2019 秋季本科时间序列

第 6 次作业参考答案

2019 年 12 月 2 日

1. 由

$$
\boldsymbol{Y}=\begin{bmatrix} X_{p+1} \\ \vdots \\ X_T \end{bmatrix}, \boldsymbol{X}=\begin{bmatrix} X_p & \cdots & X_1 \\ \vdots & \ddots & \vdots \\ X_{T-1} & \cdots & X_{T-p} \end{bmatrix}
$$

可得

$$
\begin{aligned}
\frac{1}{T}\boldsymbol{X}^{\intercal}\boldsymbol{X} &= \frac{1}{T}\begin{bmatrix} X_p & \cdots & X_{T-1} \\ \vdots & \ddots & \vdots \\ X_1 & \cdots & X_{T-p} \end{bmatrix}\begin{bmatrix} X_p & \cdots & X_1 \\ \vdots & \ddots & \vdots \\ X_{T-1} & \cdots & X_{T-p} \end{bmatrix} \\
&= \frac{1}{T}\begin{bmatrix} \sum_{t=p}^{T-1} X_t X_t & \cdots & \sum_{t=1}^{T-p} X_{t+p-1} X_t \\ \vdots & \ddots & \vdots \\ \sum_{t=1}^{T-p} X_t X_{t+p-1} & \cdots & \sum_{t=1}^{T-p} X_t X_t \end{bmatrix}
\end{aligned}
$$

因为 $X_t X_{t-k}$ 为平稳序列，又 $\mathbb{E}X_t = 0$, 由大数定律可知其样本均值收敛到其期望, 即协方差 $\sigma^2(k)$，即

$$
\frac{1}{T}\sum_{t=1}^{T} X_t X_{t-k} \xrightarrow{a.s.} \sigma^2(k)
$$

所以

$$
\begin{aligned}
\frac{1}{T}\boldsymbol{X}^{\intercal}\boldsymbol{X} &\xrightarrow{a.s.} \frac{T-p}{T}\begin{bmatrix} \sigma_X^2(0) & \cdots & \sigma_X^2(p-1) \\ \vdots & \ddots & \vdots \\ \sigma_X^2(p-1) & \cdots & \sigma_X^2(0) \end{bmatrix} \\
&\xrightarrow{a.s.} \begin{bmatrix} \sigma_X^2(0) & \cdots & \sigma_X^2(p-1) \\ \vdots & \ddots & \vdots \\ \sigma_X^2(p-1) & \cdots & \sigma_X^2(0) \end{bmatrix}
\end{aligned}
$$

同理

$$\begin{aligned}
\frac{1}{T}\boldsymbol{X}^{\intercal}\boldsymbol{Y} &= \frac{1}{T}\begin{bmatrix} X_p & \cdots & X_{T-1} \\ \vdots & \ddots & \vdots \\ X_1 & \cdots & X_{T-p} \end{bmatrix}\begin{bmatrix} X_{p+1} \\ \vdots \\ X_T \end{bmatrix} \\
&= \frac{1}{T}\begin{bmatrix} \sum_{i=1}^{T-p} X_{i+p-1}X_{i+p} \\ \vdots \\ \sum_{i=1}^{T-p} X_i X_{i+p} \end{bmatrix} \\
&\xrightarrow{a.s.} \frac{T-p}{T}\begin{bmatrix} \sigma_X^2(1) \\ \vdots \\ \sigma_X^2(p) \end{bmatrix} \\
&\xrightarrow{a.s.} \begin{bmatrix} \sigma_X^2(1) \\ \vdots \\ \sigma_X^2(p) \end{bmatrix}
\end{aligned}$$

故 AR(p) 自回归系数的 OLS 估计为

$$\begin{aligned}
\hat{\boldsymbol{\beta}}_T &= (\boldsymbol{X}\boldsymbol{X}^{\intercal})^{-1}\boldsymbol{X}^{\intercal}\boldsymbol{Y} \\
&= \Big(\frac{1}{T}\boldsymbol{X}^{\intercal}\boldsymbol{X}\Big)^{-1}\frac{1}{T}\boldsymbol{X}^{\intercal}\boldsymbol{Y} \\
&\xrightarrow{a.s.} \begin{bmatrix} \sigma_X^2(0) & \cdots & \sigma_X^2(p-1) \\ \vdots & \ddots & \vdots \\ \sigma_X^2(p-1) & \cdots & \sigma_X^2(0) \end{bmatrix}^{-1}\begin{bmatrix} \sigma_X^2(1) \\ \vdots \\ \sigma_X^2(p) \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_p \end{bmatrix}
\end{aligned}$$

由 Yule_Walker 方程可知，该 OLS 估计满足一致性

2. (a) 由已知 e_t 的概率密度函数为

$$p(e_t) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{e_t^2}{2\sigma^2}\right)$$

将 $e_t = Y_t - \boldsymbol{X}_t^{\intercal}\boldsymbol{\beta}$ 代入得

$$p(Y_t|X_t,\boldsymbol{\beta}) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(Y_t - \boldsymbol{X}_t^{\intercal}\boldsymbol{\beta})^2}{2\sigma^2}\right)$$

则似然函数为

$$\begin{aligned}
f(\boldsymbol{\beta},\sigma^2|\boldsymbol{Y},\boldsymbol{X}) &= \prod_{t=1}^{T} p(Y_t|X_t,\boldsymbol{\beta}) \\
&= \prod_{t=1}^{T}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(Y_t - \boldsymbol{X}_t^{\intercal}\boldsymbol{\beta})^2}{2\sigma^2}\right)
\end{aligned}$$

(b)

$$\begin{aligned}\log f(\boldsymbol{\beta},\sigma^2) &= \log\prod_{t=1}^{T}\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(Y_t-\boldsymbol{X}_t^\intercal\boldsymbol{\beta})^2}{2\sigma^2}\right)\\ &= \sum_{t=1}^{T}\log\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(Y_t-\boldsymbol{X}_t^\intercal\boldsymbol{\beta})^2}{2\sigma^2}\right)\\ &= T\log\frac{1}{\sqrt{2\pi\sigma^2}}-\frac{1}{2\sigma^2}\sum_{t=1}^{T}\left(Y_t-\boldsymbol{X}_t^\intercal\boldsymbol{\beta}\right)^2\\ &= -\frac{T}{2}\log 2\pi-\frac{T}{2}\log\sigma^2-\frac{1}{2\sigma^2}\sum_{t=1}^{T}\left(Y_t-\boldsymbol{X}_t^\intercal\boldsymbol{\beta}\right)^2\end{aligned}$$

则

$$\begin{aligned}\frac{\partial\log f(\boldsymbol{\beta},\sigma^2)}{\partial\boldsymbol{\beta}} &= \frac{1}{\sigma^2}\sum_{t=1}^{T}\boldsymbol{X}_t(Y_t-\boldsymbol{X}_t^\intercal\boldsymbol{\beta})\\ &= \frac{1}{\sigma^2}(\boldsymbol{X}^\intercal\boldsymbol{Y}-\boldsymbol{X}^\intercal\boldsymbol{X}\boldsymbol{\beta})\\ \frac{\partial\log f(\boldsymbol{\beta},\sigma^2)}{\partial\sigma^2} &= -\frac{T}{2\sigma^2}+\frac{1}{2\sigma^4}\sum_{t=1}^{T}(Y_t-\boldsymbol{X}_t^\intercal\boldsymbol{\beta})^2\end{aligned}$$

(c) 对 $f(\boldsymbol{\beta},\sigma^2)$ 取对数后，不影响原函数的单调性，概率的最大对数值出现在与原始概率函数相同的点上，故似然函数最大化问题的解与对数似然函数最大化问题的解等价.

(d) 由 (c) 问，似然函数最大化问题转化为对数似然函数最大化问题，且由 (b) 问

$$\frac{\partial\log f(\boldsymbol{\beta},\sigma^2)}{\partial\boldsymbol{\beta}} = \frac{1}{\sigma^2}(\boldsymbol{X}^\intercal\boldsymbol{Y}-\boldsymbol{X}^\intercal\boldsymbol{X}\boldsymbol{\beta})$$

令偏导数为 0,

$$\begin{aligned}(\boldsymbol{X}^\intercal\boldsymbol{Y}-\boldsymbol{X}^\intercal\boldsymbol{X}\boldsymbol{\beta}) &= 0\\ \boldsymbol{X}^\intercal\boldsymbol{Y} &= \boldsymbol{X}^\intercal\boldsymbol{X}\boldsymbol{\beta}\end{aligned}$$

得 $\hat{\boldsymbol{\beta}}_{ML}=(\boldsymbol{X}^\intercal\boldsymbol{X})^{-1}\boldsymbol{X}^\intercal\boldsymbol{Y}$
同样地，令

$$\frac{\partial f(\boldsymbol{\beta},\sigma^2)}{\partial\sigma^2} = -\frac{T}{2\sigma^2}+\frac{1}{2\sigma^4}\sum_{t=1}^{T}(Y_t-\boldsymbol{X}_t^\intercal\boldsymbol{\beta})^2=0$$

得

$$\begin{aligned}\frac{T}{2\sigma^2} &= \frac{1}{2\sigma^4}\sum_{t=1}^{T}(Y_t-\boldsymbol{X}_t^\intercal\boldsymbol{\beta})^2\\ \hat{\sigma}^2_{ML} &= \frac{1}{T}\sum_{t=1}^{T}(Y_t-\boldsymbol{X}_t^\intercal\boldsymbol{\beta})^2\end{aligned}$$

故线性回归模型的最大似然估计为 $\hat{\boldsymbol{\beta}}_{ML}=(\boldsymbol{X}^\intercal\boldsymbol{X})^{-1}\boldsymbol{X}^\intercal\boldsymbol{Y}$,$\hat{\sigma^2}_{ML}=\frac{1}{T}\sum_{t=1}^{T}(Y_t-\boldsymbol{X}_t^\intercal\boldsymbol{\beta})^2$
$\hat{\boldsymbol{\beta}}_{ML}$ 与 $\hat{\boldsymbol{\beta}}_{OLS}$ 一致。