Computational Methods of Heterogeneous Agent Models
Dynamics of the Distribution Function

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Contents

Introduction

- Households

$$E_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t\right), \left(c_t\right) = \frac{c_t^{1-\eta}}{1-\eta}, \quad \eta > 0 \tag{1}$$

$$\pi\left(\epsilon' \mid \epsilon\right) = \operatorname{Prob}\left\{\epsilon_{t+1} = \epsilon' \mid \epsilon_t = \epsilon\right\} = \begin{pmatrix} p_{uu} & p_{ue} \\ p_{eu} & p_{ee} \end{pmatrix} \tag{2}$$

$$a_{t+1} = \begin{cases} \left(1+(1-\tau) r_t\right) a_t + (1-\tau) w_t - c_t & \text{if } \epsilon = e \\ \left(1+(1-\tau) r_t\right) a_t + b_t - c_t & \text{if } \epsilon = u \end{cases} \tag{3}$$

$$K_t = \sum_{\epsilon_t \in \{e,u\}} \int_{a_{\min}}^{\infty} a_t f_t\left(\epsilon_t, a_t\right) d a_t$$

$$F_{t+1}\left(\epsilon_{t+1}, a_{t+1}\right) = \sum_{\epsilon_t \in \{e,u\}} \pi\left(\epsilon_{t+1} \mid \epsilon_t\right) F_t\left(\epsilon_t, a_{t+1}^{-1}\left(\epsilon_t, a_{t+1}\right)\right) \equiv G\left(F_t\right)$$

Introduction

$$w_t = w\left(K_t, N_t\right)$$
$$r_t = r\left(K_t, N_t\right)$$
$$u'\left(c_t\right) = \beta E_t\left[u'\left(c_{t+1}\right)\left(1+\left(1-\tau_{t+1}\right) r_{t+1}\right)\right]$$

- The solution consists of a time-invariant function $a'(\epsilon, a, F)$ which gives him the optimal next-period capital stock
- $$V(\epsilon, a, F) = \max_{c, a'}\left[u(c) + \beta E\left\{V\left(\epsilon', a', F'\right) \mid \epsilon, F\right\}\right]$$

  subject to the budget constraint, the government policy $\{b, \tau\}$, the stochastic process of the employment status ϵ, and the distribution dynamics

Transition Dynamics: Example

$$V(\epsilon, a, F)=\max_{c, a^{\prime}}\left[\frac{c_{t}^{1-\eta}}{1-\eta}+\beta E\left\{V\left(\epsilon^{\prime}, a^{\prime}, F^{\prime}\right) \mid \epsilon, F\right\}\right]$$

s.t.

$$a^{\prime}=\left\{\begin{array}{ll}(1+(1-\tau) r) a+(1-\tau) w-c, & \text { if } \epsilon=e \\ (1+(1-\tau) r) a+b-c, & \text { if } \epsilon=u\end{array}\right.$$

$a \geq a_{\min }$

$$\pi\left(\epsilon^{\prime} \mid \epsilon\right)=\operatorname{Prob}\left\{\epsilon_{t+1}=\epsilon^{\prime} \mid \epsilon_{t}=\epsilon\right\}=\left(\begin{array}{ll}p_{u u} & p_{u e} \\ p_{e u} & p_{e e}\end{array}\right)$$

The distribution F of (ϵ, a) is described by the following dynamics:

$$F^{\prime}\left(\epsilon^{\prime}, a^{\prime}\right)=\sum_{\epsilon \in\{e, u\}} \pi\left(\epsilon^{\prime} \mid \epsilon\right) F\left(\epsilon, a^{-1}\left(\epsilon, a^{\prime}, F\right)\right)$$

Transition Dynamics: Example

Factor prices are equal to their respective marginal products:

$$r = \alpha \left(\frac{N}{K}\right)^{1-\alpha} - \delta$$

$$w = (1-\alpha)\left(\frac{K}{N}\right)^{\alpha}$$

The aggregate consistency conditions hold:

$$K = \sum_{\epsilon \in \{e,u\}} \int_{a_{\min}}^{\infty} a f(\epsilon, a) da$$

$$C = \sum_{\epsilon \in \{e,u\}} \int_{a_{\min}}^{\infty} c f(\epsilon, a) da$$

$$T = \tau(wN + rK)$$

$$B = \int_{a_{\min}}^{\infty} b f(u, a) da$$

Transition Dynamics

- Two ways in order to approximate the dynamics of the distribution
- Krusell and Smith (1998): partial information
- shooting method

Partial Information

- We assume that agents only use partial information in order to predict the law of motion for the state variable(s) or, equivalently, are boundedly rational.
- Agents perceive the dynamics of the distribution $F' = G(F)$ in a simplified way. In particular, they characterize the distribution F by I statistics $m = (m_1, \ldots, m_I)$.
- $$m' = H_I(m)$$

$$V(\epsilon, a, m) = \max_{c,a'} \left[u(c) + \beta E\left\{V\left(\epsilon', a', m'\right) \mid \epsilon, m\right\}\right]$$

Partial Information

- $$T = \tau N^{1-\alpha} K^{\alpha} = B = (1-N)b$$
  $$b = \zeta(1-\tau)w = \zeta(1-\tau)(1-\alpha)\left(\frac{K}{N}\right)^{-\alpha}$$
- We will choose a simple parameterized functional form for $H_I(m)$ following KRUSELL and SMITH (1998)

$$\ln K' = \gamma_0 + \gamma_1 \ln K$$

Algorithm 1: Transition Dynamics with Bounded Rationality

Step 1: Choose the initial distribution of assets F_0 with mean K_0.
Step 2: Choose the order I of moments m.
Step 3: Guess a parameterized functional form for H_I and choose initial parameters of H_I
Step 4: Solve the consumer's optimization problem and compute $v(\epsilon, a, m)$
Step 5: Simulate the dynamics of the distribution.
Step 6: Use the time path for the distribution to estimate the law of motion for the moments m.
Step 7: Iterate until the parameters of H_I converge.
Step 8: Test the goodness of fit for H_I. If the fit is satisfactory, stop, otherwise increase I or choose a different functional form for H_I.

Algorithm 1

- In the first step, we assume that at time period $t = 0$, the distribution is uniform over an interval approximately equal to $[-2, 300]$.

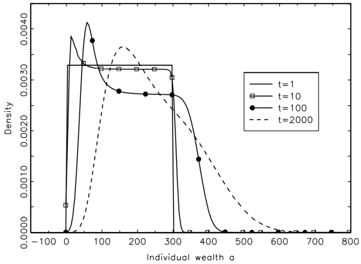

Figure 8.3: Dynamics of the Distribution Function over Time

- We can use the time path of the capital stock to update the coefficients γ_0 and γ_1 (step 6 and 7).

Algorithm 1

- $$N = \int_{a_{\min}}^{\infty} n(a; K, N) f(e, a) da$$

  In this case, individual labor supply depends on individual wealth a and, consequently, aggregate labor supply N depends on the distribution of wealth. In this case, we also need to estimate a prediction function for aggregate labor

  $$N' = J(N, K)$$

  that, for example, might take the log-linear form $\ln N' = \psi_0 + \psi_1 \ln N + \psi_2 \ln K$. The household maximizes intertemporal utility subject to the additional constraint and the value function $v(\epsilon, a, K, N)$ has the aggregate labor N as an additional argument.

Guessing a Finite Time Path for the Factor Prices

- In this section, we introduce another method for the computation of the transition path that only considers the individual variables as arguments of the value function (or policy functions).

Guessing a Finite Time Path for the Factor Prices

Step 1: Choose the number of transition periods T.
Step 2: Compute the stationary distribution $\tilde{F}$ of the new stationary equilibrium. Initialize the first-period distribution function F^1.
Step 3: Guess a time path for the factor prices r and w, unemployment compensation b, and the income tax rate τ that balances the budget. The values of these variables in both periods $t = 1$ and $t = T$ are implied by the initial and stationary distribution, respectively.
Step 4: Compute the optimal decision functions using the guess for the interest rate r, the wage income w, the tax rate τ and the unemployment compensation b. Iterate backwards in time, $t = T - 1, \ldots, 1$

Guessing a Finite Time Path for the Factor Prices

Step 5: Simulate the dynamics of the distribution with the help of the optimal policy functions and the initial distribution for the transition from $t = 1$ to $t = T$.
Step 6: Compute the time path for the interest rate r, the wage w, unemployment compensation b, and the income tax rate τ, and return to step 3, if necessary.
Step 7: Compare the simulated distribution F^T with the stationary distribution function $\tilde{F}$. If the goodness of fit is poor, increase the number of transition periods T.

Guessing a Finite Time Path for the Factor Prices

- In step 4, we compute the optimal policy functions by backward iteration. In period T, we know the new stationary distribution, optimal policy functions, and the factor prices. For periods $t = T-1, \ldots, 1$, we may compute the policy functions $c_t\left(\epsilon_t, a_t\right)$ and $a_{t+1}\left(\epsilon_t, a_t\right)$, for given policy functions $c_{t+1}\left(\epsilon_{t+1}, a_{t+1}\right)$ and $a_{t+2}\left(\epsilon_{t+1}, a_{t+1}\right)$ from the Euler equation with the help of projection methods:

$$\frac{u'(c_t(\epsilon_t,a_t))}{\beta} = E_t\left[u'\left(c_{t+1}\left(\epsilon_{t+1}, a_{t+1}\right)\right)\left(1+\left(1-\tau_{t+1}\right)r_{t+1}\right)\right]$$
$$\epsilon_t = e, u$$

where $c_t\left(e, a_t\right) = \left(1+r_t\left(1-\tau_t\right)\right)a_t + \left(1-\tau_t\right)w_t - a_{t+1}\left(e, a_t\right)$ and $c_t\left(u, a_t\right) = \left(1+r_t\left(1-\tau_t\right)\right)a_t + b_t - a_{t+1}\left(u, a_t\right)$ for the employed and unemployed worker, respectively.

Aggregate Uncertainty

- Aggregate risk is introduced by a stochastic technology level Z_t in period t. In particular, the productivity shock follows a Markov process with transition matrix $\Gamma_Z\left(Z' \mid Z\right)$, where Z' denotes next-period technology level and $\pi_{ZZ'}$ denotes the transition probability from state Z to Z'.

Example

$$V(\epsilon, a, Z, F) = \max_{c,a'} \left[\frac{c_t^{1-\eta}}{1-\eta} + \beta E \left\{ V\left(\epsilon', a', Z', F'\right) \mid \epsilon, Z, F \right\} \right]$$

s.t.

$$a' = \begin{cases} (1+(1-\tau)r)a + (1-\tau)w - c & \text{if } \epsilon = e \\ (1+(1-\tau)r)a + b - c & \text{if } \epsilon = u \end{cases}$$

$$a \geq a_{\min}$$

$$\Gamma\left(Z', \epsilon' \mid Z, \epsilon\right) = \text{Prob}\left\{ Z_{t+1} = Z', \epsilon_{t+1} = \epsilon' \mid Z_t = Z, \epsilon_t = \epsilon \right\}$$

$$= \begin{pmatrix} p_{Z_g} e Z_g e & p_{Z_g} e Z_g u & p_{Z_g} e Z_b e & p_{Z_g} e_{b_b u} \\ p_{Z_g u Z_g} e & p_{Z_g u Z_g u} & p_{Z_g u Z_b e} & p_{Z_g u Z_b u} \\ p_{Z_b e Z_g e} & p_{Z_b e Z_g u} & p_{Z_b e Z_b e} & p_{Z_b e Z_b u} \\ p_{Z_b u Z_g e} & p_{Z_b u Z_g u} & p_{Z_b u Z_b} e & p_{Z_b u Z_b u} \end{pmatrix}$$

Example

$$F'\left(\epsilon', a'; Z', K'\right) = \sum_{\epsilon} \Gamma\left(Z', \epsilon' \mid Z, \epsilon\right) F(\epsilon, a; Z, K)$$

where $a = a^{-1}\left(e, a'; Z, K\right)$ is the inverse of the optimal policy funct $a' = a'(\epsilon, a; Z, K)$ with respect to individual wealth a and

$$K' = \sum_{\epsilon} \int_{a} a' f(\epsilon, a; Z, K) da$$

Again, f denotes the density function that is associated with F. Factors prices are equal to their respective marginal products:

$$r = \alpha Z_t \left(\frac{N}{K}\right)^{1-\alpha} - \delta$$

$$w = (1-\alpha) Z \left(\frac{K}{N}\right)^{\alpha}$$

Example

The aggregate consistency conditions hold:

$$K = \sum_{\epsilon} \int_{a} af(\epsilon, a; Z, K) da$$

$$N = \int_{a} f(e, a; Z, K) da$$

$$C = \sum_{\epsilon} \int_{a} c(\epsilon, a; Z, K) f(\epsilon, a; Z, K) da$$

$$T = \tau(wN + rK)$$

$$B = \int_{a} bf(u, a; Z, K) da$$

Aggregate Uncertainty

- We will simplify the analysis further following KRUSELL and SMITH (1998). In particular, we assume that the unemployment rate takes only two values u_g and u_b in good times and in bad times, respectively, with $u_g < u_b$. In order to simplify the dynamics of aggregate employment accordingly, the following restrictions have to be imposed on the transition matrix Γ :

$$u_Z \frac{p_{ZuZ'u}}{p_{ZZ'}} + (1 - u_Z) \frac{p_{ZeZ'u}}{p_{ZZ'}} = u_{Z'}$$

  for $Z, Z' \in \{Z_g, Z_b\}$. Condition implies that unemployment is u_g and u_b if $Z' = Z_g$ and $Z' = Z_b$, respectively.

-

$$\ln K' = \begin{cases} \gamma_{0g} + \gamma_{1g} \ln K & \text{if } Z = Z_g \\ \gamma_{0b} + \gamma_{1b} \ln K & \text{if } Z = Z_b \end{cases}$$

Algorithm 3

Step 1: Compute aggregate next-period employment N as a function of current productivity $Z : N = N(Z)$
Step 2: Choose the order I of moments m.
Step 3: Guess a parameterized functional form for H_I in (8.21) and choose initial parameters of H_I.
Step 4: Solve the consumer's optimization problem and compute $V(\epsilon, a, Z, m)$
Step 5: Simulate the dynamics of the distribution function.
Step 6: Use the time path for the distribution to estimate the law of motion for the moments m.
Step 7: Iterate until the parameters of H_I converge.
Step 8: Test the goodness of fit for H_I using, for example, R^2. If the fit is satisfactory, stop, otherwise increase I or choose a different functional form for H_I.

Applications: Costs of Business Cycles with Liquidity Constraints and Indivisibilities

- The model in İMROHOROĞU (1989)
- If $\epsilon = e(\epsilon = u)$, the agent is employed (unemployed). If the agent is employed he produces $y(e) = 1$ units of income. If he is unemployed, he engages in home production and produces $y(u) = \theta$ units of consumption goods, where $0 < \theta < 1$. Furthermore, the agents cannot insure against unemployment.
- agents cannot borrow, $a \geq 0$. They can insure against fluctuations in their income by storing the asset. The budget constraint is given by:

$$a_{t+1} = a_t - c_t + y(\epsilon_t)$$

- In the second economy, the agents can borrow at rate r_b. Agents can save assets by either lending at rate $r_l = 0$ or storing them. There is an intermediation sector between borrowing and lending households. The borrowing rate r_b exceeds the lending rate $r_b > r_l$.

Applications: Computation

- In the present economy, the interest rate is given. We first compute the decision functions by value function iteration. The value function of the individual is a function of his assets a and the state s :

$$
\begin{aligned}
V(a, s) &= \max_{c, a'} \left[u(c) + \beta E \left\{ V\left(a', s'\right) \mid s \right\} \right] \\
&= \max_{c, a'} \left[u(c) + \beta \sum_{s'} \pi\left(s' \mid s\right) V\left(a', s'\right) \right]
\end{aligned}
$$

- $$f'\left(a', s'\right) = \sum_{a' = a'(a, s)} \sum_{s'} \pi\left(s' \mid s\right) f(a, s)$$

Applications: Computation

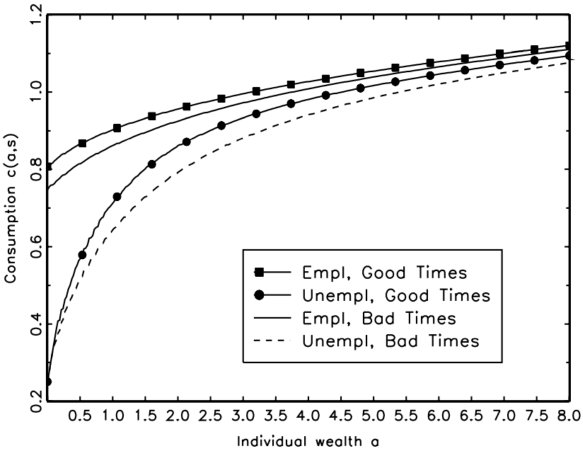

Figure 8.9: Consumption $c(a, s)$ in the Storage Economy

Applications: Computation

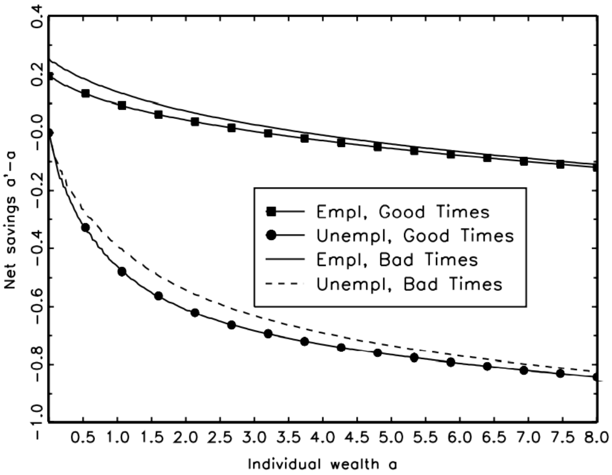

Figure 8.10: Net Savings $a' - a$ in the Storage Economy

Applications: Computation

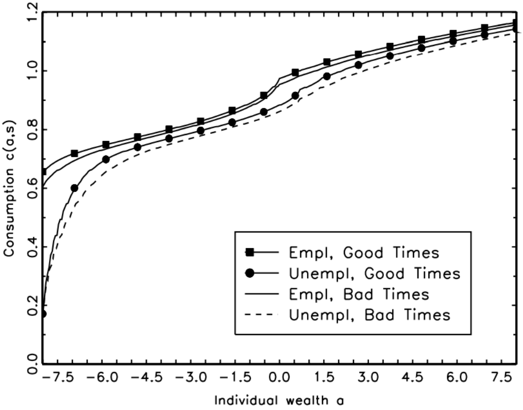

Figure 8.12: Consumption $c(a, s)$ in an Economy with Intermediation

Applications: Computation

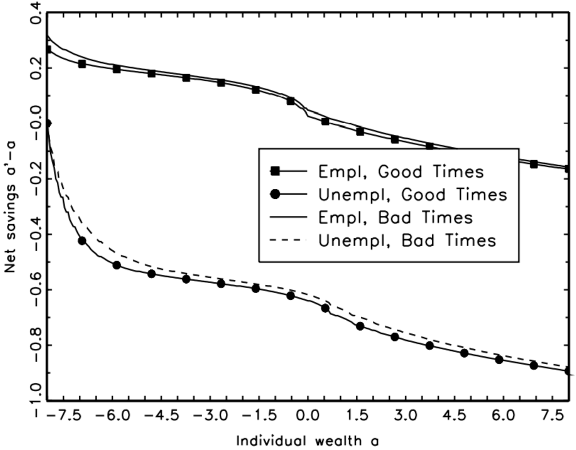

Figure 8.13: Net Savings $a' - a$ in an Economy with Intermediation

Applications: welfare effects from business cycle fluctuations

- She computes average utility in the economy with and without business cycle fluctuations
- For the benchmark calibration, the elimination of business cycles is equivalent to a utility gain corresponding to 0.3% of consumption in the economy with a storage technology.
- the fluctuations only cause a utility loss equivalent to 0.05% of consumption.

Business Cycle Dynamics of the Income Distribution

- CASTAÑEDA, DÍAZ-GIMÉNEZ, and RÍOS-RULL (1998b) explore the business cycle dynamics of the income distribution both empirically and in a theoretical computable general equilibrium model. They find that, in the US, the income share earned by the lowest quintile is more procyclical and more volatile than the other income shares. In particular, the income shares earned by the $60\% - 95\%$ group are even countercyclical, while the share earned by the top 5% is still acyclical.

The Model

- There are many infinitely lived households of mass one who differ with regard to the assets a_t, their employment status ϵ_t, and their efficiency type $i \in \{1, \ldots, 5\}$.
- In good times, agents work $h\left(Z_g\right)$ hours, and, in bad times, agents work $h\left(Z_b\right)$ hours. Let ζ_i denote the efficiency factor of a type i agent. If employed, the agent receives the labor income $h(Z)\zeta_i w$; otherwise, he produces home production $\bar{w}$.
- We will calibrate these values below so that $N_i(Z)$ is constant for $Z \in \left\{Z_g, Z_b\right\}$ and does not depend on the history of the productivity level Z, $\{Z_\tau\}_{\tau=-\infty}^{\tau=t}$.
- The aggregate labor input measured in efficiency units is given by $N(Z) = \sum_i \zeta_i h(Z) N_i(Z)$.

The Model

$$V(i,\epsilon,a;Z,F)=\max_{c,a'}\left[u(c)+\beta E\left\{V\left(i',\epsilon',a';Z',F'\right)\mid i,\epsilon,Z,F\right\}\right]$$

subject to the budget constraint:

$$a'=\begin{cases}(1+r)a+w\zeta_i h(Z)-c & \text{if } \epsilon=e\\(1+r)a+\bar{w}-c & \text{if } \epsilon=u\end{cases}$$

and subject to price, the stochastic process of the employment status ϵ and the aggregate technology Z, $\pi_i\left(Z',\epsilon'\mid Z,\epsilon\right)$ the agent's efficiency mobility as given by $\pi\left(i'\mid i\right)$, and the distribution dynamics $F'=G\left(F,Z,Z'\right)$

Computation

- Step 1 : In the first step, we choose computational parameters and compute the aggregate employment levels in good and bad times,

$$N\left(Z_g\right) = \sum_i \mu_i \zeta_i h\left(Z_g\right) N_i\left(Z_g\right)$$
$$N\left(Z_b\right) = \sum_i \mu_i \zeta_i h\left(Z_b\right) N_i\left(Z_b\right)$$

- the value function of the agents, $V(i, \epsilon, a, Z, K)$, and the consumption function, $c(i, \epsilon, a, Z, K)$
- Step 3: We impose again a very simple law of motion for the capital stock.

$$\ln K' = \begin{cases} \gamma_{0g} + \gamma_{1g} \ln K & \text{if } Z = Z_g \\ \gamma_{0b} + \gamma_{1b} \ln K & \text{if } Z = Z_b \end{cases}$$

Computation

- Step 4 : In this step, we compute the optimal next-period asset level $a'(i, \epsilon, a, Z, K)$ by value function iteration.
- Step 5 : In order to simulate the dynamics of the wealth distribution, we choose a sample of $nh = 5,000$ households. We divide the households in 10 subsamples $(i, \epsilon), i = 1, \ldots, 5, \epsilon \in \{e, u\}$
- Step 6: estimate the coefficients γ_0 and γ_1 of the equation (8.22) with the help of an OLS-regression.

Results

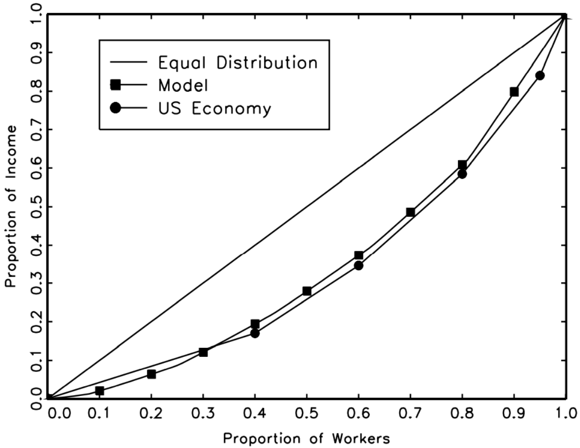

Figure 8.15: Lorenz Curve of Income

Results

Table 8.2

Income Quintile	Correlation output and income US	Correlation output and income model
lowest quintile (0-20%)	0.53	0.79
second quintile (20-40%)	0.49	0.79
third quintile (40-60%)	0.31	-0.74
fourth quintile (60-80%)	-0.29	-0.80
next 15% (80-95%)	-0.64	-0.80
top 5% (95-100%)	0.00	-0.78

Results

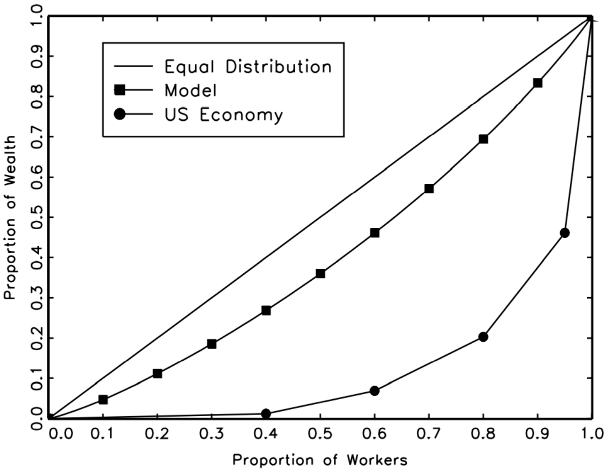

Figure 8.16: Lorenz Curve of Wealth

Thank You!