2018 秋季本科时间序列

第五次作业答案

2018 年 11 月 22 日

1. (a) z_1, z_2 为 $A(z) = 1 - \phi_1 z - \phi_2 z^2$ 的零点，代入有

$$\begin{cases} 1 - \phi_1 z_1 - \phi_2 z_1^2 = 0 \\ 1 - \phi_1 z_2 - \phi_2 z_2^2 = 0 \end{cases}$$

得 $\phi_1 = \frac{z_1+z_2}{z_1 z_2}$，$\phi_2 = -\frac{1}{z_1 z_2}$, 从而有

$$\phi_2 + \phi_1 = \frac{z_1 + z_2 - 1}{z_1 z_2} = \frac{1}{z_1} + \frac{1}{z_2} - \frac{1}{z_1 z_2} = 1 - \left(1 - \frac{1}{z_1}\right)\left(1 - \frac{1}{z_2}\right)$$

$$\phi_2 - \phi_1 = -\frac{z_1 + z_2 + 1}{z_1 z_2} = -\frac{1}{z_1} - \frac{1}{z_2} - \frac{1}{z_1 z_2} = 1 - \left(1 + \frac{1}{z_1}\right)\left(1 + \frac{1}{z_2}\right)$$

又 $|z_1|, |z_2| > 1$ 有

$$|\phi_2| = \frac{1}{|z_1 z_2|} < 1 \qquad \left(1 \pm \frac{1}{z_1}\right)\left(1 \pm \frac{1}{z_2}\right) > 0$$

所以 ϕ_1, ϕ_2 满足限制条件：

$$\begin{cases} |\phi_2| < 1 \\ \phi_2 \pm \phi_1 < 1 \end{cases}$$

(b)

$$\begin{aligned} detA &= \phi_1[(-\phi_1\phi_2) - (\phi_1)] - (\phi_2 - 1)[(-\phi_2^2 - 1)] \\ &= -\phi_1^2\phi_2 - \phi_1^2 + \phi_2^3 - \phi_2^2 + \phi_2 - 1 \\ &= (\phi_2 + 1)(\phi_2^2 - 2\phi_2 - \phi_1^2 + 1) \\ &= (\phi_2 + 1)(\phi_2 + \phi_1 - 1)(\phi_2 - \phi_1 - 1) \end{aligned}$$

由 (a) 结论

$$|\phi_2| < 1 \qquad \phi_2 \pm \phi_1 < 1$$

有

$$(\phi_2+\phi_1-1)<0 \qquad (\phi_2-\phi_1-1)<0 \qquad (\phi_2+1)>0$$

所以

$$\det A>0$$

(c) 根据 Gramer 法则有

$$A^{-1}=\frac{A^*}{\det A}$$

其中 A^* 为伴随矩阵

$$A^*=\begin{bmatrix} \begin{vmatrix} \phi_1 & -1 \\ -\phi_1 & -\phi_2 \end{vmatrix} & -\begin{vmatrix} \phi_2-1 & 0 \\ -\phi_1 & -\phi_2 \end{vmatrix} & \begin{vmatrix} \phi_2-1 & 0 \\ \phi_1 & -1 \end{vmatrix} \\ -\begin{vmatrix} \phi_2 & -1 \\ 1 & -\phi_2 \end{vmatrix} & \begin{vmatrix} \phi_1 & 0 \\ 1 & -\phi_2 \end{vmatrix} & -\begin{vmatrix} \phi_1 & 0 \\ \phi_2 & -1 \end{vmatrix} \\ \begin{vmatrix} \phi_2 & \phi_1 \\ 1 & -\phi_1 \end{vmatrix} & -\begin{vmatrix} \phi_1 & \phi_2-1 \\ 1 & -\phi_1 \end{vmatrix} & \begin{vmatrix} \phi_1 & \phi_2-1 \\ \phi_2 & \phi_1 \end{vmatrix} \end{bmatrix}$$

进而有

$$\begin{bmatrix} \sigma_x^2(0) \\ \sigma_x^2(1) \\ \sigma_x^2(2) \end{bmatrix}=\frac{A^*}{\det A}\begin{bmatrix} 0 \\ 0 \\ \sigma_\varepsilon^2 \end{bmatrix}=\begin{bmatrix} \frac{(1-\phi_2)\sigma_\varepsilon^2}{(\phi_2+1)(\phi_2^2-2\phi_2-\phi_1^2+1)} \\ \frac{\phi_1\sigma_\varepsilon^2}{(\phi_2+1)(\phi_2^2-2\phi_2-\phi_1^2+1)} \\ \frac{(\phi_1^2-\phi_2^2+\phi_2)\sigma_\varepsilon^2}{(\phi_2+1)(\phi_2^2-2\phi_2-\phi_1^2+1)} \end{bmatrix}$$

也可以由 Gramer 法则直接计算

$$\sigma_x^2(0)=\frac{\det A_1}{\det A}=\frac{\begin{vmatrix} 0 & \phi_2-1 & 0 \\ 0 & \phi_1 & -1 \\ \sigma_\varepsilon^2 & -\phi_1 & -\phi_2 \end{vmatrix}}{\det A}=\frac{(1-\phi_2)\sigma_\varepsilon^2}{(\phi_2+1)(\phi_2^2-2\phi_2-\phi_1^2+1)}$$

$$\sigma_x^2(1)=\frac{\det A_2}{\det A}=\frac{\begin{vmatrix} \phi_1 & 0 & 0 \\ \phi_2 & 0 & -1 \\ 1 & \sigma_\varepsilon^2 & -\phi_2 \end{vmatrix}}{\det A}=\frac{\phi_1\sigma_\varepsilon^2}{(\phi_2+1)(\phi_2^2-2\phi_2-\phi_1^2+1)}$$

$$\sigma_x^2(2) = \frac{\det A_3}{\det A} = \frac{\begin{vmatrix} \phi_1 & \phi_2 - 1 & 0 \\ \phi_2 & \phi_1 & 0 \\ 0 & -\phi_1 & \sigma_\varepsilon^2 \end{vmatrix}}{\det A} = \frac{(\phi_1^2 - \phi_2^2 + \phi_2)\sigma_\varepsilon^2}{(\phi_2 + 1)(\phi_2^2 - 2\phi_2 - \phi_1^2 + 1)}$$

(d)

$$\begin{aligned}
\sigma_x^2(k) &= \mathrm{cov}(X_t, X_{t-k}) \\
&= \mathrm{cov}(\phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t, X_{t-k}) \\
&= \phi_1 \mathrm{cov}(X_{t-1}, X_{t-k}) + \phi_2 \mathrm{cov}(X_{t-2}, X_{t-k}) + \mathrm{cov}(\varepsilon_t, X_{t-k}) \\
&= \phi_1 \sigma_x^2(k-1) + \phi_2 \sigma_x^2(k-2) + \mathrm{cov}(\varepsilon_t, X_{t-k})
\end{aligned}$$

由作业 4 附加题 (a) 的结论有

$$X_{t-k} = \sum_{i=0}^{\infty} \left[\frac{1}{1 - \frac{z_1}{z_2}} \left(\frac{1}{z_1} \right)^i + \frac{1}{1 - \frac{z_2}{z_1}} \left(\frac{1}{z_2} \right)^i \right] \varepsilon_{t-k-i} = \sum_{i=0}^{\infty} \theta_i \varepsilon_{t-k-i}$$

根据白噪声的性质

$$\mathrm{cov}(\varepsilon_t, X_{t-k}) = \sum_{i=0}^{\infty} \theta_i \mathrm{cov}(\varepsilon_t, \varepsilon_{t-k-i}) = 0$$

所以

$$\sigma_x^2(k) = \phi_1 \sigma_x^2(k-1) + \phi_2 \sigma_x^2(k-2)$$