2019 秋季本科时间序列

第 4 次作业参考答案

2019 年 11 月 12 日

1. (a)

由题目可知 $Y_t = \frac{1}{1-\frac{\mathscr{L}}{\lambda}} X_t$，且 $|\lambda| > 1$，故可得 $Y_t = \sum_{i=0}^{\infty} \left(\frac{1}{\lambda^i}\right) X_{t-i}$

故 $E(Y_t) = \sum_{i=0}^{\infty} \left(\frac{1}{\lambda^i}\right) E(X_{t-i}) = \frac{1}{1-\frac{1}{\lambda}} E(X_{t-i})$，又知 X_t 为平稳时间序列，则 $E(Y_t)$ 与 t 无关

$$
\begin{aligned}
cov(Y_t, X_{t-j}) &= cov(\sum_{i=0}^{\infty} \left(\frac{1}{\lambda^i}\right) X_{t-i}, X_{t-j}) \\
&= \sum_{i=0}^{\infty} \left(\frac{1}{\lambda^i}\right) cov(X_{t-i}, X_{t-j}) \\
&= \sum_{i=0}^{\infty} \left(\frac{1}{\lambda^i}\right) \sigma_X^2(j-i) \\
cov(Y_t, Y_{t-k}) &= cov(Y_t, \sum_{m=0}^{\infty} \left(\frac{1}{\lambda^m}\right) X_{t-k-m}) \\
&= \sum_{m=0}^{\infty} \left(\frac{1}{\lambda^m}\right) cov(Y_t, X_{t-k-m}) \\
&= \sum_{m=0}^{\infty} \left(\frac{1}{\lambda^m}\right) \sum_{i=0}^{\infty} \left(\frac{1}{\lambda^i}\right) \sigma_X^2(k+m-i)
\end{aligned}
$$

由上式易知 $\sigma_Y^2(k) = cov(Y_t, Y_{t-k})$ 与 t 无关

即可证 Y_t 为平稳时间序列

(b)

对 $AR(p)$ 过程有

$$
\begin{aligned}
X_t &= \sum_{i=1}^{p} \phi_i X_{t-i} + \varepsilon_t \\
\left(1 - \sum_{i=1}^{p} \phi_i \mathscr{L}^i\right) X_t &= \varepsilon_t
\end{aligned}
$$

已知 $z_1 \ldots z_p$ 为特征多项式 $A(z)$ 的根

$A(z) = 1 - \phi_1 z - \cdots - \phi_p z^p = (1 - \frac{z}{z_1})(1 - \frac{z}{z_2}) \ldots (1 - \frac{z}{z_p})$

把 $\mathscr{L}$ 带入 $A(z)$, 则有 $A(\mathscr{L})=(1-\frac{\mathscr{L}}{z_1})(1-\frac{\mathscr{L}}{z_2})\dots(1-\frac{\mathscr{L}}{z_p})$，则有如下

$$\begin{aligned}X_t &= \frac{\varepsilon_t}{A(\mathscr{L})}\\ &= \frac{1}{(1-\frac{\mathscr{L}}{z_1})(1-\frac{\mathscr{L}}{z_2})\dots(1-\frac{\mathscr{L}}{z_p})}\varepsilon_t\\ &= \frac{1}{(1-\frac{\mathscr{L}}{z_1})}\frac{1}{(1-\frac{\mathscr{L}}{z_2})}\cdots\frac{1}{(1-\frac{\mathscr{L}}{z_p})}\varepsilon_t\end{aligned}$$

其中 ε_t 为白噪声，故其为平稳时间序列。由 (a) 中结论可知，当 $|z_p|>1$ 时，$\frac{1}{(1-\frac{\mathscr{L}}{z_p})}\varepsilon_t$ 为平稳时间序列，依次类推可知当 $z_1\dots z_p$ 的模长均大于 1 时，X_t 为平稳时间序列，即 $AR(p)$ 过程平稳

2. (a)
已知特征多项式 $A(z)=1-\phi_1 z-\phi_2 z^2$，$A(0)=1>0$ 恒成立
当 λ 和 η 为实数时，若两实根模长大于 1，则必有

$$\begin{cases} \mid \lambda\eta \mid = \mid -\frac{1}{\phi_2} \mid > 1 \Rightarrow \mid \phi_2 \mid < 1 \\ \Delta = \phi_1^2 + 4\phi_2 > 0 \end{cases}$$

分情况讨论 (1) 当 $-1<\phi_2<0$ 时，函数开口向上，若使两实根模长大于 1，则必有

$$\begin{cases} -1<\phi_2<0 \\ \Delta = \phi_1^2 + 4\phi_2 > 0 \\ A(1)>0 \\ A(-1)>0 \end{cases}$$

(2) 当 $0<\phi_2<1$ 时，函数开口向下，若使两实根模长大于 1，则必有

$$\begin{cases} 0<\phi_2<1 \\ \Delta = \phi_1^2 + 4\phi_2 > 0 \\ A(1)>0 \\ A(-1)>0 \end{cases}$$

解得此时 ϕ_1 和 ϕ_2 的取值满足

$$\begin{cases} \phi_2 < 1 \pm \phi_1 \\ \phi_1^2 > -4\phi_2 \end{cases}$$

当 λ 和 η 为复数时，有

$$\lambda = \frac{\phi_1 + i\sqrt{-\phi_1^2 - 4\phi_2}}{-2\phi_2}$$

$$\eta = \frac{\phi_1 - i\sqrt{-\phi_1^2 - 4\phi_2}}{-2\phi_2}$$

两复根共轭，故模长相等，$\mid \lambda\eta \mid > 1$ 即可说明两根模长均大于 1，于是，若存在两个模长大于 1 的复根，则以下条件成立

$$\begin{cases} \mid \lambda\eta \mid & = \mid \frac{\phi_1^2 + (-\phi_1^2 - 4\phi_2)}{4\phi_2^2} \mid = \mid \frac{1}{\phi_2} \mid > 1 \\ \Delta & = \phi_1^2 + 4\phi_2 < 0 \end{cases}$$

解得此时 ϕ_1 和 ϕ_2 的取值满足

$$\begin{cases} |\phi_2| < 1 \\ \phi_1^2 < -4\phi_2 \end{cases}$$

(b)

系数矩阵 $\Phi = \begin{bmatrix} \phi_1 & \phi_2 - 1 & 0 \\ \phi_2 & \phi_1 & -1 \\ 1 & -\phi_1 & -\phi_2 \end{bmatrix}$

$$|\Phi| = (\phi_2 + 1)(\phi_1 + \phi_2 - 1)(\phi_2 - \phi_1 - 1)$$

结合 (a) 可知，当满足平稳性条件时，$|\Phi| \neq 0$，系数矩阵可逆

$$\Phi^{-1} = \frac{\Phi^*}{|\Phi|}$$

$$\Phi^* = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

$$\Phi \begin{bmatrix} \sigma_X^2(0) \\ \sigma_X^2(1) \\ \sigma_X^2(2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sigma_\varepsilon^2 \end{bmatrix}$$

故有

$$\begin{bmatrix} \sigma_X^2(0) \\ \sigma_X^2(1) \\ \sigma_X^2(2) \end{bmatrix} = \Phi^{-1} \begin{bmatrix} 0 \\ 0 \\ \sigma_\varepsilon^2 \end{bmatrix} = \frac{\Phi^*}{|\Phi|} \begin{bmatrix} 0 \\ 0 \\ \sigma_\varepsilon^2 \end{bmatrix} = \frac{\sigma_\varepsilon^2}{(\phi_2 + 1)(\phi_1 + \phi_2 - 1)(\phi_2 - \phi_1 - 1)} \begin{bmatrix} 1 - \phi_2 \\ \phi_1 \\ \phi_1^2 - \phi_2^2 + \phi_2 \end{bmatrix}$$

(c)

令

$$|\mu \boldsymbol{I} - \boldsymbol{A}| = \begin{vmatrix} \mu - \phi_1 & -\phi_2 \\ -1 & \mu \end{vmatrix} = \mu^2 - \phi_1\mu - \phi_2 = 0$$

即 $1 - \phi_1 \frac{1}{\mu} - \phi_2 \frac{1}{\mu^2} = 0$, 又 $\lambda, \eta \neq 0$, 故得到特征值为 $\mu_1 = \frac{1}{\lambda}, \mu_2 = \frac{1}{\eta}$

当 $\mu = \mu_1 = \frac{1}{\lambda}$ 时

$$(\mu \boldsymbol{I} - \boldsymbol{A})\boldsymbol{x} = \begin{bmatrix} \frac{1}{\lambda} - \phi_1 & -\phi_2 \\ -1 & \frac{1}{\lambda} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

又 $(\frac{1}{\lambda} - \phi_1)\frac{1}{\lambda} - \phi_2 = \frac{1}{\lambda^2} - \phi_1 \frac{1}{\lambda} - \phi_2 = 0$

故取

$$\boldsymbol{x} = \begin{bmatrix} \frac{1}{\lambda} \\ 1 \end{bmatrix}$$

单位化得特征向量

$$\xi_1 = \begin{bmatrix} \frac{1}{\sqrt{1+\lambda^2}} \\ \frac{\lambda}{\sqrt{1+\lambda^2}} \end{bmatrix}$$

同理特征向量 $\xi_2 = \begin{bmatrix} \frac{1}{\sqrt{1+\eta^2}} \\ \frac{\eta}{\sqrt{1+\eta^2}} \end{bmatrix}$，特征向量矩阵为 $\boldsymbol{S} = \begin{bmatrix} \frac{1}{\sqrt{1+\lambda^2}} & \frac{1}{\sqrt{1+\eta^2}} \\ \frac{\lambda}{\sqrt{1+\lambda^2}} & \frac{\eta}{\sqrt{1+\eta^2}} \end{bmatrix}$

特征向量矩阵的逆矩阵为

$$\begin{aligned} \boldsymbol{S}^{-1} &= \frac{1}{|\boldsymbol{S}|} \begin{bmatrix} \frac{\eta}{\sqrt{1+\eta^2}} & -\frac{1}{\sqrt{1+\eta^2}} \\ -\frac{\lambda}{\sqrt{1+\lambda^2}} & \frac{1}{\sqrt{1+\lambda^2}} \end{bmatrix} \\ &= \frac{1}{\eta-\lambda} \begin{bmatrix} \eta\sqrt{1+\lambda^2} & -\sqrt{1+\lambda^2} \\ -\lambda\sqrt{1+\eta^2} & \sqrt{1+\eta^2} \end{bmatrix} \end{aligned}$$

则

$$\begin{aligned} \boldsymbol{A}^k &= \boldsymbol{S}\Lambda^k\boldsymbol{S}^{-1} \\ &= \frac{1}{\eta-\lambda} \begin{bmatrix} \frac{1}{\sqrt{1+\lambda^2}} & \frac{1}{\sqrt{1+\eta^2}} \\ \frac{\lambda}{\sqrt{1+\lambda^2}} & \frac{\eta}{\sqrt{1+\eta^2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\lambda^k} & 0 \\ 0 & \frac{1}{\eta^k} \end{bmatrix} \begin{bmatrix} \eta\sqrt{1+\lambda^2} & -\sqrt{1+\lambda^2} \\ -\lambda\sqrt{1+\eta^2} & \sqrt{1+\eta^2} \end{bmatrix} \\ &= \frac{1}{\eta-\lambda} \begin{bmatrix} \frac{1}{\lambda^k\sqrt{1+\lambda^2}} & \frac{1}{\eta^k\sqrt{1+\eta^2}} \\ \frac{1}{\lambda^{l-1}\sqrt{1+\lambda^2}} & \frac{1}{\eta^{k-1}\sqrt{1+\eta^2}} \end{bmatrix} \begin{bmatrix} \eta\sqrt{1+\lambda^2} & -\sqrt{1+\lambda^2} \\ -\lambda\sqrt{1+\eta^2} & \sqrt{1+\eta^2} \end{bmatrix} \\ &= \frac{1}{(\lambda\eta)^k(\eta-\lambda)} \begin{bmatrix} \eta^{k+1}-\lambda^{k+1} & \lambda^k-\eta^k \\ \lambda\eta(\eta^k-\lambda^k) & \lambda\eta(\lambda^{k-1}-\eta^{k-1}) \end{bmatrix} \end{aligned}$$

(d)

当 $\lambda = \eta$ 时，$\boldsymbol{A}$ 只有一个特征值 $\mu = \frac{\phi_1}{2}$（二重代数重数），有 $\phi_1^2 + 4\phi_2 = 0$

$$\begin{aligned} (\mu\boldsymbol{I}-\boldsymbol{A}) &= \begin{bmatrix} -\frac{\phi_1}{2} & -\phi_2 \\ -1 & \frac{\phi_1}{2} \end{bmatrix} \\ &\to \begin{bmatrix} -\frac{\phi_1}{2} & -\phi_2 \\ \frac{\phi_1}{2} & -\frac{\phi_1^2}{4} \end{bmatrix} \\ &\to \begin{bmatrix} -\frac{\phi_1}{2} & -\phi_2 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

得 $rank(\lambda\boldsymbol{I}-\boldsymbol{A}) = 1$，$\boldsymbol{A}$ 无两个线性无关的特征向量。

故其 Jordan 标准型有 1 一个 Jordan 块，其 Jordan 标准型为

$$\boldsymbol{J} = \begin{bmatrix} \frac{\phi_1}{2} & 1 \\ 0 & \frac{\phi_1}{2} \end{bmatrix}$$

由 $(\mu\boldsymbol{I}-\boldsymbol{A})\boldsymbol{z} = 0$ 可得 $\boldsymbol{t}_1 = \begin{bmatrix} \frac{\phi_1}{2} \\ 1 \end{bmatrix}$ 为链首

记 $\boldsymbol{T} = (\boldsymbol{t}_1, \boldsymbol{t}_2)$,$\boldsymbol{t}_2 = (t_{21}, t_{22})^{\mathrm{T}}$ 有

$$(\boldsymbol{A}\boldsymbol{t}_1, \boldsymbol{A}\boldsymbol{t}_2) = (\boldsymbol{t}_1, \boldsymbol{t}_2) \begin{bmatrix} \frac{\phi_1}{2} & 1 \\ 0 & \frac{\phi_1}{2} \end{bmatrix}$$

有

$$
\begin{cases}
\frac{\phi_1}{2}t_{21} + \phi_2 t_{22} = \frac{\phi_1}{2} \\
t_{21} = 1 + \frac{\phi_1}{2}t_{22}
\end{cases}
$$

故取 $t_2 = (1, 0)^{\mathrm{T}}$, 变换矩阵则为 $T = \begin{bmatrix} \frac{\phi_1}{2} & 1 \\ 1 & 0 \end{bmatrix}$

其逆矩阵为

$$
\boldsymbol{T}^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -\frac{\phi_1}{2} \end{bmatrix}
$$

故

$$
\begin{aligned}
\boldsymbol{A}^k &= \boldsymbol{T}\boldsymbol{J}^k\boldsymbol{T}^{-1} \\
&= \begin{bmatrix} \frac{\phi_1}{2} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\phi_1^k}{2^k} & \frac{k\phi_1^{k-1}}{2^{k-1}} \\ 0 & \frac{\phi_1^k}{2^k} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -\frac{\phi_1}{2} \end{bmatrix} \\
&= \frac{\phi_1^k}{2^k} \begin{bmatrix} 1+k & -k \\ \frac{\phi_1}{2}k & 1-k \end{bmatrix}
\end{aligned}
$$