2018 秋季本科时间序列

第四次作业答案

2018 年 11 月 13 日

1. (a) $\{\theta_i\}_{i=0}^{\infty}$ 绝对和收敛从而有部分和序列 $T_k = \sum_{i=0}^{k} |\theta_i|$ 为 Cauchy 序列。
   对 $\forall \frac{\epsilon}{\bar{a}} > 0$，$\exists K \in \mathbb{N}$，使得 $\forall m > n \geq K$，有

$$|T_m - T_n| = \sum_{i=n+1}^{m} |\theta_i| \leq \frac{\epsilon}{\bar{a}}$$

   从而，对 $S_k = \sum_{i=0}^{k} a_i \theta_i$，且 $|a_i| \leq \bar{a}$，有

$$|S_m - S_n| = |\sum_{i=n+1}^{m} a_i \theta_i| \leq \sum_{i=n+1}^{m} |a_i \theta_i| \leq \bar{a} \sum_{i=n+1}^{m} |\theta_i| \leq \bar{a} \frac{\epsilon}{\bar{a}} = \epsilon$$

   因此，部分和序列 S_k 为 Cauchy 序列，所以级数 $\sum_{i=0}^{\infty} a_i \theta_i$ 收敛

   (b) 因为 $\{Y_i\}_{i=0}^{\infty}$ 为平稳序列
   对 $\forall t$, 有 $\mathbb{E} Y_t = \mu_Y$
   且 μ_Y 与 t 无关。从而有

$$\begin{aligned}
\mathbb{E} X_t &= \mathbb{E}\left[\sum_{i=0}^{\infty} \theta_i Y_{t-i}\right] = \sum_{i=0}^{\infty} \mathbb{E}\left[\theta_i Y_{t-i}\right] \\
&= \sum_{i=0}^{\infty} \theta_i \mathbb{E}[Y_{t-i}] \\
&= \mu_Y \sum_{i=0}^{\infty} \theta_i
\end{aligned}$$

   θ_i 与 t 无关，所以 $\mathbb{E} X_t$ 与 t 无关

(c)

$$\begin{aligned}\text{cov}(X_t, Y_{t-j}) &= \text{cov}\left(\sum_{i=0}^{\infty}\theta_i Y_{t-i}, Y_{t-j}\right)\\ &= \sum_{i=0}^{\infty}\theta_i \text{cov}\left(Y_{t-i}, Y_{t-j}\right)\\ &= \sum_{i=0}^{\infty}\theta_i \sigma_Y^2(j-i)\end{aligned}$$

θ_i 和 $\sigma_Y^2(j-i)$ 与 t 无关，所以 $\text{cov}(X_t, Y_{t-j})$ 与 t 无关
对 $\forall k$, $|\sigma_Y^2(k)| \le \sigma_Y^2$, 所以序列 $\{\sigma_Y^2(j-i)\}_{i=0}^{\infty}$ 有界
由题 (a) 结论，级数

$$\text{cov}(X_t, Y_{t-j}) = \sum_{i=0}^{\infty}\sigma_Y^2(j-i)\theta_i$$

收敛

(d)

$$\begin{aligned}\text{cov}(X_t, X_{t-k}) &= \text{cov}\left(X_t, \sum_{i=0}^{\infty}\theta_i Y_{t-k-i}\right)\\ &= \sum_{i=0}^{\infty}\theta_i \text{cov}(X_t, Y_{t-k-i})\\ &= \sum_{i=0}^{\infty}\theta_i\left[\sum_{j=0}^{\infty}\sigma_Y^2(k+i-j)\theta_j\right]\\ &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\theta_i\theta_j\sigma_Y^2(k+i-j).\end{aligned}$$

同理，θ_i，θ_j 和 $\sigma_Y^2(k+i-j)$ 均与 t 无关
所以 $\text{cov}(X_t, X_{t-k})$ 与 t 无关。
由上述推导，有 $\sum_{j=0}^{\infty}\theta_j\sigma_Y^2(k+i-j)$ 有界，$\{\theta_i\}_{i=0}^{\infty}$ 收敛，所以级数

$$\text{cov}(X_t, X_{t-k}) = \sum_{i=0}^{\infty}\theta_i\sum_{j=0}^{\infty}\theta_j\sigma_Y^2(k+i-j)$$

收敛

2. 附加题

(a) λ, η 为 $A(z) = 1 - \phi_1 z - \phi_2 z^2$ 的零点，代入有

$$\begin{cases} 1 - \phi_1\lambda - \phi_2\lambda^2 = 0 \\ 1 - \phi_1\eta - \phi_2\eta^2 = 0 \end{cases}$$

得 $\phi_1 = \frac{\lambda+\eta}{\lambda\eta}$，$\phi_2 = -\frac{1}{\lambda\eta}$, 从而有

$$A^{-1}(z) = \frac{1}{1 - \phi_1 z - \phi_2 z^2} = \frac{1}{1 - \frac{z}{\lambda}} \cdot \frac{1}{1 - \frac{z}{\eta}}$$

若

$$A^{-1}(z) = \frac{a}{1 - \frac{z}{\lambda}} + \frac{b}{1 - \frac{z}{\eta}}$$

对应可解

$$a = \frac{1}{1 - \frac{\lambda}{\eta}}, \qquad b = \frac{1}{1 - \frac{\eta}{\lambda}}$$

进而

$$\begin{aligned} X_t &= A^{-1}(\mathscr{L})\varepsilon_t \\ &= \frac{1}{1 - \frac{\lambda}{\eta}} \cdot \frac{1}{1 - \frac{\mathscr{L}}{\lambda}}\varepsilon_t + \frac{1}{1 - \frac{\eta}{\lambda}} \cdot \frac{1}{1 - \frac{\mathscr{L}}{\eta}}\varepsilon_t \\ &= \frac{1}{1 - \frac{\lambda}{\eta}} \sum_{i=0}^{\infty} \left(\frac{1}{\lambda}\right)^i \mathscr{L}^i \varepsilon_t + \frac{1}{1 - \frac{\eta}{\lambda}} \sum_{j=0}^{\infty} \left(\frac{1}{\eta}\right)^j \mathscr{L}^j \varepsilon_t \\ &= \sum_{i=0}^{\infty} \left[\frac{1}{1 - \frac{\lambda}{\eta}} \left(\frac{1}{\lambda}\right)^i + \frac{1}{1 - \frac{\eta}{\lambda}} \left(\frac{1}{\eta}\right)^i\right] \varepsilon_{t-i} \\ &= \sum_{i=0}^{\infty} \theta_i \varepsilon_{t-i} \end{aligned}$$

令 $\mathrm{var}(\varepsilon_t)=\sigma^2$, 进而计算 X_t 的方差

$$
\begin{aligned}
\mathrm{var}(X_t) &= \mathrm{var}\left(\sum_{i=0}^{\infty}\theta_i\varepsilon_{t-i}\right)\\
&= \sigma^2\sum_{i=0}^{\infty}\theta_i^2\\
&= \sigma^2\sum_{i=0}^{\infty}\left[\frac{1}{1-\frac{\lambda}{\eta}}\left(\frac{1}{\lambda}\right)^i+\frac{1}{1-\frac{\eta}{\lambda}}\left(\frac{1}{\eta}\right)^i\right]^2\\
&= \sigma^2\left[\left(\frac{\eta}{\eta-\lambda}\right)^2\sum_{i=0}^{\infty}\left(\frac{1}{\lambda^2}\right)^i-\frac{2\lambda\eta}{(\eta-\lambda)^2}\sum_{i=0}^{\infty}\left(\frac{1}{\lambda\eta}\right)^i+\left(\frac{\lambda}{\eta-\lambda}\right)^2\sum_{i=0}^{\infty}\left(\frac{1}{\eta^2}\right)^i\right]\\
&= \sigma^2\left[\left(\frac{\eta}{\eta-\lambda}\right)^2\lim_{i\to\infty}\left(\frac{1-\frac{1}{\lambda^{2i}}}{1-\frac{1}{\lambda^2}}\right)-\frac{2\lambda\eta}{(\eta-\lambda)^2}\lim_{i\to\infty}\left(\frac{1-\frac{1}{\lambda\eta^i}}{1-\frac{1}{\lambda\eta}}\right)+\left(\frac{\lambda}{\eta-\lambda}\right)^2\lim_{i\to\infty}\left(\frac{1-\frac{1}{\eta^{2i}}}{1-\frac{1}{\eta^2}}\right)\right]\\
&= \sigma^2\left(\frac{\lambda\eta}{\eta-\lambda}\right)^2\left[\frac{1}{\lambda^2-1}-\frac{2}{\lambda\eta-1}+\frac{1}{\eta^2-1}\right]\\
&= \sigma^2\left(\frac{\lambda\eta}{\eta-\lambda}\right)^2\left[\frac{\lambda^3\eta+\lambda\eta^3+\lambda^2+\eta^2-2\lambda\eta-2\lambda^2\eta^2}{(\lambda^2-1)(\lambda\eta-1)(\eta^2-1)}\right]\\
&= \frac{\sigma^2(\lambda\eta+1)(\lambda\eta)^2}{(\lambda\eta-1)[\lambda^2\eta^2-(\lambda^2+\eta^2)+1]}
\end{aligned}
$$

(b) 根据 X_t 表达式，有：

$$
\begin{aligned}
\mathrm{var}(X_t) &= \mathrm{cov}(X_t,X_t)\\
&= \mathrm{cov}\left(\frac{1}{(1-\mathscr{L}/\lambda)}Y_t,\frac{1}{(1-\mathscr{L}/\lambda)}Y_t\right)\\
&= \mathrm{cov}\left(\sum_{i=0}^{\infty}\left(\frac{1}{\lambda}\right)^iY_{t-i},\sum_{j=0}^{\infty}\left(\frac{1}{\lambda}\right)^jY_{t-j}\right)\\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(\frac{1}{\lambda}\right)^{i+j}\mathrm{cov}(Y_{t-i},Y_{t-j})
\end{aligned}
$$

又 $Y_t=\frac{1}{(1-\mathscr{L}/\lambda)}\varepsilon_t$，所以 Y_t 为 $AR(1)$ 过程：

$$
Y_t=\frac{1}{\lambda}Y_{t-1}+\varepsilon_t
$$

所以有：

$$
\sigma_Y(k)=\mathrm{cov}(Y_t,Y_{t-k})=\left(\frac{1}{\lambda}\right)^k\frac{1}{1-(1/\lambda)^2}\sigma^2
$$

所以有 $\sigma_Y^2(j-i) = \left(\frac{1}{\lambda}\right)^{j-i} \frac{1}{1-(1/\lambda)^2}\sigma^2$, 代入得:

$$\begin{aligned}
\text{var}(X_t) &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(\frac{1}{\lambda}\right)^{i+j}\text{cov}(Y_{t-i}, Y_{t-j}) \\
&= \frac{1}{1-(1/\lambda)^2}\sigma^2\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(\frac{1}{\lambda}\right)^{i+j}\left(\frac{1}{\lambda}\right)^{|j-i|} \\
&= \frac{1}{1-(1/\lambda)^2}\sigma^2\sum_{i=0}^{\infty}\left[\sum_{j=0}^{i}\left(\frac{1}{\lambda}\right)^{2i}+\sum_{j=i+1}^{\infty}\left(\frac{1}{\lambda^2}\right)^{j}\right] \\
&= \frac{1}{1-(1/\lambda)^2}\sigma^2\left[\sum_{i=0}^{\infty}\frac{i}{\lambda^{2i}}+\frac{\lambda^2}{(\lambda^2-1)^2}\right] \\
&= \frac{1}{1-(1/\lambda)^2}\sigma^2\left[\frac{\lambda^4}{(\lambda^2-1)^2}+\frac{\lambda^2}{(\lambda^2-1)^2}\right] \\
&= \frac{\lambda^4(\lambda^2+1)}{(\lambda^2-1)^3}
\end{aligned}$$

(c) 当 $\lambda \neq \eta$

$$\text{var}(X_t) = \frac{\sigma^2(\lambda\eta+1)(\lambda\eta)^2}{(\lambda\eta-1)[\lambda^2\eta^2-(\lambda^2+\eta)^2+1]} = \frac{\sigma^2(\lambda\eta+1)(\lambda\eta)^2}{(\lambda\eta-1)[\lambda^2\eta^2-(\lambda+\eta)^2+2\lambda\eta+1]}$$

因为 λ, η 是方程 $1-\phi_1 z-\phi_2 z^2=0$ 的两个解，从而有：

$$\lambda+\eta = -\frac{\phi_1}{\phi_2}, \qquad \lambda\eta = -\frac{1}{\phi_2}$$

代入原式得：

$$\text{var}(X_t) = \frac{(1-\phi_2)\sigma^2}{(\phi_2+1)(\phi_2^2-2\phi_2-\phi_1^2+1)}$$

当 $\lambda = \eta$ 时,$\lambda = -\frac{2}{\phi_1}$ 代入得：

$$\text{var}(X_t) = \frac{16(4+\phi_1^2)}{(4-\phi_1^2)^3}$$