned}
$$

Solutions: matrix quadratic equation.

Execute with *Toolkit 4.1*.

Solve with Toolkit 4.1

Cast the log-linearized equations into the system.

An $m \times 1$ endogenous state vector x_t, $\{\hat{k}_t\}$.
An $n \times 1$ vector of other endogenous variables y_t, $\{\hat{c}_t, \hat{r}_t, \hat{y}_t\}$.
A $k \times 1$ vector of exogenous stochastic processes z_t: $\{\hat{z}_t\}$.

1. $\hat{r}_t = [1 - \beta(1-\delta)] \left[\hat{z}_t - (1-\alpha)\hat{k}_{t-1}\right]$
2. $\hat{c}_t = \frac{\bar{Y}}{\bar{C}}\hat{z}_t + \frac{\bar{K}}{\bar{C}}\bar{R}\hat{k}_{t-1} - \frac{\bar{K}}{\bar{C}}\hat{k}_t$
3. $\hat{y}_t = \hat{z}_t + \alpha\hat{k}_{t-1}$
4. $0 = E_t\left[\sigma(\hat{c}_t - \hat{c}_{t+1}) + \hat{r}_{t+1}\right]$
5. $\hat{z}_t = \psi\hat{z}_{t-1} + \varepsilon_t$

Solve with Toolkit 4.1

Cast these equations into three blocks

1) The first block

$$0 = Ax_t + Bx_{t-1} + Cy_t + Dz_t$$

$$\begin{aligned}
0 &= [1-\beta(1-\delta)]\,\hat{z}_t - [1-\beta(1-\delta)]\,(1-\alpha)\hat{k}_{t-1} - \hat{r}_t \\
0 &= \frac{\bar{Y}}{\bar{C}}\hat{z}_t + \frac{\bar{K}}{\bar{C}}\bar{R}\hat{k}_{t-1} - \frac{\bar{K}}{\bar{C}}\hat{k}_t - \hat{c}_t \\
0 &= \hat{z}_t + \alpha\hat{k}_{t-1} - \hat{y}_t
\end{aligned}$$

2) The second block

$$\begin{aligned}
0 &= E_t[Fx_{t+1} + Gx_t + Hx_{t-1} + Jy_{t+1} + Ky_t + Lz_{t+1} + Mz_t] \\
0 &= E_t\left[\sigma(\hat{c}_t - \hat{c}_{t+1}) + \hat{r}_{t+1}\right]
\end{aligned}$$

3) The third block $z_t = Nz_{t-1} + \epsilon_{t+1};\;\; E_t[\epsilon_{t+1}] = 0$

$$\hat{z}_t = \psi\hat{z}_{t-1} + \varepsilon_t$$

RBC models with sunspot equilibria

- Factor-generated externalities (Farmer and Guo, 1994, JET; Wen, 1998, JME)

$$Y_t = Z_t K_{t-1}^{1-\alpha} N_t^{\alpha} \text{ and } Z_t = z_t \overline{K}_{t-1}^{\alpha\eta} \overline{N}_t^{(1-\alpha)\eta},$$

  where $\overline{K}_{t-1}$ and N_t are the social average level of capital and labor inputs.
- Taxation (Schmitt-Grohe and Uribe, 1997, JPE)

$$K_t = (1 - \tau_t) w_t N_t + r_t K_t + (1 - \delta) K_{t-1} - C_t,$$

  and the government holds balanced-budget rule as

$$G = \tau_t w_t N_t,$$

  where G is a constant.
- More references see Benhabib and Farmer (1999).

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