2019 秋季本科时间序列

第 9 次作业参考答案

2020 年 1 月 11 日

1. (a) α,β 的 OLS 估计量 $\hat{\alpha},\hat{\beta}$ 使得 $\sum\epsilon_t^2$ 取得最小值 $\min\sum\epsilon_t^2$
$\min\sum\epsilon_t^2=\min\sum(y_t-\alpha-\beta x_t)^2$ 的一阶条件为

$$\begin{cases}-2\sum(y_t-\alpha-\beta x_t)=0\\ -2\sum x_t(y_t-\alpha-\beta x_t)=0\end{cases}$$

解得

$$\begin{bmatrix}\hat{\alpha}\\ \hat{\beta}\end{bmatrix}=\begin{bmatrix}\frac{1}{T}\sum y_t-\frac{1}{T}\hat{\beta}\sum x_t\\ \frac{\sum x_ty_t-\frac{1}{T}\sum x_t\sum y_t}{\sum x_t^2-\frac{1}{T}(\sum x_t)^2}\end{bmatrix}$$

(b) 由 a. 问得

$$\begin{aligned}\lim_{T\to\infty}\hat{\beta}&=\lim_{T\to\infty}\frac{\sum x_ty_t-\frac{1}{T}\sum x_t\sum y_t}{T\sum x_t^2-\frac{1}{T}(\sum x_t)^2}\\ &=\lim_{T\to\infty}\frac{\frac{1}{T}\sum x_ty_t-\frac{1}{T}\sum x_t\frac{1}{T}y_t}{\frac{1}{T}\sum x_t^2-(\frac{1}{T}\sum x_t)^2}\end{aligned}$$

由大数定律知,$T\to\infty$ 时

$$\begin{aligned}\lim_{T\to\infty}\hat{\beta}&=\frac{\mathbb{E}(x_ty_t)-\mathbb{E}(x_t)\mathbb{E}(y_t)}{\mathbb{E}(x_t^2)-\mathbb{E}^2(x_t)}\\ &=\frac{\mathrm{cov}(x_t,y_t)}{\mathrm{var}(x_t)}\\ &=\frac{\mathrm{cov}(x_t,\alpha+\beta x_t+\epsilon_t)}{\mathrm{var}(x_t)}\\ &=\frac{\mathrm{cov}(x_t,\alpha)+\beta\mathrm{cov}(x_t,x_t)+\mathrm{cov}(x_t,\epsilon_t)}{\mathrm{var}(x_t)}\\ &=\beta+\frac{\rho\sigma_x\sigma_\epsilon}{\sigma_x^2}\\ &=\beta+\rho\frac{\sigma_\epsilon}{\sigma_x}\end{aligned}$$

$$\begin{aligned}\lim_{T\to\infty}\hat{\alpha} &= \lim_{T\to\infty}(\frac{1}{T}\sum y_t - \frac{1}{T}\hat{\beta}\sum x_t)\\ &= \mathbb{E}(y_t) - \mathbb{E}(x_t)(\beta + \rho\frac{\sigma_\epsilon}{\sigma_x})\\ &= \mathbb{E}(\alpha + \beta x_t + \epsilon_t) - \mathbb{E}(x_t)\beta - \mathbb{E}(x_t)\rho\frac{\sigma_\epsilon}{\sigma_x}\\ &= \alpha - \mathbb{E}(x_t)\rho\frac{\sigma_\epsilon}{\sigma_x}\end{aligned}$$

故当 $\rho > 0, \hat{\beta} > \beta$, 即 β 的 OLS 估计值大于真实值; 若 $\mathbb{E}x_t \geq 0, \hat{\alpha} < \alpha$, 即 α 的 OLS 估计值小于等于真实值, 若 $\mathbb{E}x_t < 0, \hat{\alpha} > \alpha$, 即 α 的 OLS 估计值大于真实值。
当 $\rho < 0, \hat{\beta} < \beta$, 即 β 的 OLS 估计值小于真实值; 若 $\mathbb{E}x_t \leq 0, \hat{\alpha} > \alpha$, 即 α 的 OLS 估计值大于等于真实值, 若 $\mathbb{E}x_t > 0, \hat{\alpha} < \alpha$, 即 α 的 OLS 估计值小于真实值。

(c)

$$\begin{aligned}\mathbb{E}(y_t) &= \mathbb{E}(\alpha + \beta x_t + \epsilon_t)\\ &= \alpha + \beta\mathbb{E}(x_t)\\ \mathbb{E}(y_t z_t) &= \mathbb{E}(\alpha z_t) + \beta\mathbb{E}(x_t z_t) + \mathbb{E}(\epsilon_t z_t)\\ &= \alpha\mathbb{E}(z_t) + \beta\mathbb{E}(x_t z_t)\end{aligned}$$

解得

$$\begin{cases}\alpha = \mathbb{E}(y_t) - \beta\mathbb{E}(x_t)\\ \beta = \frac{\mathbb{E}(y_t z_t) - \mathbb{E}(y_t)\mathbb{E}(z_t)}{\mathbb{E}(x_t z_t) - \mathbb{E}(x_t)\mathbb{E}(z_t)}\end{cases}$$

故 α, β 的一致性估计为

$$\begin{bmatrix}\hat{\alpha}\\ \hat{\beta}\end{bmatrix} = \begin{bmatrix}\frac{1}{T}\sum y_t - \frac{1}{T}\beta\sum x_t\\ \frac{\frac{1}{T}\sum y_t z_t - \frac{1}{T}\sum y_t \frac{1}{T}\sum z_t}{\frac{1}{T}\sum x_t z_t - \frac{1}{T}\sum x_t \frac{1}{T}\sum z_t}\end{bmatrix}$$

2. (a) 矩阵 $\boldsymbol{A}$ 的特征方程为

$$\begin{aligned}|\boldsymbol{A} - \lambda\boldsymbol{I}| &= \begin{vmatrix}0.8-\lambda & 0.1\\ 0.3 & 0.6-\lambda\end{vmatrix}\\ &= \lambda^2 - 1.4\lambda + 0.48 - 0.03\\ &= (\lambda - 0.5)(\lambda - 0.9) = 0\end{aligned}$$

解得特征值 $\lambda_1 = 0.5, \lambda_2 = 0.9$, 即 $|\lambda_1|, |\lambda_2| < 1$, 上述 VAR(1) 模型满足平稳性条件。

(b) 当 $\lambda = 0.5$ 时

$$(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{x} = \begin{bmatrix}0.3 & 0.1\\ 0.3 & 0.1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

解得 $\boldsymbol{x}_1 = [1, -3]^\intercal$
当 $\lambda = 0.9$ 时

$$(\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{x} = \begin{bmatrix}-0.1 & 0.1\\ 0.3 & -0.3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

解得 $\boldsymbol{x}_2 = [1, 1]^\top$
故

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{C}\boldsymbol{\Lambda}\boldsymbol{C}^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & \\ & 0.9 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}^{-1} \end{aligned}$$

由 Cramer 法则可得

$$\boldsymbol{C}^{-1} = \frac{1}{|\boldsymbol{C}|}\boldsymbol{C}^* = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}$$

故

$$\begin{aligned} \boldsymbol{A}^j &= \boldsymbol{C}\boldsymbol{\Lambda}\boldsymbol{C}^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 0.5^j & \\ & 0.9^j \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \frac{1}{4} \\ &= \frac{1}{4}\begin{bmatrix} 0.5^j + 3\times 0.9^j & -0.5^j + 0.9^j \\ -3\times 0.5^j + 3\times 0.9^j & 3\times 0.5^j + 0.9^j \end{bmatrix}, j = 1, 2... \end{aligned}$$

(c) 对任意 j, 有

$$\begin{aligned} \begin{bmatrix} x_{t+j} \\ y_{t+j} \end{bmatrix} &= \boldsymbol{A}\begin{bmatrix} x_{t+j-1} \\ y_{t+i-1} \end{bmatrix} + \begin{bmatrix} \epsilon_{xt+j} \\ \epsilon_{yt+j} \end{bmatrix} \\ &= \boldsymbol{A}\left(\boldsymbol{A}\begin{bmatrix} x_{t+j-2} \\ y_{t+j-2} \end{bmatrix} + \begin{bmatrix} \epsilon_{xt+j-1} \\ \epsilon_{yt+j-1} \end{bmatrix}\right) + \begin{bmatrix} \epsilon_{xt+j} \\ \epsilon_{yt+j} \end{bmatrix} \\ &\cdots \\ &= \boldsymbol{A}^{j+1}\begin{bmatrix} x_{t-1} \\ y_{t-1} \end{bmatrix} + \sum_0^j \boldsymbol{A}^i \begin{bmatrix} \epsilon_{xt+i} \\ \epsilon_{yt+i} \end{bmatrix} \end{aligned}$$

故

$$\begin{aligned} \frac{\partial x_{t+j}}{\partial \epsilon_{xt}} &= \frac{1}{4}(0.5^j + 3\times 0.9^j) \\ \frac{\partial x_{t+j}}{\partial \epsilon_{yt}} &= \frac{1}{4}(-0.5^j + 0.9^j) \\ \frac{\partial y_{t+j}}{\partial \epsilon_{xt}} &= \frac{1}{4}(-3\times 0.5^j + 3\times 0.9^j) \\ \frac{\partial y_{t+j}}{\partial \epsilon_{xt}} &= \frac{1}{4}(3\times 0.5^j + 0.9^j) \end{aligned}$$