2019 秋季本科时间序列

第 4 次作业附加题参考答案

2019 年 11 月 9 日

1.

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\begin{aligned}
X_t &= \frac{1}{(1-\frac{\mathscr{L}}{z_1})}\frac{1}{(1-\frac{\mathscr{L}}{z_2})}\cdots\frac{1}{(1-\frac{\mathscr{L}}{z_p})}\varepsilon_t \\
&= \frac{1}{(1-\frac{\mathscr{L}}{z_1})}\frac{1}{(1-\frac{\mathscr{L}}{z_2})}\cdots\frac{1}{(1-\frac{\mathscr{L}}{z_{p-1}})}\left(\sum_{i=0}^{\infty}\frac{\mathscr{L}^i}{z_p^i}\right)\varepsilon_t \\
&= \frac{1}{(1-\frac{\mathscr{L}}{z_1})}\frac{1}{(1-\frac{\mathscr{L}}{z_2})}\cdots\frac{1}{(1-\frac{\mathscr{L}}{z_{p-1}})}\left(\varepsilon_t+\frac{1}{z_p}\varepsilon_{t-1}+\dots\right) \\
&= \frac{1}{(1-\frac{\mathscr{L}}{z_1})}\frac{1}{(1-\frac{\mathscr{L}}{z_2})}\cdots\left(1+\frac{\mathscr{L}}{z_{p-1}}+\dots\right)\left(\varepsilon_t+\frac{1}{z_p}\varepsilon_{t-1}+\dots\right) \\
&= \frac{1}{(1-\frac{\mathscr{L}}{z_1})}\frac{1}{(1-\frac{\mathscr{L}}{z_2})}\cdots\frac{1}{(1-\frac{\mathscr{L}}{z_{p-2}})}\left(\varepsilon_t+(\frac{1}{z_{p-1}}+\frac{1}{z_p})\varepsilon_{t-1}+\dots\right) \\
&= \frac{1}{(1-\frac{\mathscr{L}}{z_1})}\frac{1}{(1-\frac{\mathscr{L}}{z_2})}\cdots\frac{1}{(1-\frac{\mathscr{L}}{z_{p-3}})}\left(\varepsilon_t+(\frac{1}{z_{p-2}}+\frac{1}{z_{p-1}}+\frac{1}{z_p})\varepsilon_{t-1}+\dots\right)
\end{aligned}
$$

依此类推，可得 ε_{t-1} 的系数为 $(\frac{1}{z_1}+\frac{1}{z_2}\dots\frac{1}{z_p})$ 即 $\sum_{i=1}^{p}\frac{1}{z_p}$

2.

$$
\begin{aligned}
E(\hat{\sigma_T}^2) &= E\left(\frac{1}{T-1}\sum_{t=1}^{T}(X_t-\hat{\mu}_T)^2\right) \\
&= \frac{1}{T-1}E\left(\sum_{t=1}^{T}\left[(X_t-\mu)+(\mu-\hat{\mu}_T)\right]^2\right) \\
&= \frac{1}{T-1}E\left(\sum_{t=1}^{T}(X_t-\mu)^2+2\sum_{t=1}^{T}(X_t-\mu)(\mu-\hat{\mu}_T)+\sum_{t=1}^{T}(\mu-\hat{\mu}_T)^2\right) \\
&= \frac{1}{T-1}E\left(\sum_{t=1}^{T}(X_t-\mu)^2-2T(\mu-\hat{\mu}_T)^2+T(\mu-\hat{\mu}_T)^2\right) \\
&= \frac{1}{T-1}\left(\sum_{t=1}^{T}E(X_t-\mu)^2-TE(\mu-\hat{\mu}_T)^2\right)
\end{aligned}
$$

由于 $E(\mu-\hat{\mu}_T)^2=var(\hat{\mu}_T)=\frac{var(\sum_{t=1}^{T}X_t)}{T^2}$，故 $E(\hat{\sigma_T}^2)=\frac{1}{T-1}[T\sigma^2-\frac{var(\sum_{t=1}^{T}X_t)}{T}]$

当 X_t 为 iid 样本时，$var(\sum_{t=1}^{T} X_t) = T\sigma^2$，此时 $E(\hat{\sigma_T}^2) = \frac{1}{T-1}[T\sigma^2 - \sigma^2] = \sigma^2$，即证 $\hat{\sigma_T}^2$ 是 σ^2 的无偏估计

若 X_t 有序列相关性,则 $var(\sum_{t=1}^{T} X_t) = T\sigma^2 + 2\sigma^2 \sum_{i\neq j} \rho_{ij}$,此时 $E(\hat{\sigma_T}^2) = \frac{1}{T-1}\left[(T-1)\sigma^2 - \frac{2\sigma^2 \sum_{i\neq j} \rho_{ij}}{T}\right] = \sigma^2 - \frac{2\sigma^2 \sum_{i\neq j} \rho_{ij}}{T(T-1)}$

3.

$$
\begin{aligned}
\hat{\sigma}_T^2(1) &= \frac{1}{T-1}\sum_{t=2}^{T}(X_t - \hat{\mu}_T)(X_{t-1} - \hat{\mu}_T) \\
&= \frac{1}{T-1}\sum_{t=2}^{T}\Big(\left[(X_t - \mu) + (\mu - \hat{\mu}_T)\right] \left[(X_{t-1} - \mu) + (\mu - \hat{\mu}_T)\right] \Big) \\
&= \frac{1}{T-1}\sum_{t=2}^{T}(X_t - \mu)(X_{t-1} - \mu) + \frac{1}{T-1}(\mu - \hat{\mu}_T)\sum_{t=2}^{T}(X_t + X_{t-1} - 2\mu) + (\mu - \hat{\mu}_T)^2
\end{aligned}
$$

根据大数定律，当 $T \to \infty$ 时，$\frac{1}{T}\sum_{t=1}^{T} X_t \xrightarrow{a.s.} E(X_t)$，即 $\hat{\mu}_T \xrightarrow{a.s.} \mu$

故结合上述结果可得 $\lim_{T\to\infty} \hat{\sigma}_T^2(1) = \sigma^2(1)$ ，即证 $\hat{\sigma}_T^2(1) \xrightarrow{a.s.} \sigma^2(1)$