2019 秋季本科时间序列

第 3 次作业参考答案

2019 年 10 月 26 日

1. 已知 $U \sim \mathcal{U}\big([-\pi, \pi]\big)$, 则 U 的密度函数为

$$f(u) = \frac{1}{2\pi}, u \sim [-\pi, \pi]$$

故 X_t 的期望为

$$\begin{aligned}
\mathbb{E}X_t &= \int_{-\pi}^{\pi} \cos(\pi t + u) f(u) \mathrm{d}u \\
&= \int_{-\pi}^{\pi} \cos(\pi t + u) \frac{1}{2\pi} \mathrm{d}u \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(\pi t + u) \mathrm{d}u \\
&= \frac{1}{2\pi} \sin(\pi t + u)\Big|_{-\pi}^{\pi} \\
&= 0
\end{aligned}$$

故 X_t 的自协方差为

$$\begin{aligned}
\sigma_k^2 &= \mathrm{cov}(X_{t+k}, X_t) \\
&= \mathbb{E}\Big[\cos\big(\pi t + u\big)\cos\big(\pi(t+k) + u\big)\Big] \\
&= \int_{-\pi}^{\pi} \Big[\cos\big(\pi t + u\big)\cos\big(\pi(t+k) + u\big)\frac{1}{2\pi}\Big] \mathrm{d}u \\
&= \frac{1}{4\pi} \int_{-\pi}^{\pi} \Big[\cos\big(2\pi t + \pi k + 2u\big) + \cos\big(\pi k\big)\Big] \mathrm{d}u \\
&= \frac{1}{4\pi} \Big[\frac{1}{2}\sin\big(2\pi t + \pi k + 2u\big)\Big|_{-\pi}^{\pi} + \cos\big(\pi k\big) u\Big|_{-\pi}^{\pi}\Big] \\
&= \frac{1}{2}\cos\big(\pi k\big)
\end{aligned}$$

且 $\sigma_{k+2}^2 = \frac{1}{2}\cos\big(\pi(k+2)\big) = \frac{1}{2}\cos\big(\pi k + 2\pi\big) = \frac{1}{2}\cos\big(\pi k\big)$
故 σ_k^2 具有周期性, 且周期为 2

2. (a) 若 $X_0=0$, 则

$$\begin{aligned} X_t &= \rho X_{t-1}+\varepsilon_t \\ &= \rho^2 X_{t-2}+\rho\varepsilon_{t-1}+\varepsilon_t \\ &\vdots \\ &= \sum_{i=0}^{t-1}\rho^i\varepsilon_{t-i}+\rho^t X_0 \\ &= \sum_{i=0}^{t-1}\rho^i\varepsilon_{t-i} \end{aligned}$$

由 $\varepsilon_t \overset{\text{iid}}{\sim} N(0,\sigma_\varepsilon^2)$ 可知，其线性组合也服从正态分布
又 $\mathbb{E}(X_t)=0$,$\text{var}(X_t)=\sum_{i=0}^{t-1}\rho^{2i}\sigma_\varepsilon^2=\frac{1-\rho^{2t}}{1-\rho^2}\sigma_\varepsilon^2$,
即 $X_t\sim N(0,\frac{1-\rho^{2t}}{1-\rho^2}\sigma_\varepsilon^2)$ 则 X_t 的分布为

$$F_t(x)=\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi\frac{1-\rho^{2t}}{1-\rho^2}\sigma_\varepsilon^2}}e^{-\frac{x^2}{2\frac{1-\rho^{2t}}{1-\rho^2}\sigma_\varepsilon^2}}\,\mathrm{d}x$$

(b) 因 $|\rho|<1$, 所以

$$\begin{aligned} \lim_{t\to\infty}\sum_{i=0}^{t-1}\rho^{2i}\sigma_\varepsilon^2 &= \lim_{t\to\infty}\frac{1-\rho^{2t}}{1-\rho^2}\sigma_\varepsilon^2 \\ &= \frac{1}{1-\rho^2}\sigma_\varepsilon^2 \end{aligned}$$

此时 $X_t\sim N(0,\frac{1}{1-\rho^2}\sigma_\varepsilon^2)$
即 $F_t(x)$ 收敛，其极限分布为

$$F(x)=\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi\frac{1}{1-\rho^2}\sigma_\varepsilon^2}}e^{-\frac{x^2}{2\frac{1}{1-\rho^2}\sigma_\varepsilon^2}}\,\mathrm{d}x$$

(c) 由已知 $X_0\sim N(0,\frac{1}{1-\rho^2}\sigma_\varepsilon^2)$,
当 $X_t\sim N(0,\frac{1}{1-\rho^2}\sigma_\varepsilon^2)$ 时,

$$\begin{aligned} \mathbb{E}X_{t+1} &= \rho\mathbb{E}X_t+\mathbb{E}\varepsilon_{t+1} \\ &= 0 \\ \text{var}(X_{t+1}) &= \text{var}(\rho X_t)+\text{var}(\varepsilon_{t+1}) \\ &= (\frac{\rho^2}{1-\rho^2}+1)\sigma_\varepsilon^2 \\ &= \frac{1}{1-\rho^2}\sigma_\varepsilon^2 \end{aligned}$$

即 $X_{t+1}\sim N(0,\frac{1}{1-\rho^2}\sigma_\varepsilon^2)$, 分布不变
即 X_t 的分布均为该极限分布 ($t\geq 1$)，该分布为 $\{X_t\}$ 的平稳分布