Lecture Notes in Microeconomics: General Equilibrium

Pradeep Dubey
Stony Brook University

Yan Liu[1]
Wuhan University

Comments welcome
First version: Spring, 2010
Latest Revision: September 19, 2017

[1]Send comments to Yan Liu at yanliu.ems@whu.edu.cn.

Contents

Chapter 1

Pure Exchange Economy

▶ Feb.1, 2010

Throughout these lecture notes, we maintain the following notations: Denote the non-negative orthant of a Euclidean space of dimension n by $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n | x_i \geq 0, i = 1, \ldots, n\}$; $\mathbb{R}^n_+ \setminus \{0\}$ represents the non-negative orthant without the origin; for any $x \in \mathbb{R}^n$, $x \gg 0$ means $x_i > 0$, $\forall\, i$, $x \geq 0$ means $x_k \geq 0$, $\forall\, i$, and $x \gneq 0$ means $x \geq 0$, and $x_i > 0$ for some i; and denotes the dot product of two vector $x, y \in \mathbb{R}^n$ by $x \cdot y \equiv \sum_i x_i y_i$. Moreover, for all $x \in \mathbb{R}^n$, we define the *vector norm* of x as $||x|| = \max_{1 \leq i \leq n}\{|x_i|\}$.

1.1 Pure Exchange Economy $\mathscr{E}$

Let $H = \{1, \ldots, H\}$ be a set of households, and $L = \{1, \ldots, L\}$ be a set of commodities, in which labor, but not food, might be the most important one. Naturally, we use $\mathbb{R}^L_+$ to represent the *commodity space*, in which each point (vector) represents a *consumption bundle*. For each household $h \in H$, two most important characteristics are its preference and its initial endowment. For the former one, a utility function $u^h : \mathbb{R}^L_+ \to \mathbb{R}$ is always assumed to represent the preference of h, and up to now, no particular properties of preferences, e.g., convexity, monotonicity, etc., are assumed; for the latter one, $e^h = (e^h_1, \ldots, e^h_L) \in \mathbb{R}^L_+$ denotes the initial endowment of h.

To sum up, a pure exchange economy is defined as a collection of the set of households with their preferences and endowments, and is denoted by $\mathscr{E} = (e^h, u^h)_{h \in H}$.

1.2 Competitive Equilibrium in $\mathscr{E}$

Given a pure exchange economy $\mathscr{E} = (e^h, u^h)_{h \in H}$, we could define the concept of *competitive equilibria*.

Before proceeding to the formal definition of competitive equilibria, it is worth to point out several basic assumptions underlying the concept of competitive equilibria. First, we presume that there exists a specific market for each commodity $\ell \in L$, in which a unique price p_ℓ is to be announced. Second, all the commodities are private goods and perfectly divisible, the latter of which also gives the rationale of assuming $\mathbb{R}^L_+$ to be the commodity space. Third, there is no tax or transaction cost in this model. Forth, households in this model only take into account of the prices in the markets, that is, they don't care about how many people are there in the markets, or what is the total amount of a certain commodity available in the economy, and in fact they don't even need to know all this sort of information. By using the word "price" here, there is no necessity of including *money* into this model economy, and actually, there is *no* money in such an economy, *nor* anywhere else in these lecture notes. We will talk about price recurrently, but in no way has it to do with money. After all, what we will concentrate on are merely exchange rates, or to say, relative prices.

Given a *price system*, specified by a *price vector* $p = (p_1, \dots, p_L) \in \mathbb{R}^L_+ \setminus \{0\}$,[1] we can always normalize p to have $\sum_{\ell \in L} p_\ell = 1$. Doubling all prices in this model doesn't matter at all, since it makes no change of the budget set. Of course, in real life, this can not be true, since it's always the case that all prices go up while the income remains the same.

Despite the mild normative tone, as if there were a superb social planner, we shall still use the term *allocation*, whenever we are referring to a set of consumption bundles $(x^h)_{h \in H}$ for all households in the economy which satisfies the additional condition that the aggregate bundle is feasible given the aggregate endowment, i.e., $\sum_{h \in H} x^h \leq \sum_{h \in H} e^h$. Just to mention it a little bit, an allocation $(x^h)_{h \in H}$ could also be viewed as a vector in the product commodity space $\mathbb{R}^{HL}_+ = \mathbb{R}^L_+ \times \cdots \times \mathbb{R}^L_+$.

We make two more assumptions on initial endowments and preferences in this economy.

[1] More precisely, by assuming all commodities are *goods* — this is indeed what we are going to assume throughout these lecture notes — it suffices to restrict the analysis to the case of $p \in \mathbb{R}^L_+ \setminus \{0\}$, ignoring the trivial case of $p = 0$. However, if some commodities are *bads*, then perhaps we should extend the domain of price vectors to $\mathbb{R}^L \setminus \{0\}$, at least in the case that we want to treat these bads in the same way as goods.

A.1. Each household has some positive amount of endowment for some commodities, or in symbol, $e^h \in \mathbb{R}^L_+ \setminus \{0\}$; moreover, $\sum_{h\in H} e^h \gg 0$, that is every named commodity does exist in the economy.

A.2. For all $h \in H$ and $x, y \in \mathbb{R}^L_+$, if $x \gg y$ then $u^h(x) > u^h(y)$, and in addition, if $x \geq y$, then $u^h(x) \geq u^h(y)$. This is also termed as *weak monotonicity* of utility (preference).[2]

Now we are at the stage of stating the formal definition of competitive equilibria in $\mathscr{E}$.

Definition 1.1. *A collection of a price system and an allocation of commodities* $\langle p, x^1, \ldots, x^H \rangle$, *is a* competitive equilibrium $(C.E.)$ *in a pure exchange economy* $\mathscr{E} = (e^h, u^h)_{h\in H}$ *if the following conditions are satisfied:*

- Utility maximization. *For each household* $\forall h \in H$, x^h *maximizes its utility over the budget set, i.e.,*
$$x^h \in \underset{y\in B^h(p)}{\operatorname{argmax}}\, u^h(y)$$
where $B^h(p) \equiv \{y \in \mathbb{R}^L_+ : p \cdot y \leq p \cdot e^h\}$ *is the budget set of* h *defined by the equilibrium price system* p.[3]

- Market clearing. *The equilibrium allocation clears the market, i.e.,*
$$\sum_{h\in H}(x^h - e^h) = 0.$$

Remark 1.1. In the definition of the budget set, each point $y \in B^h(p)$ is an affordable (final) consumption bundle, and the inequality constraint $p \cdot y \leq p \cdot e^h$ can be interpreted in two ways: (i) a two-step procedure, that is first sell out all the endowment, and then buy what is desired; and (ii) a supply-demand procedure, that is first transform the inequality into $\sum_{\ell\in L} p_\ell(y_\ell - e^h_\ell) = p \cdot (y - e^h) \leq 0$, and whenever $y_\ell - e^h_\ell < 0$, h supplies some ℓ to the market, while whenever $y_\ell - e^h_\ell > 0$, h demands some ℓ from the market.

[2]The standard definition of weak monotonicity is just the second half of what stated here. By including the first half into our definition of weak monotonicity, we are essentially assuming *locally non-satiated* preference without explicitly mentioning this term. Obviously, the first half in the definition implies local non-satiation.

[3]For a function $f : Z \to \mathbb{R}$, where $Z \subset \mathbb{R}^n$, define $\operatorname{argmax}_{z\in Z} f(z)$ as the set of all points in Z that maximize $f(\cdot)$ over Z; if the maximum of $f(\cdot)$ over Z does not exist, then define $\operatorname{argmax}_{z\in Z} f(z)$ as empty set.

Remark 1.2. If we assume complete freedom for households on choosing the utility maximizing bundles by themselves, then the equilibrium might fail to exist because the self-selected allocation might violate the market clearing condition, even if the same price system does constitute a competitive equilibrium with some other allocation (which is compatible with this price system). Conceptually, we economists choose the specific utility maximizing bundles for the households, and that's why we present the equilibrium allocation at the very beginning in defining C.E.

1.3 The Core of $\mathscr{E}$

For any subset $S \subset H$, where the number of the elements is also denoted by S, we call $(x^h)_S \in \mathbb{R}_+^{LS}$ an S-allocation, if $\sum_{h \in S} x^h \leq \sum_{h \in S} e^h$; and an H-allocation just indicates an allocation in the economy.

Definition 1.2. *An H-allocation $(x^h)_{h \in H}$ in $\mathscr{E} \equiv (e^h, u^h)_{h \in H}$ is called a core allocation of $\mathscr{E}$, if there exists no S-allocation $(y^h)_{h \in S}$ where S is a subset of H, such that*

$$\begin{aligned} &u^h(y^h) \geq u^h(x^h), \quad \forall\, h \in S; \\ &u^h(y^h) > u^h(x^h), \quad \textit{for at least one } h \in S. \end{aligned}$$

The set of all core allocations is simply defined as the core of $\mathscr{E}$

When $S = H$, this definition coincides with *Pareto optimality*, thus each core allocation is Pareto optimal. However, the converse is not true. A simplest example goes like follows. Consider an allocation of assigning all the endowment to one household and leaving nothing to everyone else. Obviously this allocation is Pareto optimal yet not a core allocation.

Definition 1.3. *$\langle x^1, \ldots, x^H \rangle$ is a competitive allocation of $\mathscr{E}$, if there exists a price system p such that $\langle p, x^1, \ldots, x^H \rangle$ is a competitive equilibrium in $\mathscr{E}$.*

The desired welfare property of competitive equilibrium is justified by the following theorem, from which we could directly get the famous *first theorem of welfare economics* as a corollary. Note assumption A.2 listed in the precede section is maintained.

Theorem 1.1. *Every competitive allocation of $\mathscr{E}$ is a core allocation.*

Corollary 1.2. *Every competitive allocation is Pareto optimal.*

▶ Feb.3, 2010

Proof of Theorem 1.1. Let $\langle p, x^1, \ldots, x^H \rangle$ be a C.E. of $\mathscr{E}$. Suppose in converse, the competitive allocation is not a core allocation, then there exists an S-allocation $(y^h)_{h \in S}$ such that $u^h(y^h) \geq u^h(x^h)$ for any $h \in S$ with strict inequality for some h.

First, we claim that $p \cdot y^h \geq p \cdot e^h$, $\forall h \in S$. If not, there exists an $h \in S$ such that $p \cdot y^h < p \cdot e^h$. Then we can find $\epsilon > 0$ small enough such that $y^h \ll y^h_\epsilon \triangleq y^h + (\epsilon, \ldots, \epsilon)$ and $p \cdot y^h_\epsilon < p \cdot e^h$, i.e. $y^h_\epsilon \in B^h(p)$. Since $y^h_\epsilon \gg y^h$ and by the assumption of weak monotonicity, we have $u^h(y^h_\epsilon) > u^h(y^h)$. By the definition of S-allocation, we have $u^h(y^h) \geq u^h(x^h)$. Thus $u^h(y^h_\epsilon) > u^h(x^h)$. But this contradicts the fact that x^h maximizes h's utility over $B^h(p)$ since y^h_ϵ is also in the budget set.

Second, we claim that there exists an h such that $p \cdot y^h > p \cdot e^h$. By definition of S-allocation, there must be at least one $h \in S$ such that $u^h(y^h) > u^h(x^h)$. For this h, if $p \cdot y^h \leq p \cdot e^h$, then $y^h \in B^h(p)$. But again, this contradicts the fact that x^h maximizes $u^h(\cdot)$ over $B^h(p)$.

Combining the two claims, we must have $\sum_{h \in S} p \cdot y^h > \sum_{h \in S} p \cdot e^h$. However, since $(y^h)_{h \in S}$ is an S-allocation, we have $\sum_{h \in S} y^h \leq \sum_{h \in S} e^h$, and multiply each side by p, we have $\sum_{h \in S} p \cdot y^h \leq \sum_{h \in S} p \cdot e^h$. This contradiction proves our initial assumption is not true and thus complete the proof. ∎

Remark 1.3. As shown in the proof, it suffices to assume that the preferences satisfy

$$\text{if } x \gg y \text{ then } u^h(x) > u^h(y),$$

which is the first half of our definition of weak monotonicity.

Now, let us think about a five sentences proof for a slightly modified version of theorem 1.1, which is known as *Debreu's proof*.[4] To this end, we first impose two more assumptions about the preference, strong monotonicity, i.e., whenever $x \gneq y$ then $u^h(x) > u^h(y)$, and continuity. Notice that under these assumptions, if an S-allocation makes it invalid for a competitive allocation to be in the core, then we can always slightly reallocate the S-allocation

[4] Somebody said Debreu demonstrated this result and gave an five sentences proof in his Nobel Prize speech. However, there has no such an argument in neither Debreu's speech nor his prize lecture.

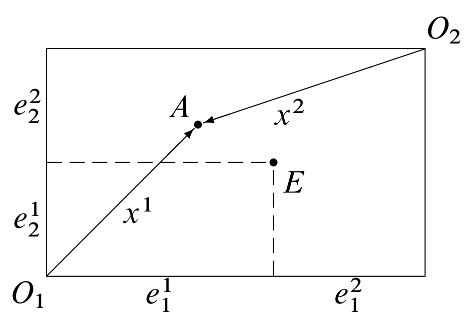
such that every household in S has a strictly higher utility than that in the competitive allocation.[5]

With this fact, the *Debreu's proof* can be showed as follows: First, notice whenever there is a red S-allocation which makes the blue C.E. not in the core, we can simply assume every household in S is strictly better off than in the blue C.E.; but if so, each household would have a higher total value of consumption bundle in this S-allocation than that of its endowment, which were to imply that the aggregate value of the S-allocation exceeds that of the total endowments in S.

The term "core" originates from von Neumann and Morgenstern (1947), where they defined a closely related concept "stable set", something located at the center of the relevant geometrical structure. However, the idea of core allocation dates back to Edgeworth.

1.4 Edgeworth Box

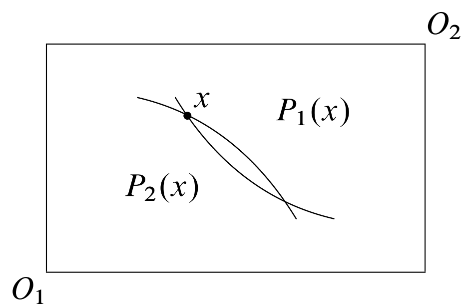
Consider the simplest pure exchange economy with two households and two commodities. Let $H = \{1, 2\}$ and $L = \{1, 2\}$, together with the utility $u^1(x_1^1, x_2^1),\ u^2(x_1^2, x_2^2)$, as well as initial endowments $e^1 = (e_1^1, e_2^1),\ e^2 = (e_1^2, e_2^2)$. Let $T = [0, e_1^1 + e_1^2] \times [0, e_2^1 + e_2^2]$ be the set of all feasible consumption bundles of each household in this economy. We can use a rectangular to represent the set T, as in the following figure. And we call this rectangular a *Edgeworth box*.



[5]There exists some $h \in S$ such that $u^h(y^h) > u^h(x^h)$, and use the same argument as in the proof of theorem 1.1, we have $p \cdot y^h > p \cdot e^h$. Moreover, this implies that h could distribute some commodities in its consumption bundle evenly across all other households in S by a sufficiently small amount, and by the continuity of utility, keep its utility in the S-allocation still higher than that in the competitive allocation. Now, every household else in S will achieve a higher utility by the strong monotonicity of utility.

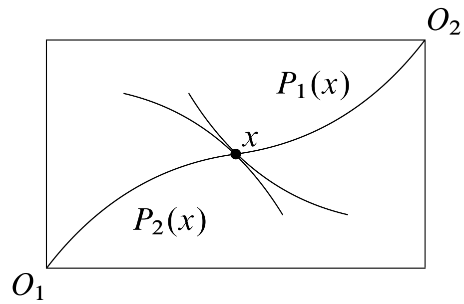
In this box, E represents the initial endowments of the two households, and A represents an allocation which exhausts the total endowments. It turns out that the Edgeworth box is a of great use in illustrating the connection of three concepts, the Pareto optimal allocation, the competitive allocation and the core allocation, in a rather intuitive way.

Let us focus on the simplest case, namely, the utility functions of the two households are strictly quasi-concave, strongly monotone and continuously differentiable. Let $P_h(x) = \{y \in T | u^h(y) \geq u^h(x)\}$ be the set of consumption bundles for each $h \in H$ with which h is at least as happy as having x. Observe, for instance from the following box, that if the intersection of $P_1(x)$ and $P_2(x)$ has a non-empty interior region, then the allocation x can not be a Pareto optimal.[6]

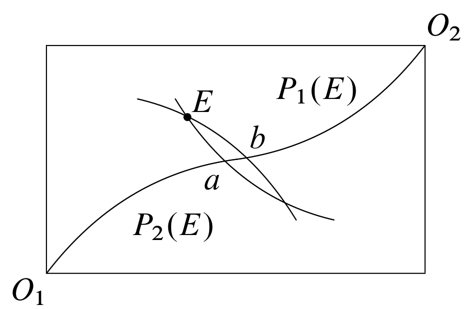


We conclude that Pareto optimality can be achieved only in the case that two indifference curves intersect with each other at a single point, i.e., they are tangent with each other at this point. From our assumptions about the utility functions, the set of allocations which are Pareto optimal constitutes a continuous curve connecting the two origins. We call this curve the *contract curve* of the economy. Also notice that at each Pareto optimal allocation x, there exists a unique straight line that is tangent with $P_1(x)$ and $P_2(x)$ at x. We call this line the *separating line* of the two households.

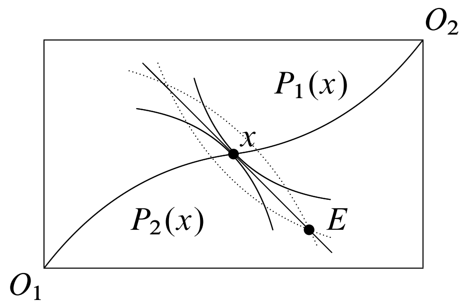
[6]Recall that an allocation x is a set of consumption bundles $(x^h)_{h \in H}$. As a slight abuse of notations, we will simply write down $P_h(x)$ indicating the exact form $P_h(x_h)$.



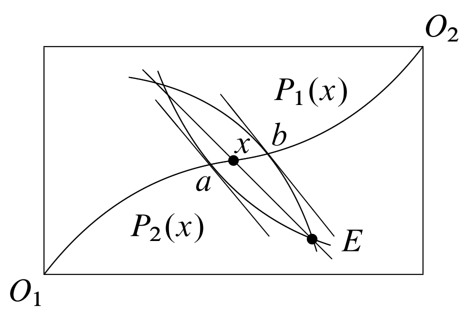
Given the initial endowments $E = (e^1, e^2)$, we can find out the core allocations in this economy. Note in this case, the possible S-allocations consist of only three sets, two households respectively and H as a whole. Therefore once an allocation satisfies both Pareto optimality condition and rationality condition, i.e., each household with this allocation is no worse than with its initial endowment, then this allocation must be a core allocation. It can be easily seen that a fraction of the contract curve constitutes the set of core allocations in this economy. For example, in the following box, the set of core allocations is the fraction of the contract curve between a and b.



Next, consider the competitive allocations in this economy. In an Edgeworth box, a competitive equilibrium $\langle p, x\rangle$ can be viewed geometrically as defining a separating line which goes through the initial endowments E and the allocation point x, while the latter one is also the tangent point of both $P_h(x)$, $h = 1, 2$, and the separating line. Moreover, the price vector is the normal vector of the separating line. Following this geometrical interpretation, it is obvious that the competitive allocation x is Pareto optimal (the separating line and the contract curve intersect at x), thus x is also a core allocation, as shown in the following Edgeworth box.



The discussion above also provides some geometrical insight of the first theorem of welfare economics. Moreover, by using the Edgeworth box, we can give a heuristic yet intuitive proof the existence of C.E. in this 2 by 2 economy. Through the initial endowment point E, we can draw two indifference curves for households 1 and 2, which intersect with the contract curve at a and b respectively. Since the preferences are convex (utility functions are quasi-concave), the tangent line to 1's indifference curve at a is on the left side of E, while for 2 is on the right side. By the assumption of continuous differentiability of utility functions, the separating lines at each point of the contract curve between a and b will change continuously, hence one of them, say the separating line at x must go through E. Therefore, the allocation x together with the normal vector of the separating line joining x and E consist of a competitive equilibrium.



Chapter 2

Nash Equilibrium

► Feb.8, 2010

2.1 N.E. in A Game

Let $N \equiv \{1, \dots, N\}$ be a set of players, and for each $n \in N$, define $S^n \equiv$ strategy set of player n. Put $S \equiv S^1 \times \dots \times S^N$, and define $\pi^n : S \to \mathbb{R}$ as payoff function for n. A game is given by $\Gamma = (S^n, \pi^n)_{n \in N}$.

Given $s \equiv (s^1, \dots, s^N) \in S$ and $t \in S^n$, $(s|_n t) \equiv (s^1, \dots, s^{n-1}, t, s^{n+1}, \dots, s^N) \in S$ denotes the *unilateral deviation* of player n from s^n to t at s.

Definition 2.1. *Define* $\beta^n(s) = \operatorname{argmax}_{t \in S^n} \pi^n(s|_n t)$ *as the best reply set of* n *at* s.

Definition 2.2. *A strategy profile* $s = (s^1, \dots, s^N)$ *is called a Nash equilibrium* $(N.E.)$ *of* $\Gamma = (S^n, \pi^n)_{n \in N}$ *if* $s^n \in \beta^n(s)$ $\forall n \in N$.

Definition 2.3. *Let the best reply correspondence* $\beta : S \rightrightarrows S$ *of* Γ *be defined in the following way,* $\forall s \in S$

$$\begin{array}{ccccccc} \beta(s) \equiv & \beta^1(s) & \times & \cdots & \times & \beta^N(s) & \\ & \cap & & \cdots & & \cap & \\ & S^1 & \times & \cdots & \times & S^N & \equiv S, \end{array}$$

where $\rightrightarrows$ *represents correspondence (point-to-set map).*

With this notation, the strategy profile s is a *N.E.* iff $s \in \beta(s)$, i.e., a fixed point of the best reply correspondence. In order to prove the existence of *N.E.*, we need following mathematical preliminaries. And throughout the remaining of this chapter, we restrict to Euclidean space.

We have mentioned "correspondence" several times, now let's give a formal definition.

Definition 2.4. *We call φ as a* correspondence *from X to Y, where $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$, if $\forall x \in X$, $\varphi(x)$ is a non-empty subset of $\mathbb{R}^n$, and this correspondence is denoted as $\varphi : X \rightrightarrows Y$.*

Since each subset of Y can be regarded as a point in the power set of Y,[1] the correspondence φ is nothing but a map from X to the power set of Y.[2] Moreover, if $\forall\, x \in X$, $\varphi(x)$ is a singleton, then φ is an ordinary map (function) from X to Y.

As in the case of function, some kind of continuity concept for correspondences is always desirable. It turns out that following definitions are most useful for economic applications. Several concepts are involved in defining the continuity for correspondences.

Definition 2.5. *Let $\varphi : X \rightrightarrows Y$ be a correspondence, the set $G_\varphi = \{(x, y) \in X \times Y | y \in \varphi(x)\}$ is called the* graph *of φ.*

Definition 2.6. *We call a correspondence φ to be upper semi-continuous $(u.s.c.)$ on X if G_φ is a closed set in $X \times Y$, which is equivalent to either one of the following conditions:*

- *We say φ to be u.s.c. at $x \in X$, if $(x_n, y_n) \to (x, y) \in X \times Y$ and $(x_n, y_n) \in G_\varphi$, then $(x, y) \in G_\varphi$; and we say φ to be u.s.c. on X, if φ is u.s.c. $\forall\, x \in X$.*
- *We say φ to be u.s.c. at $x \in X$, if $x_n \to x$, $y_n \to y \in Y$ and $y_n \in \varphi(x_n)$, then $y \in \varphi(x)$; we say φ to be u.s.c. on X, if φ is u.s.c. $\forall\, x \in X$.*

We will use the term *upper hemi-continuous* (u.h.c) in the same meaning of u.s.c.

Definition 2.7. *We call a correspondence φ to be lower semi-continuous $(l.s.c.)$ at $x \in X$ if for any sequence $x_n \to x$ and for any point $y \in \varphi(x)$, there exists $y_n \in \varphi(x_n)$ for each n s.t. $y_n \to y$; if φ is l.s.c. $\forall\, x \in X$, then we say φ to be l.s.c. on X.*

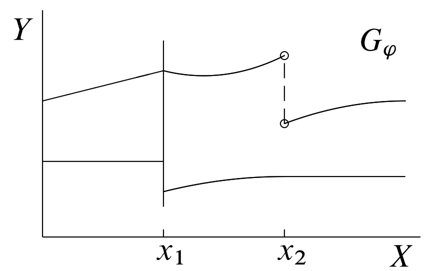
[1] The power set of a set A is the collection of all the subsets of A, which defines a new set as well.

[2] Mathematically, a *map* always refers to a point-to-point relationship between two sets.

We will also use the term *lower hemi-continuous* (l.h.c) in the same meaning of l.s.c.

Definition 2.8. *We say φ to be continuous if it is both u.s.c. and l.s.c.*

The graph below is an illustration for these concepts. φ is u.s.c. at x_1 but not l.s.c., and on the contrary, l.s.c. at x_2 but not u.s.c. Intuitively, u.s.c. says that a correspondence can not "shrink" while l.s.c. says that it can not "expand".



The following propositions demonstrate some basic properties and applications of the concept of correspondences. In particular, theorem 2.1 characterizes the fundamental property of constrained optimization in the language of correspondences. It has been labeled as *Berge's maximal theorem* in the literature.

Theorem 2.1. *Let $f : Y \to \mathbb{R}$ be a continuous function, and $\varphi : X \rightrightarrows Y$ be continuous and compact valued at $x \in X$. Then $\eta : X \rightrightarrows Y$ is a u.s.c. and compact valued correspondence at x where $\beta(x) = \operatorname{argmax}_{y\in\varphi(x)} f(y)$.*

Proof. First of all, since $\varphi(x)$ is compact and $f(\cdot)$ is continuous on $\varphi(x)$, there exists $y \in \varphi(x)$ that maximizes $f(\cdot)$ by Weierstrass theorem. Thus $\beta(x)$ is non-empty.

Let $(x_n, y_n) \to (x, y) \in (X \times Y)$ where $y_n \in \beta(x_n)\ \forall n$, we want to show that $y \in \beta(x)$. First, by u.h.c of φ and $y_n \in \beta(x_n) \subset \varphi(x_n)$, $y \in \varphi(x)$. Second, by l.h.c of φ, $\forall z \in \varphi(x)$, there exists a sequence of $\{z_n\}$ that goes to z and $z_n \in \varphi(x_n)$. Since $y_n \in \beta(x_n)$ maximizes $f(\cdot)$ in $\varphi(x_n)$, it follows $f(y_n) \geq f(z_n)\ \forall n$. Then we have $f(y) = \lim_n f(y_n) \geq \lim_n f(z_n) = f(z)$ from the continuity of $f(\cdot)$. Moreover, this implies $y \in \beta(x)$, so that β is u.s.c.

In order to prove $\beta(x)$ is compact, we only need to show that $\beta(x)$ is closed since it is a subset of the compact set $\varphi(x)$. Let $\{w_n\}$ be an arbitrary sequence in $\beta(x)$ which converges to $w \in \varphi(x)$. By definition, $f(w_n) = \max_{t\in\varphi(x)} f(t) \equiv M$ for all n, hence $f(y) = M$, which implies $y \in \beta(x)$. Therefore $\beta(x)$ is compact. ∎

A more general form of theorem 2.1 is given as a corollary.[3]

Corollary 2.2. *Let $f : X\times Y \to \mathbb{R}$ be a continuous function, and $\varphi : X \rightrightarrows Y$ be continuous and compact valued. Then*

- *$\beta : X \rightrightarrows Y$ is a u.s.c. correspondence where $\beta(x) = \operatorname{argmax}_{y\in\varphi(x)} f(x, y)$;*
- *the maximal function $g : X \to \mathbb{R}$ is continuous, where $g(x) = \max_{y\in\varphi(x)} f(x, y)$.*

Lemma 2.3. *Finite product of u.s.c. ($l.s.c.$) correspondences is also a u.s.c. ($l.s.c.$) correspondence. Product of correspondences is defined as follows: for each $k = 1, \ldots, K$ we have a correspondence $\varphi_k : X \rightrightarrows Y_k$, then the product correspondence is defined as $\varphi : X \rightrightarrows Y = \prod_{k=1}^{K} Y_k$, where $\varphi(x) = \prod_{k=1}^{K} \varphi_k(x) \subset Y$.*

Now we give two useful fixed point theorems. The first one is Brouwer's fixed point theorem.

Theorem. *Let D be a compact convex set in an Euclidean space,[4] and let $f : D \to D$ be a continuous function, then $\exists\, x \in D$, s.t. $f(x) = x$.*

From this theorem we can easily prove the following Kakutani's fixed point theorem, which plays a key role in establishing the existence of equilibria.

Theorem 2.4. *Let $\varphi : X \rightrightarrows X$ be u.s.c. and convex valued, where X is a compact convex set in an Euclidean space, then $\exists\, x \in X$ s.t. $x \in \varphi(x)$.*

---

[3] In fact, this is the primitive form of *Berge's maximal theorem*. However the essence of the proofs in both theorems are exactly the same.

[4] From a mathematical point of view, the convexity condition is neither intuitive nor critical for the existence of a fixed point. A much better one could be that there's no "hole" in the space, or equivalently, the space to be contractible, both of which, in more advanced mathematical terms (algebraic topology), are defined as the homotopy groups induced by the space are all trivial. Furthermore, this conclusion is in turn a simple corollary of the result proved by Eilenberg and Montgomery (1946), in which the fundamental property of a space admitting a fixed point is to be acyclic (that is the reduced homology groups are all trivial). However, convexity is somehow a common assumption in economic models, and in particular, preferences of players in a game can be easily and reasonably (or unreasonably) formulated in a way satisfying the convexity assumption, based on which mathematical tools apply in a convenient way. For this reason we maintain this convexity condition in the statements of Brouwer's fixed point theorem and Kakutani's theorem.

By now, we are ready to prove the existence of Nash equilibria.

Theorem 2.5. *Let $\Gamma = (S^n, \pi^n)_{n \in N}$ be a game satisfying:*

- *The strategy set S^n is a compact convex set in $\mathbb{R}^{k(n)}$,[5] for each $n \in N$.*
- *The payoff function $\pi^n : S \to \mathbb{R}$ is continuous on $S \equiv \prod S^n$ for each $n \in N$.*
- *For each $s \in S$ the payoff function $\pi^n(s|_n t)$ of unilateral deviation is quasi-concave in $t \in S^n$, for each $n \in N$.*

Then, there exists a Nash equilibrium.

Proof. Define $\varphi_n(s) = S^n$ for all $s \in S$ and $n \in N$, and obviously φ_n is continuous and compact valued. Then by theorem 2.1 $\beta^n(s)$ is u.s.c. in S. And by lemma 2.3, $\beta(s) = \beta^1(s) \times \cdots \times \beta^N(s)$ is also u.s.c. From quasi-concavity of $\pi^n(S|_n t)$, β^n hence β is convex valued. Therefore, by theorem 2.4, $\exists\, s^*$ s.t. $s^* \in \beta(s^*)$. ∎

▶ Feb.15, 2010

2.2 N.E. in A Generalized Game

In order to prove the existence of competitive equilibrium, we introduce the concept of *generalized game* (pseudo game) in this section.

Let $N \equiv \{1, \ldots, N\}$ be a set of players. For each $n \in N$, $S^n \equiv$ the "underlying" strategy set, and assume S^n to be a compact convex subset of some Euclidean space. Let $S \equiv S^1 \times \cdots \times S^N$, and define $\pi^n : S \to \mathbb{R}$ as the payoff function for n. Let $\varphi_n : S \rightrightarrows S^n$ be a correspondence where $\varphi_n(S) \subset S^n$ for each $n \in N$.[6] Define $\widetilde{\Gamma} = (S^n, \pi^n, \varphi_n)_{n \in N}$ as a generalized game.

[5] Strategy spaces of different players may have different dimensions.

[6] By writing $\varphi_n(S) \subset S^n$, we mean $\varphi_n(S) = \bigcup_{s \in S} \varphi_n(s) \subset S$, i.e., the union of all the image (set) of φ_n over S, is possibly a *proper subset* of S^n. That is, the correspondence φ_n represents a certain restriction on the feasible strategy that player n could take when the strategy profile other than n is fixed at $s_{-n} = (s^1, \ldots, s^{n-1}, s^{n+1}, \ldots, s^N)$. In a more precise way, the only relevant domain for defining φ_n is merely $S_{-n} = S^1 \times \cdots \times S^{n-1} \times S^{n+1} \times \cdots \times S^N$, or put in another way, it suffices to define $\varphi_n(s^n, s_{-n}) \equiv \widetilde{\varphi}_n(s_{-n})$, where the latter one is a correspondence from S_{-n} to S^n. However, for notational simplicity, we still use $\varphi_n(s)$ rather than $\widetilde{\varphi}_n(s_{-n})$. One also notes that when $\varphi_n(s) = S^n$ for all $s \in S$, then $\widetilde{\Gamma}$ is no more than a standard game.

Let $\beta^n : S \rightrightarrows S^n$ be *best reply correspondence* of n for $\widetilde{\Gamma}$, where

$$\beta^n(s) = \operatorname*{argmax}_{t \in \varphi^n(s)} \pi^n(s|_n t),$$

and $\beta = \beta^1 \times \cdots \times \beta^N$. Define $s \in S$ as a *Nash equilibrium* of $\widetilde{\Gamma}$ if $s \in \beta(s)$.

The following theorem plays a fundamental role in proving the existence of competitive equilibrium, following the approach of Arrow and Debreu (1954).

Theorem 2.6. *Let $\widetilde{\Gamma} = (S^n, \pi^n, \varphi_n)_{n \in N}$ be a generalized game.If for each n, π^n is continuous in s and quasi-concave in s^n, and φ_n is continuous, compact and convex valued, then there exists a N.E. of $\widetilde{\Gamma}$.*

Proof. The proof of this theorem is exactly the same as theorem 2.5. ∎

Chapter 3

Existence of C.E. in Pure Exchange Economy

3.1 A Benchmark Case

In this section we prove the existence of competitive equilibrium in a pure exchange economy $\mathscr{E} = (e^h, u^h)_{h \in H}$. The approach, due to Arrow and Debreu (1954), is based on generalized game. We first consider the case in which strictly positive endowments of all households are assumed, and in the next section, we'll see how this condition can be weakened.

Theorem 3.1. *Let $\mathscr{E} = (e^h, u^h)_{h \in H}$ be a pure exchange economy such that*

- *for all $h \in H$, $e^h \gg 0$;*
- *for all $h \in H$, u^h is continuous, quasi-concave and weakly monotone;*
- *for each $\ell \in L$, there is some h who likes ℓ, that is $u^h(x) > u^h(y)$ whenever $x \geq y$ and $x_\ell > y_\ell$.*

Then, there exists a competitive equilibrium in $\mathscr{E}$.

Proof. Let $\triangle$ = price simplex $\equiv \{p \in \mathbb{R}^L_+ | \sum_{\ell \in L} p_\ell = 1\}$. Choose $M > \max_{\ell \in L} \sum_{h \in H} e^h_\ell$, and define $\square = \{x \in \mathbb{R}^L_+ | x_j \leq M\}$. Define strategy set $S = \triangle \times \underbrace{\square \times \cdots \times \square}_{H}$, which is obviously a compact and convex set in an Euclidean space.

Define H households as H players with payoff functions

$$\pi^h(p, x^1, \dots, x^h, \dots, x^H) = u^h(x^h), \quad x^h \in \Box.$$

And define one special player pr called *price player* with payoff function

$$\pi^{pr}(p, x^1, \dots, x^H) = p \cdot (\sum x^h - \sum e^h), \quad p \in \triangle.$$

For each $h \in H$, define a correspondence $\varphi^h : S \rightrightarrows \Box$ as follows

$$\varphi^h(p, x^1, \dots, x^H) = B^h(p) \cap \Box \equiv \bar{B}^h(p),$$

By lemma 3.2 (following this proof), φ^h is continuous.[1]

Moreover, define $\varphi^{pr} : S \rightrightarrows \triangle$ for pr as $\varphi^{pr}(p, x^1, \dots, x^H) = \triangle$, which is a constant valued correspondence, thus is continuous obviously.

Now let $N = \{1, \dots, H, pr\}$, $S^h = \Box$, $\forall\, h$, and $S^{pr} = \triangle$. Then $\widetilde{\Gamma} = (S^n, \pi^n, \varphi_n)_{n \in N}$ is a generalized game satisfying the conditions of theorem 2.6, thus a N.E. $s = \langle p, x^1, \dots, x^H \rangle \in S$ exists. We will verify this s is a competitive equilibrium in $\mathscr{E}$.

Step 1: $\sum x^h \leq \sum e^h$.

For all h, since $x^h \in \bar{B}^h(p) \subset B^h(p)$, we have $p \cdot x^h \leq p \cdot e^h$. Summing over h, we have

$$p \cdot (\sum x^h - \sum e^h) \leq 0. \tag{3.1}$$

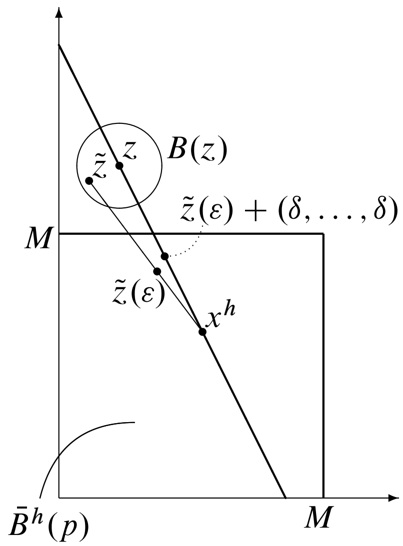
But if $\sum x^h_\ell - \sum e^h_\ell > 0$ for some ℓ, then by choosing

$$p = 1_\ell \equiv (0, \dots, 0, \underset{\ell\text{th}}{1}, 0, \dots, 0) \in \triangle,$$

the price player can get positive payoff, contradicting (3.1).

Step 2: $\forall\, h$, x^h maximizes u^h on $B^h(p)$, not just on $\bar{B}^h(p)$.

[1] In order to prove φ^h is l.h.c, the first condition in the theorem is necessary. For u.h.c, that is not necessary.



The main idea is illustrated in above figure. By step 1, we have

$$x_\ell^h < M, \quad \forall\, \ell \in L. \tag{3.2}$$

Suppose instead there exists $z \in B^h(p) \setminus \square$ s.t. $u^h(z) > u^h(x^h)$. By continuity of $u^h(\cdot)$, there exists a small open ball $B(z)$ s.t. $\forall\, y \in B(z)$, $u^h(y) > u^h(x^h)$. Choose one point $\tilde{z} \in B(z)$ with $p \cdot \tilde{z} < p \cdot e^h$. By (3.2), x^h will never reach the bound of $\square$, thus $\exists\, \varepsilon > 0$ small enough such that $\tilde{z}(\varepsilon) \equiv (1-\varepsilon)x^h + \varepsilon\tilde{z}$ lies in the interior region of $\bar{B}^h(p)$. Therefor, we can choose $\delta > 0$ small enough s.t. $z^* \equiv \tilde{z}(\varepsilon) + (\delta, \ldots, \delta) \in \bar{B}^h(p)$, and by weak monotonicity, $u^h(z^*) > u^h(\tilde{z}(\varepsilon))$. Since quasi-concavity of $u^h(\cdot)$ implies $u^h(\tilde{z}(\varepsilon)) \geq \min\{u^h(\tilde{z}), u^h(x^h)\} \geq u^h(x^h)$, we have $u^h(z^*) > u^h(x^h)$, which contradicts that x^h maximizes $u^h(\cdot)$ on $\bar{B}^h(p)$.

Step 3: $p \gg 0$.

Suppose $p_\ell = 0$ for some $\ell \in L$. By the third assumption of the theorem, there is an h who likes ℓ. It follows that $x^h + 1_\ell \in B^h(p)$, and by weak monotonicity there is $u^h(x^h + 1_\ell) > u^h(x^h)$, an contradiction to step 2.

Step 4: $p \cdot x^h = p \cdot e^h$, $\forall\, h \in H$.

Suppose $p \cdot x^h < p \cdot e^h$ (notice that $x^h \in B^h(p)$, i.e. $p \cdot x^h \leq p \cdot e^h$). From step 3 and the first assumption of the theorem, there is $p \cdot e^h > 0$, and it follows that $x^h + (\delta, \ldots, \delta) \in B^h(p)$ for small $\delta > 0$. But $u^h(x^h + (\delta, \ldots, \delta)) > u^h(x^h)$ by weak monotonicity. Contradiction again.

Step 5: $\sum_h x_\ell^h = \sum_h e_\ell^h, \forall \ell \in L$.

By step 1 we have $\sum_h x_\ell^h \leq \sum_h e_\ell^h, \forall \ell \in L$. So the only possible violation is $\sum_h x_\ell^h < \sum_h e_\ell^h$ for some ℓ. But since $p \gg 0$, this implies $p \cdot (\sum_h x^h - \sum_h e^h) = \sum_\ell p_\ell \cdot \sum_h (x_\ell^h - e_\ell^h) < 0$, which contradicts step 4. ∎

Lemma 3.2. *Assume $e^h \gg 0$, then the correspondence $B^h(p)$ is continuous on the price simplex $\triangle$, where $B^h(p) = \{x \in \mathbb{R}_+^L | p \cdot x \leq p \cdot e^h\}$.*

Proof. First we show such a correspondence is u.h.c. Let $(p_n, x_n) \to (p^*, x) \in \triangle \times \mathbb{R}_+^L$ with $x_n \in B^h(p_n)$, then by definition we have $p_n \cdot x_n \leq p_n \cdot e^h$. Since inner product is a continuous function, there is $p^* \cdot x \leq p^* \cdot e^h$, i.e. $x \in B^h(p^*)$.

Second we show such a correspondence is also l.h.c. Let $\{p_n\} \in \triangle$ and $p_n \to p^* \in \triangle$, and fix an (arbitrary) $x \in B^h(p^*)$, we want to find a sequence $\{x_n\} \in B^h(p_n)$ satisfying $x_n \to x$.

Case 1. Suppose $p^* \cdot x = 0$. Since $p \cdot x$ is a continuous function of p, we have $p_n \cdot x \to 0$. Observe $p \cdot e^h$ is continuous on $\triangle$ and $e^h \gg 0$, we have $m \equiv \min_{p \in \triangle} p \cdot e^h > 0$. Therefore, for n large enough we have $p_n \cdot x < m \leq p_n \cdot e^h$, i.e. $x \in B^h(p_n)$. So, it suffices to choose $x_n = x$ for large n and set the beginning elements of the sequence as e^h.

Case 2. Suppose $p^* \cdot x > 0$. Let $t(p) = \frac{p \cdot e^h}{p \cdot x}$ be a scalar function of $p \in \triangle$. Obviously, $t^* \equiv t(p^*) \geq 1$ and $t(p)$ is continuous at p^*, hence $t_n \equiv t(p_n) \to t(p^*)$. Define $x_n = \frac{t_n}{t^*} x$. Observe $p_n \cdot x_n = p_n \cdot \frac{t_n}{t^*} x \leq t(p_n) p_n \cdot x = p_n \cdot e^h$, we have $x_n \in B^h(p_n)$. Moreover, $\lim_n x_n = \frac{1}{t^*} x \lim_n t_n = x$. This completes the proof. ∎

▶ Feb.17, 2010

3.2 A Generalization

The positive endowment assumption $e^h \gg 0$ in the previous theorem 3.1 is too strong and not realistic at all — obviously, not everyone has everything. In this section, we will weaken this assumption, and prove the existence of C.E. in $\mathscr{E}$ under a new assumption about the endowment. To proceed, we first define some necessary concepts.

Definition 3.1. *We say household $h \in H$ want's commodity $\ell \in L$, if $u^h(x + \delta 1_\ell) > u^h(x)$, $\forall x \in \mathbb{R}_+^L$ and $\delta > 0$. And say h has ℓ if $e_\ell^h > 0$.*

Definition 3.2. *For ℓ, $k \in L$, we say* directed arc (ℓ, k) *exists if there is a household $h \in H$ who has ℓ and likes k. Moreover, commodity i is connected to commodity j if there is a directed path from i to j.*[2]

The commodity set L can be viewed as a directed graph, with the directed arcs defined by the above *having-wanting* property. In this sense, L could also be viewed as a *having-wanting graph.*

Definition 3.3. *Graph L is connected, if for all pairs $(i, j) \in L^2$, i is connected with j.*

Theorem 3.3. *Let $\mathscr{E} = (e^h, u^h)_{h \in H}$ be an pure exchange economy such that*

- *for each h, u^h is continuous, concave and weakly monotonic;*
- *the having-wanting graph of $\mathscr{E}$ is connected.*

Then there exists a competitive equilibrium in $\mathscr{E}$.

Remark 3.1. The first and third assumptions in theorem 3.1 together ensure that L is a connected graph. More directly, those two assumptions read everyone has every commodity and every commodity is liked by someone. So, in this sense the theorem stated above is a generalization of theorem 3.1.

Remark 3.2. The intuition of substituting the positive endowment assumption in theorem 3.1 by the connected graph condition for ensuring the existence of a competitive equilibrium is straightforward. Let (i, j) be an arbitrary pair of commodities in L, there exist a path from i to j, i.e. there are $\{\ell_1, \dots, \ell_m\} \in L$ s.t. $i \to \ell_1 \to \cdots \to \ell_m \to j$; this further implies there are $m + 1$ household $\{h^1, \dots, h^{m+1}\}$ such that h^1 has i and likes ℓ_1, …, h^{m+1} has ℓ_m and likes j. At the same time there's another path from j to i, thus you can imagine an buying-selling process from i to j, e.g h^1 sells i to who likes it and buys ℓ_1, …, h^{m+1} sells ℓ_m to h^m and buys j, …. By such an exchang procedure, all households will be satisfied and no commodity will be discarded, i.e., all prices will be positive, a fact which suffices to ensure the continuity of the budget correspondence (see lemma 3.4).

Proof of the theorem. For any $\varepsilon > 0$, define a perturbed pure exchange economy $\mathscr{E}(\varepsilon) \equiv (e^h(\varepsilon), u^h)_{h \in H}$, where $e^h(\varepsilon) = e^h + \varepsilon(1, \dots, 1)_{1 \times L}$. By theorem 3.1, there exists a C.E.

[2] That is there exist several commodities $\ell_1, \dots, \ell_n$, and the path consists of the corresponding directed arcs $(i, \ell_1), (\ell_1, \ell_2), \dots, (\ell_n, j)$.

$\langle p(\varepsilon), (x^h(\varepsilon))_{h\in H}\rangle$ for each ε. Observe that $\forall\, \varepsilon$, $p(\varepsilon) \in \triangle$, and $(x^h(\varepsilon))_{h\in H} \in \underbrace{\square \times \cdots \times \square}_{H} \equiv \square^H$ which are compact sets,[3] thus there exists a sequence ε_n converging to 0 such that

$$\langle p(\varepsilon_n), (x^h(\varepsilon_n))_{h\in H}\rangle \to \langle p, (x^h)_{h\in H}\rangle,$$

where $p \in \triangle$ and $(x^h)_{h\in H} \in \square^H$.

We're going to prove $\langle p, (x^h)_{h\in H}\rangle$ is the C.E of $\mathscr{E}$.

Step 1: $p \gg 0$.

Suppose there is an $\ell \in L$ s.t. $p_\ell = 0$. Since $p \in \triangle$, there exists $k \in L$ s.t. $p_k > 0$. By the connected graph assumption, there is a path from k to ℓ

$$k \equiv \ell_0 \to \ell_1 \to \cdots \to \ell_m \to \ell \equiv \ell_{m+1}.$$

Starting from k, let ℓ_{j+1} denote the first commodity on the path of price 0. It follows that $p_{\ell_j} > 0$, hence $p_{\ell_{j+1}}(\varepsilon_n)/p_{\ell_j}(\varepsilon_n) \to p_{\ell_{j+1}}/p_{\ell_j} = 0$. Note ℓ_j and ℓ_{j+1} are two commodities connected by a directed arc, thus there exists $h^* \in H$ such that it has ℓ_j and likes ℓ_{j+1}.

We claim that, for large n, $x^{h^*}(\varepsilon_n)$ does not maximize $u^{h^*}(\cdot)$ over $B^{h^*}(p(\varepsilon_n))$. Hence, if we can prove this claim, then the induced contradiction to the fact that $x^{h^*}(\varepsilon_n)$ is h^*'s equilibrium consumption would imply that p_ℓ can not be zero, i.e., $p \gg 0$.

For each $z \in \square$, let $H_z : \mathbb{R}^L_+ \to \mathbb{R}$ be the family of supporting hyperplanes of the graph of $u^{h^*}(\cdot)$ at the point $(z, u^{h^*}(z))$.[4] For h^*, define a modified utility function $\tilde{u}^{h^*}(x) = \inf_{z\in\square} H_z(x)$. Note that a so defined utility function $\tilde{u}^{h^*}(\cdot)$ coincides with $u^{h^*}(\cdot)$ over $\square$, i.e. $\tilde{u}^{h^*}(x) = u^{h^*}(x)$, $\forall\, x \in \square$.

Consider the family of modified economies $\tilde{\mathscr{E}}(\varepsilon) \equiv (e^h(\varepsilon), u^h)_{h\in H\setminus\{h^*\}} \cup (e^{h^*}, \tilde{u}^{h^*})$. Since all C.E. allocations of $\tilde{\mathscr{E}}(\varepsilon)$ must locate in $\square^H$,[5] and $\tilde{u}^{h^*}(x) = u^{h^*}(x)$ over $\square$, we conclude that for each ε, $\tilde{\mathscr{E}}(\varepsilon)$ has the same C.E. as $\mathscr{E}(\varepsilon)$. Therefore $x^{h^*}(\varepsilon_n)$ must also maximize $\tilde{u}^{h^*}(\cdot)$ over $B^{h^*}(p(\varepsilon_n))$.

---

[3] In fact, for each ε, the scale of the $\square_\varepsilon$ is associated with $e^h(\varepsilon)$. However, since $e^h(\varepsilon) \to e^h$, we can choose a $\square$ large enough, s.t. $\square_\varepsilon \subset \square\ \forall\, \varepsilon$.

[4] Note we only assume the utility function to be continuous and concave, therefore there may be more than one supporting hyperplane at a given point. In addition, the supporting hyperplane could also be viewed as a function.

[5] Note that $\square$ does not depend on utility function, but only on endowment.

Since we know $e^{h^*}_{\ell_j} > 0$ and $p_{\ell_j} > 0$, we can choose a point $y^{h^*} \in \mathbb{R}^L_+$ s.t. $y^{h^*} \in B^{h^*}(p(\varepsilon_n))\ \forall\, n$.[6] Let $d = \sup_n \tilde{u}^{h^*}(x^{h*}(\varepsilon_n)) - \tilde{u}^{h^*}(y^{h^*})$. Given $x^{h*}(\varepsilon_n) \to x^{h^*}$ and $\tilde{u}^{h^*}(\cdot)$ being continuous over $\Box$, it follows that $d < \infty$. Define $\hat{y}^{h^*} = (y^{h^*}_1, \ldots, y^{h^*}_{\ell_{j+1}-1}, \hat{y}^{h^*}_{\ell_{j+1}}, y^{h^*}_{\ell_{j+1}+1}, \ldots, y^{h^*}_L) \in \mathbb{R}^L_+$. Observe the budget set of h can be written in the following form

$$\sum_{\ell \neq \ell_{j+1}} \frac{p_\ell}{p_{\ell_j}} z_{\ell_j} + \frac{p_{\ell_{j+1}}}{p_{\ell_j}} z_{\ell_{j+1}} \leq \sum_{\ell \neq \ell_j} \frac{p_\ell}{p_{\ell_j}} e^{h^*}_\ell + e^{h^*}_{\ell_j},$$

and provided that $p_{\ell_{j+1}}(\varepsilon_n)/p_{\ell_j}(\varepsilon_n) \to 0$, as well as $e^{h^*} > 0$, we could choose $\hat{y}^{h^*}_{\ell_{j+1}}(\varepsilon_n) \to \infty$ such that $y^{h^*}(\varepsilon_n) = (y^{h^*}_1, \ldots, y^{h^*}_{\ell_{j+1}}, y^{h^*}_L) \in B^{h^*}(p(\varepsilon_n))$ for all n.

However, since $u^{h^*}(\cdot)$ is strictly increasing in its ℓ^{th}_{j+1} argument, by lemma 3.7 (following this proof), there exists $\kappa > 0$ s.t. $\tilde{u}^{h^*}(\hat{y}^{h^*}(\varepsilon_n)) \geq \tilde{u}^{h^*}(y^{h^*}) + \kappa(\hat{y}^{h^*}_{\ell_{j+1}}(\varepsilon_n) - y^{h^*}_{\ell_{j+1}})$. Obviously, for large n, $\tilde{u}^{h^*}(\hat{y}^{h^*}(\varepsilon_n)) - \tilde{u}^{h^*}(y^{h^*}) > d$, which implies $\tilde{u}^{h^*}(\hat{y}^{h^*}(\varepsilon_n)) > \tilde{u}^{h^*}(x^{h^*}(\varepsilon_n))$. Observe that $\hat{y}^{h^*}(\varepsilon_n) \in B^{h^*}(p(\varepsilon_n))$, the claim is proved.

Step 2: x^h maximizes $u^h(\cdot)$ over $B^h(p)$, $\forall\, h \in H$.

The basic idea for proving $x^h \in \operatorname{argmax}_{y \in B^h(p)} u^h(y)$ is to use Berge's maximal theorem (theorem 2.1). To this end, we introduce some notations.

For all $h \in H$, define $\varphi^h : \triangle \times \Box \rightrightarrows \Box$ where

$$\varphi^h(q, w^h) = B^h(q, w^h) \cap \Box \equiv \{y^h \in \mathbb{R}^L_+ | q \cdot y^h \leq q \cdot w^h\} \cap \Box,$$

in which q and w denote the price and the endowment, and y denotes a possible consumption bundle. By lemma 3.4 (following this proof), φ^h is continuous at (p, e^h). Let $\beta^h(q, w^h) = \operatorname{argmax}_{z \in \varphi^h(q,w^h)} u^h(z)$ be the best reply correspondence from $\triangle \times \Box$ to $\Box$. Since $u^h(\cdot)$ is continuous and concave, by theorem 2.1, β^h is u.h.c at (p, e^h). Therefore, given $x^h(\varepsilon_n) \in \beta^h(p(\varepsilon_n), e^h(\varepsilon_n))$ and $\big((p(\varepsilon_n), e^h(\varepsilon_n)), x^h(\varepsilon_n)\big) \to \big((p, e^h), x^h\big)$, we have $x^h \in \beta^h(p, e^h)$. Moreover, an analogous argument as step 2 in the proof of theorem 3.1 establishes that x^h maximizes $u^h(\cdot)$ over $B^h(p)$, not only over $B^h(p) \cap \Box$.

Step 3: $\sum_h x^h = \sum_h e^h$

Since $(x^h(\varepsilon_n))_{h \in H}$ is competitive allocation in $\mathscr{E}(\varepsilon_n)\ \forall, n$, there is $\sum_h x^h(\varepsilon_n) = \sum_h e^h(\varepsilon_n)$. Take the limit on both sides, we get $\sum_h x^h = \sum_h e^h$. ∎

Lemma 3.4. *Suppose $p \gg 0$, then the correspondence $\varphi^h : \triangle \times \Box \rightrightarrows \Box$ is continuous at (p, e^h) for all $h \in H$.*

[6] For instance, we can choose $y^{h^*} = (0, \ldots, e^{h^*}_{\ell_j}, \ldots, 0)$.

Proof. Since $\varphi^h = B^h \cap \Box$ and $\Box$ could be viewed as a constant valued correspondence from $\triangle \times \Box$ to $\Box$, so we only need to show B^h is a continuous correspondence at (p, e^h). We employ similar devices as in lemma 3.2 to prove the continuity of B^h at (p, e^h).

First we show B^h is u.h.c at (p, e^h). Suppose $\big((p_n, e_n^h), y_n^h\big) \to \big((p, e^h), y^h\big)$ with $y_n^h \in B^h(p_n, e_n^h)$. Since $p_n \cdot y_n^h \le p_n \cdot e_n^h$, $\forall\, n$, and inner product is a continuous function, there is $p \cdot y^h \le p \cdot e^h$, i.e. $y^h \in B^h(p, e^h)$.

Second we show B^h is l.h.c at (p, e^h). Let $(p_n, e_n^h) \to (p, e^h)$ and $\forall\, y^h \in B^h(p, e^h)$, we want to find a sequence $\{y_n^h\}$ s.t. $y_n^h \in B^h(p_n, e_n^h)$ and $y_n^h \to y^h$.

Case 1. Suppose $p \cdot y^h = 0$. Since $p \gg 0$, $y^h = (0, \ldots, 0)$. Thus, setting $y_n^h = (0, \ldots, 0)$, $\forall\, n$, is enough.

Case 2. Suppose $p \cdot y^h > 0$. Define $t(q, w^h) = \frac{q \cdot w^h}{q \cdot x^h}$, and evidently $t(q, w^h)$ is continuous at (p, e^h). Obviously, $t^* \equiv t(p, e^h) \ge 1$. Let $t_n = t(p_n, e_n^h)$ and $y_n^h = \frac{t_n}{t^*} y^h$, then $t_n \to t^*$ and $y_n^h \to y^h$. Observe $p_n \cdot y_n^h = \frac{t_n}{t^*} p_n \cdot y^h \le t_n p_n \cdot y^H = p_n \cdot e_n^h$, i.e. $y_n^h \in B^h(p_n, e_n^h)$. So $\{y_n^h\}$ fulfill our requirement. ∎

In order to prove that $\tilde{u}^h(\cdot)$ satisfies the desired increasing condition (lemma 3.7), we first prove a basic property for one variable concave function.

Lemma 3.5. *Let $f(x)$ be a concave function on $\mathbb{R}$, then $\forall\, a < b < c$,*

$$\frac{f(b) - f(a)}{b - a} \ge \frac{f(c) - f(a)}{c - a} \ge \frac{f(c) - f(b)}{c - b}.$$

Proof. Let $t = \frac{c-b}{c-a}$, we have $0 < t < 1$. By concavity of $f(x)$, there is $f(b) = f(ta + (1-t)c) \ge t f(a) + (1-t) f(b)$. Thus we have following inequalities,

$$\begin{aligned}
\frac{f(b) - f(a)}{b - a} &\ge \frac{[t f(a) + (1-t) f(c)] - f(a)}{b - a} \\
&= \frac{f(c) - f(a)}{c - a} \\
&= \frac{f(c) - [t f(a) + (1-t) f(c)]}{c - b} \\
&\ge \frac{f(c) - f(b)}{c - b}.
\end{aligned}$$ ∎

Corollary 3.6. *Let $f(x)$ be a concave function on $\mathbb{R}$, then its left and right derivatives exist* *respectively for all $x \in \mathbb{R}$ and*

$$f'_-(x) \equiv \lim_{\epsilon \to 0+} \frac{f(x) - f(x-\epsilon)}{\epsilon} \geq \lim_{\epsilon \to 0+} \frac{f(x+\epsilon) - f(x)}{\epsilon} \equiv f'_+(x).$$

Moreover, $\forall\, x < y$, there is

$$f'_-(x) \geq f'_+(x) \geq f'_-(y) \geq f'_+(y).$$

Proof. For any sequence $\{\epsilon_n\}$ with $\epsilon_n \downarrow 0$[7], observe that $\frac{f(x)-f(x-\epsilon_n)}{\epsilon_n}$ is an decreasing sequence bounded from below by precede lemma, thus the limit exists. Similarly, the limit of $\frac{f(x+\epsilon_n)-f(x)}{\epsilon_n}$ exists. Moreover, observe for any n, $\frac{f(x)-f(x-\epsilon_n)}{\epsilon_n} \geq \frac{f(x+\epsilon_n)-f(x)}{\epsilon_n}$, thus $\lim_{\epsilon\to 0+} \frac{f(x)-f(x-\epsilon)}{\epsilon} \geq \lim_{\epsilon \to 0+} \frac{f(x+\epsilon)-f(x)}{\epsilon}$.

For the second part, observe that there exists z s.t. $x < z < y$, thus for large n,

$$\frac{f(x+\epsilon_n) - f(x)}{\epsilon_n} \geq \frac{f(z) - f(x)}{z - x} \geq \frac{f(y) - f(z)}{y - z} \geq \frac{f(y) - f(y - \epsilon_n)}{\epsilon_n}.$$

Taking the limit, we get the desired result. ∎

Lemma 3.7. *Let a function $u : \mathbb{R}^L_+ \to \mathbb{R}$ be continuous, concave and strictly increasing in its first argument, i.e. $u(x + \delta(1, 0, \ldots, 0)) > u(x)$ $\forall x \in \mathbb{R}^L_+$ and $\delta > 0$. Define $\square = \{x \in \mathbb{R}^L_+ | 0 \leq x_\ell \leq M, \ell = 1, \ldots, L\}$ where $M > 0$ is a given number. For each $z \in \square$, let $H_z = \{H^i_z : \mathbb{R}^L_+ \to \mathbb{R}, i \in I_z\}$ be the set of supporting hyperplane of the graph of $u(\cdot)$ at $(z, u(z))$ where I_z is the index set,*[8] *and define $\tilde{u}(x) = \inf_{z\in\square}\{H^i_z(x), i \in I_z\}$, $\forall\, x \in \mathbb{R}^L_+$. Then, there exists $\kappa > 0$ s.t. $\tilde{u}(y_1, x_2, \ldots, x_L) \geq \tilde{u}(x_1, x_2, \ldots, x_L) + \kappa(y_1 - x_1)$, $\forall\, x = (x_1, \ldots, x_L) \in \mathbb{R}^L_+$ and $\forall\, y_1 > x_1$.*

Proof. Let $\hat{x} = (x_2, \ldots, x_L)$ be the last $L - 1$ coordinates of x, and $\forall\, z \in \square$, define $h(z) = \inf_{i\in I_z}\{\partial_1 H^i_z\}$.[9]

Step 1: $\exists\, \kappa > 0$ s.t. $h(z) > \kappa$, $\forall\, z \in \square$ and $\forall\, i \in I_z$.

First, $\forall\, \hat{x} \in \widehat{\square} = \{x \in \mathbb{R}^{L-1}_+ | 0 \leq x_\ell \leq M, \ell = 2, \ldots, L\}$, let $y = (y_1, \hat{x})$, then $f_{\hat{x}}(y_1) \equiv u(y) = u(y_1, \hat{x})$ is concave and strictly increasing in y_1, $\forall\, y_1 \in [0, \infty)$. Hence

$$\kappa(\hat{x}) \equiv \frac{f_{\hat{x}}(M + \delta) - f_{\hat{x}}(M)}{\delta} > 0$$

[7]"$\downarrow$" means converging from above.

[8]Since we only assume $u(\cdot)$ to be continuous, at each point z, there may be more than one supporting hyperplane, thus we use $i \in I$ to label all these hyperplane. Note, this index set I may depend on z.

[9]Here $\partial_1 = \dfrac{\partial}{\partial x_1}$. Moreover, H^i_z is a linear function for each i, hence $\partial_1 H^i_z$ is a constant.

with $\delta > 0$ fixed for all $\hat{x} \in \hat{\Box}$. By the precede corollary, we know $f'_{\hat{x}-}(y_1) \geq f'_{\hat{x}-}(M) \geq \kappa(\hat{x}) > 0$, $\forall\, y_1 \in [0, M]$. Let $y = (y_1, \hat{x})$, observe $h(y) = f'_{\hat{x}+}(y_1)$, thus $\forall\, i \in I_y$, $h(y) \geq \kappa(\hat{x})$.

Second, let $\kappa = \inf_{\hat{x}\in\widehat{\Box}} \kappa(\hat{x})$. Obviously, $\kappa \geq 0$. Suppose $\kappa = 0$, i.e. there is a sequence $\{\hat{x}_n\}$ s.t. $\kappa(\hat{x}_n) \to 0$. Since $\{\hat{x}_n\} \in \widehat{\Box}$ which is a compact set, hence there is a subsequence $\{\hat{x}_{n_k}\}$ s.t. $\hat{x}_{n_k} \to \hat{x}^* \in \widehat{\Box}$. Observe $\kappa(\hat{x}) = \delta^{-1}[u(M + \delta, \hat{x}) - u(M, \hat{x})]$ is a continuous function of $\hat{x}$, therefore $\kappa(\hat{x}^*) = \lim_k \kappa(\hat{x}_{n_k}) = 0$. However, since $\hat{x}^* \in \widehat{\Box}$, there is $\kappa(\hat{x}^*) > 0$. This contradiction implies $\kappa > 0$.

Third, $\forall\, z = (z_1, \hat{z}) \in \Box$ and $\forall\, i \in I_z$, since $h(z) \geq \kappa(\hat{z})$, there is $h(z) \geq \kappa > 0$.

Step 2: $\tilde{u}(x)$ is concave on $\mathbb{R}^L_+$.

Observe $\tilde{u}(x) = \inf_{z\in\Box}\{H^i_z(x), i \in I_z\} = \inf_{z\in\Box} \inf_{i\in I_z}\{H^i_z(x)\}$, and on the same hyperplane $H^i_z(\cdot)$ there is $H^i_z(tx + (1-t)y) = tH^i_z(x) + (1-t)H^i_z(y)$, $\forall\, x, y \in \mathbb{R}^L_+$ and $\forall\, t \in [0, 1]$.

Therefore,

$$\begin{aligned}
\tilde{u}(tx + (1-t)y) &= \inf_{z\in\Box} \inf_{i\in I_z} \{H^i_z(tx + (1-t)y)\} \\
&= \inf_{z\in\Box} \inf_{i\in I_z} \{tH^i_z(x) + (1-t)H^i_z(y)\} \\
&\geq \inf_{z\in\Box} \left\{t \inf_{i\in I_z} \{H^i_z(x)\} + (1-t) \inf_{i\in I_z} \{H^i_z(y)\}\right\} \\
&\geq t \inf_{z\in\Box} \inf_{i\in I_z} \{H^i_z(x)\} + (1-t) \inf_{z\in\Box} \inf_{i\in I_z} \{H^i_z(y)\} \\
&= t\tilde{u}(x) + (1-t)\tilde{u}(y).
\end{aligned}$$

Thus $\tilde{u}(x)$ is concave.

Step 3: Increasing condition.

$\forall\, x = (x_1, \hat{x}) \in \mathbb{R}^L_+$ and $\forall\, y > x_1$, $g(y) \equiv \tilde{u}(y, \hat{x})$ is concave in y. Let $y_1 > x_1$ be fixed, by lemma 3.5 and its corollary, there is

$$\frac{g(y_1) - g(x_1)}{y_1 - x_1} \geq g'_-(y_1).$$

By definition, $g(y_1) = \tilde{u}(y_1, \hat{x}) = \inf_{z\in\Box} \inf_{i\in I_z}\{H^i_z(y_1, \hat{x})$, thus $g'_-(y_1) \geq \inf_{z\in\Box} h(z)$. In conjunction with step 1, we have

$$\frac{g(y_1) - g(x_1)}{y_1 - x_1} \geq \kappa.$$

Rearrange this last expression, we have

$$\tilde{u}(y_1, x_2, \ldots, x_L) \geq \tilde{u}(x_1, x_2, \ldots, x_L) + \kappa(y_1 - x_1). \qquad \blacksquare$$

Remark 3.3. The proof of above lemma can be significantly simplified if we assume $u^h(\cdot)$ is continuously differentiable. Under this assumption, at each point $z \in \Box$, the only supporting hyperplane is also tangent plane, and $\partial_1 H_z = \partial_1 u^h(z) > 0$. Observe $\partial_1 u^h(z)$ is a continuous function in $\Box$, thus the minimum of $\partial_1 u^h(z)$ can be achieved, then there is $\kappa \equiv \min_{z \in \Box} \partial_1 u^h(z) > 0$.

The main logic of the theorem of connected graph is straightforward and can be summarized as follows.

First, since the endowment may have zero components, we can not use a generalized game framework to prove the existence of N.E equilibrium. The difficulty lies in the fact that the budget set correspondence may not be l.h.c.

However, by perturbing of the primitive economy slightly, we can get a sequence of C.E.s in the perturbed economies, and then taking a limit of this sequence of C.E.s, we get a limiting price vector and a limiting allocation. Therefore, all we need to do is to prove the limiting price and limiting allocation consist of a C.E. in the primitive economy.

It's trivial that the limiting allocation also satisfies the market clearing condition, but it is not easy to prove the allocation maximizes households' utilities under the limiting price. The difficulty again arises from the lack of l.h.c for the budget set correspondence.

To overcome this difficulty, the most prominent observation is that if the limiting price vector is strictly positive, then the continuity of the correspondence can be guaranteed even with zero components of endowment.Under the assumption of connectedness of having-wanting graph of this economy, we can indeed show (not quite uneasy) the limiting price vector is strictly positive. The technical difficulty for this part is to modified the primitive utility function a little bit by taking the infimum of its supporting hyperplanes on a compact set, and it turns out that this specific modified utility function has a particular increasing property (no less than a linear increasing) along a particular direction.

Example 3.1. The first example is an illustration of the necessity of connected graph of an economy for the existence of C.E. Consider a 2×2 economy with $e^1 = (1, 1)$, $e^2 = (2, 0)$, and $u^1(x, y) = y$, $u^2(x, y) = x$.

- If $p = (p_1, p_2) \gg 0$, then $\operatorname{argmax}_{B^1(p)} u^1$ is $(0, \frac{p_1}{p_2} + 1)$ and $\operatorname{argmax}_{B^2(p)} u^2$ is $(2, 0)$, thus market is not clear.

- If $p = (1, 0)$, then $\operatorname{argmax}_{B^1(p)} u^1$ is (s, t) with $t \to \infty$ and $s \in [0, 1]$, while $\operatorname{argmax}_{B^2(p)} u^2$ is $(2, r)$ with $r \geq 0$, thus market is not clear too.

- If $p = (0, 1)$, then $\operatorname{argmax}_{B^1(p)} u^1$ is $(t, 1)$ with $t \geq 0$, and $\operatorname{argmax}_{B^2(p)} u^2$ is $(r, 0)$ with $r \to \infty$, thus market is not clear once more.

Therefore, no C.E. exists. It is clear that the second commodity is not connected with the first commodity, since the only household (household 1) who has the second commodity doesn't like it.

Example 3.2. The second example shows how the result of this theorem is stronger than theorem 3.1. Consider a 2×2 economy with $e^1 = (1, 0)$, $e^2 = (0, 2)$, and $u^1(x, y) = y$, $u^2(x, y) = x$. Since initial endowment is not strictly positive, theorem 3.1 can not ensure the existence of a C.E. in this economy. However, as household 1 has commodity 1 and likes commodity 2, while household 2 has commodity 2 and likes commodity 1, the graph of this economy is connected, thus by theorem 3.3 there is a C.E. It can be easily verified that $\langle p, x^1, x^2 \rangle = \langle (\frac{2}{3}, \frac{1}{3}), (0, 2), (1, 0) \rangle$ is a C.E.

Chapter 4

An Economy with Production

4.1 Production: A Basic Formulation

► Feb.22, 2010

In this chapter, we turn to consider competitive equilibria in an economy with production. The first thing we need to do is to formulate *production* properly, making it suitable for an analysis of the equilibrium in an abstract economy, *à la* pure exchange economy in precede chapters.

It turns out to be more convenient to use a more general formulation called *production set* to characterize production instead of using production function. We assume all production activities are carried out by some production units, each of which is called a *firm*. For each firm, we use a production set, i.e., a set of all feasible *production plans*, to characterize the technological properties associated with this firm. A production plan of a firm, analogous to a consumption bundle of a consumer, can be viewed as a point (vector) in the enlarged commodity space $\mathbb{R}^L$ with L denoting the number of commodities in the economy. Each coordinate of a production plan represents a quantity of the corresponding commodity. Conventionally, a commodity is called an *output* in a given production plan if the corresponding quantity in this plan is positive, and *input* if negative. In addition, if the quantity is zero, then this commodity is irrelevant to the production plan.

On one hand, we restrict our analysis to the production plans only, ignoring the concrete producing process. Any issues of management is absent in this setup. Actually, no human being is needed, while all we need is a machine which merely executes the production plan automatically. On the other hand, we assume fully private ownership in this economy, i.e, all firms are completely owned by the households in the economy in terms of equity and dividends.

Let the household set $H = \{1, \ldots, H\}$ and the firm set $J = \{1, \ldots, J\}$, both of which are finite set. Let θ_j^h be the number of shares of firm j owned by household h. A fully private ownership requires $\theta_h^j \geq 0$ and $\sum_h \theta_h^j = 1, \forall j \in J$. Let Y_j denote the production set of firm j, where $Y_j \subset \mathbb{R}^L$. By specifying preferences and endowments for all households, *a economy with production* is summarized by the following tuple $\mathscr{E} = \big((e^h, u^h, \theta^h)_{h\in H}, (Y^j)_{j\in J}\big)$.

We presume that the objective for each firm is to maximize its profit by choosing a proper production plan in its production set. Under the notation of production set, the profit resulted from a specific production plan y^j of firm j is simply $p \cdot y^j$, where p is a price system (vector).

Of course not an arbitrary subset of $\mathbb{R}^L$ viewed as a production set would serve our purpose of investigating the equilibrium of an economy entailing at least some realistic sense. For our purpose, it would be desirable to have production sets satisfy following assumptions.

A.1. $Y^j \cap \mathbb{R}_+^L = \{0\}$, for all $j \in J$.
It means no firm can produce anything from nothing, however they are allowed to do nothing by choosing $y^j = 0$.

A.2. Y^j is convex and closed.
Since all commodities are assumed to be perfect divisible, it seems reasonable to assume the production set to be closed. However, convexity is a really strong assumption, as it implies (combined with the first assumption) that no firm could display increasing return to scale.

A.3. Y^j is comprehensive, i.e. $Y^j + \mathbb{R}_-^L \subset Y^j$, where $\mathbb{R}_-^L = -\mathbb{R}_+^L$.[1]
The essence of this assumption is free disposal, which means any excess supply in the economy can just be thrown away. Yet in reality, it could be very costly to dispose things, especially some by-products.

[1]Let $A, B \subset \mathbb{R}^L$, $A + B$ is defined as $\{a + b : a \in A, b \in B\}$, and $-A$ is defined as $\{-a : a \in A\}$.

A.4. $Y \cap \mathbb{R}^L_+ \equiv Y^1 + \cdots + Y^J \cap \mathbb{R}^L_+ = \{0\}$.
Here Y represents the *aggregate production set* in this economy. This assumption is not implied by the first assumption. It means all firms together can not do arbitrage.

A.5. $Y \cap (-Y) = \{0\}$.
Intuitively, if y is a possible aggregate production plan, $-y$ is a production plan which reverses the whole production process. This assumption excludes this possibility.

Before proceeding to the discussion of the competitive equilibria in this economy, we first show some interesting and important properties of the aggregate production set implied by the above assumptions.

Lemma 4.1. *Let $Y^1, \ldots, Y^J$ be production sets satisfying assumption A.2 to A.5, then the aggregate production set $Y = Y^1 + \cdots + Y^J$ is convex and closed.*

Proof. Convexity of Y follows directly from A.2 that each Y^j is convex. To prove the second part, suppose there is an sequence $\{y^n\}$ where $y^n = \sum_{j\in J} y^n_j \in Y$, and $y^n \to y \in \mathbb{R}^L$, we need to show that $y \in Y$.

First, we claim that $\{y^n_j\}_{n=1}^\infty$ is bounded for each $j \in J$. Suppose in converse, that for some $j \in J$, $\{y^n_j\}$ is unbounded. Let $K = \{j \in J \mid \{y^n_j\}_{n=1}^\infty \text{ is unbounded}\}$, then K is nonempty. Define

$$d(n) \equiv \max_{j\in J} ||y^n_j|| = \max_{j\in K} ||y^n_j||$$

where $||\cdot||$ denotes the vector norm. Thus $d(n) \to \infty$, $n \to \infty$, and without loss of generality, we could assume $d(n) > 1$ for all n. Moreover, let

$$\begin{aligned} z^n \equiv \frac{y^n}{d(n)} &= \sum_{j\in K} \frac{y^n_j}{d(n)} + \sum_{j\in J\setminus K} \frac{y^n_j}{d(n)} \\ &\equiv \sum_{j\in K} z^n_j + \sum_{j\in J\setminus K} z^n_j, \end{aligned} \tag{4.1}$$

then for each j and n, $z^n_j \in Y^j$, since Y^j is convex, $0 \in Y^j$ and $d(n) > 1$. On one hand, observe that y^n converges to y which is finite, hence z^n converges to 0 as $d(n)$ goes to infinity; and since for $j \in J \setminus K$, $\{y^n_j\}$ is bounded, it follows that $z^n_j \to 0$. On the other hand, observe that for some $j \in K$, there must be $||y^n_j|| = d(n)$ for infinitely many times,[2] hence

[2]Note that $d(n)$ must equal to some $||y^n_j||$, $j \in K$. Since the sequence of $\{d(n)\}$ is infinite, it's impossible for all $j \in K$ that there are only *finite* times in which $d(n) = ||y^n_j||$.

we could find a sequence $\{n(m)\}$ of index numbers such that $z_j^{n(m)} \to z_j$ with $||z_j|| = 1$ as $m \to \infty$, where $z_j \in Y^j$ since Y^j is closed. Therefore, we can assume without loss of generality that there is a nonempty set $K' \subset K$ such that z_j^n converges to $z_j \in Y^j$ which is not 0 if and only if $j \in K'$. Now, by taking limit in n on both sides of (4.1), we have $0 = \sum_{j \in K'} z_j$, or equivalently

$$z_j = -\sum_{j' \in K' \setminus \{j\}} z_{j'}.$$

Note that the LHS of this equation belongs to Y and the RHS belongs to $-Y$. However, by A.5, we conclude z_j must be 0, which is a contradiction.

Next, given that whenever $y^n = \sum_{j \in J} y_j^n \to y \in \mathbb{R}^L$ there is $\{y_j^n\}_{n=1}^{\infty}$ being bounded for all j, it follows that we could choose a sequence $\{n(m)\}$ of index numbers such that $y_j^{n(m)}$ converges to y_j for each j respectively.[3] Since each Y^j is closed by A.2, we conclude that $y^{n(m)} \to \sum_{j \in J} y^j \in Y$ as $m \to \infty$. Note that $y = \lim_{m \to \infty} y^{n(m)}$ as well, thus we have $y \in Y$. ∎

Lemma 4.2. *Let $Y^1, \dots, Y^J$ be production sets satisfying assumption A.2 to A.5, and let $e \in \mathbb{R}_+^L$ be nonzero, then $(Y + e) \cap (-Y - e)$ is non-empty and bounded.*

Proof. First we show $(Y + e) \cap (-Y - e)$ is non-empty. By A.3, $Y + \mathbb{R}_-^L \subset Y$, hence by A.4 $-e = 0 + (-e) \in Y$. So we have $0 = (-e) + e \in Y + e$. Same argument shows $0 = e + (-e) \in (-Y - e)$. Therefore $(Y + e) \cap (-Y - e)$ is non-empty.

Second we show it is bounded. Suppose in converse, there is a sequence $\{y^k\} \subset (Y + e) \cap (-Y - e)$ with $||y^k|| \to \infty$. Obviously, $y^k - e \in Y$ and $y^k + e \in (-Y)$. Without loss of generality, assume $||y^k|| > 1$, so $0 < 1/||y^k|| < 1$. By the precede lemma, Y is convex, thus $y^k/||y^k|| - e/||y^k|| \in Y$ and $y^k/||y^k|| + e/||y^k|| \in (-Y)$ are uniformly bounded, hence there are subsequence for each sequence that converge to the same point y with $||y|| = 1$. Also by the precede lemma Y is closed, thus $y \in Y$ and $y \in (-Y)$, and therefore by assumption A.4 there is $y = 0$, contradicting. ∎

[3] This sequence $\{n(m)\}$ is not necessarily the same as the previous one. In addition, this sequence can be constructed as follows: Since for $j = 1$, $\{y_1^n\}$ is bounded, it follows that there is a converging subsequence with the index sequence denoted by $n_1(m)$; then since $\{y_2^{n_1(m)}\}$ is also bounded, we could find a converging subsequence with the index sequence denoted as $n_2(m)$. Continue on this procedure, and we could find a index sequence $n(m)$ such that for each j, $\{y_j^{n(m)}\}$ is converging.

4.2 Existence of C.E. in An Economy with Production

Given the formulation of production in the previous section, we now turn to define competitive equilibria in such an economy.

Definition 4.1. *A collection of a price system and an allocation including a set of production plans* $\langle p, (x^h)_{h \in H}, (y^j)_{j \in J} \rangle$ *consists of a competitive equilibrium* $(C.E.)$ *in an economy with production* $\mathscr{E} = \big((e^h, u^h, \theta^h)_{h \in H}, (Y^j)_{j \in J}\big)$ *if following conditions are satisfied:*

- *Profit maximization. For each firm* $j \in J$*, the production plan* y^j *maximizes the profit over the production set* Y^j *given* p*, i.e.,*

$$y^j \in \operatorname*{argmax}_{z \in Y^j} p \cdot z.$$

- *Utility maximization. For each household* $h \in H$*, the consumption bundle* x^h *maximizes the utility over the budget set defined by the endowment* e^h *and its shares of all firms* $\{\theta_j^h\}_{j \in J}$ *given* p *and* $\{y^j\}$*, i.e.,*

$$x^h \in \operatorname*{argmax}_{z \in B^h(p, (y^j)_{j \in J})} u^h(z),$$

where

$$B^h(p, (y^j)_{j \in J}) = \left\{ x \in \mathbb{R}_+^L \,\middle|\, p \cdot x \leq p \cdot e^h + \sum_{j \in J} \theta_j^h \, p \cdot y^j \right\}.$$

- *Market clearing.*

$$\sum_h (x^h - e^h) - \sum_j y^j = 0.$$

We stress here that in the above definition, the term *allocation* no longer refers merely to a set of consumption bundles which satisfies the feasibility condition as discussed in section 1.2, chapter 1. Rather, an allocation in an economy with production is perceived as including a set of production plans $\{y^j\}$, in addition to a set of consumption bundles $\{x^h\}$, and all of them together satisfy the feasibility condition specified by $\sum_h x^h \leq \sum_h e^h + \sum_j y^j$.

Theorem 4.3. *Let* $\mathscr{E} = \big((e^h, u^h, \theta^h)_{h \in H}, (Y^j)_{j \in J}\big)$ *be an economy with production, if*

- *for each* $h \in H$*,* $e^h \gg 0$*;*

- *for each $h \in H$, $u^h(\cdot)$ is continuous, quasi-concave and weakly monotone;*
- *for each $\ell \in L$, there exists an h who likes it;*[4]
- *each production set satisfies assumptions A.1 to A.5 in the previous section;*

Then there exists a competitive equilibrium in $\mathscr{E}$.

Remark 4.1. The original statement of this theorem is in Arrow and Debreu (1954: theorem 1). Regardless of slightly different assumptions with respect to preference and production, the essence of the proof proposed here coincides with the original, both of which are based on the existence of a N.E. in a generalized game.

Proof. Let $e = \sum_h e^h$ and $Y = \sum_j Y^j$. By lemma 4.2 $(Y + e) \cap (-Y - e)$ is non-empty and bounded, thus we can choose $M > \max\{\|z\| : z \in (Y + e) \cap (-Y - e)\}$. Let $\Box = \{x \in \mathbb{R}^L | x_\ell \in [-M, M], \forall\, \ell \in L\}$ and $\triangle$ be the price simplex. Define $S = \triangle \times \Box^{H+J}$, and a strategy in S takes the form of $s = \big(p, (x^h)_{h\in H}, (y^j)_{j\in J}\big)$. Each household h is regarded as a player with a payoff function $\pi^h(s) = u^h(x^h)$, and each firm is also regarded as a player with a payoff function $\pi^j(s) = p \cdot y^j$. Define an additional player called price player with a payoff function $\pi^{pr}(s) = p \cdot \big(\sum_h (x^h - e^h) - \sum_j y^j\big)$. Moreover, for each household, define a modified budget set[5]

$$\tilde{B}^h(p, (y^j)_{j\in J}) = \left\{ x^h \in \mathbb{R}^L_+ \,\middle|\, p \cdot x^h \leq p \cdot e^h + \max\left\{0, \sum_j \theta^h_j p \cdot y^j \right\} \right\}.$$

Now, for all $s \in S$ define following correspondences

$$\begin{aligned} \varphi^h(s) &= \tilde{B}^h(p, (y^j)_{j\in J}) \cap \Box, \quad \forall\, h; \\ \varphi^j(s) &= Y^j \cap \Box, \quad \forall\, j; \\ \varphi^{pr}(s) &= \triangle. \end{aligned}$$

[4]That is, whenever $x \geq y$ and $x_\ell > y_\ell$, there is $u^h(x) > u^h(y)$.

[5]Introducing such a budget set is a purely technical trick to overcome the difficulty of the continuity of the budget set correspondence would we encounter if we were to use the primitive form of the budget set. This difficulty arises from the fact that, without additional restriction, $e^h + \sum_j \theta^h_j y^j$ may not be strictly positive. As summarized at the end of the proof of theorem 3.3, this backward leads to the lack of l.h.c. of the budget correspondence.

By lemma 4.4(following this theorem), φ^h is continuous, and the continuity of φ^j and φ^{pr} is a triviality. By now, we have a well-defined generalized game satisfying theorem 2.6, thus an N.E. $\langle p, (x^h)_{h\in H}, (y^j)_{j\in J}\rangle$ exists. We shall verify this is a C.E. in $\mathscr{E}$.

Step 1: $p \cdot y^j \geq 0$, for all j.

Since $0 \in Y^j \cap \Box$, zero profit is always possible, thus $p \cdot y^j \geq 0$. This also shows that $\tilde{B}^h(p, (y^j)_{j\in J}) = B^h(p, (y^j)_{j\in J})$, since $\sum_j \theta^h_j p \cdot y^j$ is non-negative.

Step 2: $\sum_h (x^h - e^h) - \sum_j y^j \leq 0$.

Observe $x^h \in B^h(p, (y^j)_{j\in J})$, hence $p \cdot x^h \leq p \cdot e^h + \sum_{j\in J} \theta^h_j \, p \cdot y^j$, and sum across h together with the last condition gives us

$$p \cdot \left(\sum_h (x^h - e^h) - \sum_j y^j \right) \leq 0.$$

Thus the optimizing by price player implies the desired result.

Step 3: x^h maximizes $u^h(\cdot)$ on $B^h(p, (y^j)_{j\in J})$.

First observe that $x^h \leq \sum_h x^h \leq e + \sum_j y^j \subset Y + e$, and $x^h \in \mathbb{R}^L_+ \subset (-Y - e)$, thus x^h is an interior point of $\Box$.

Suppose there is $z \in B^h(p, (y^j)_{j\in J}) \setminus \Box$ with $u^h(z) > u^h(x^h)$, then by continuity of utility there is a small ball $B(z)$ centering at z within which each point has a higher utility than x^h. Thus we could choose a $\hat{z}$ to be an interior point of $B^h(p, (y^j)_{j\in J})$ with $u^h(\hat{z}) > u^h(x^h)$. Let $\epsilon > 0$ small enough s.t. $\hat{z}(\epsilon) = (1-\epsilon)x^h + \epsilon\hat{z}$ is also an interior point of $B^h(p, (y^j)_{j\in J}) \cap \Box$, hence $u^h(\hat{z}(\epsilon)) \geq u^h(x^h)$. So we can further choose a small $\delta > 0$ s.t. $y \equiv \hat{z}(\epsilon) + (\delta, \ldots, \delta) \in B^h(p, (y^j)_{j\in J}) \cap \Box$ and $u^h(y) > u^h(x^h)$. Contradiction.

Step 4: $p \gg 0$.

Suppose $p_\ell = 0$, then by the third condition there is an h who likes ℓ, thus $u^h(x^h + 1_\ell) > u^h(x^h)$. Observe $x^h + 1_\ell \in B^h(p, (y^j)_{j\in J})$, which leads to a contradiction with step 3.

Step 5: $\sum_h (x^h - e^h) - \sum_j y^j = 0$.

First observe that by step 2 and 4, there is $p \cdot x^h = p \cdot e^h + \sum_j \theta^h_j p \cdot y^j$ for all h. Otherwise, for sufficient small $\delta > 0$, $x^h + (\delta, \ldots, \delta) \in B^h(p, (y^j)_{j\in J})$ which yields a higher utility.

Summing over h follows $p \cdot \left(\sum_h (x^h - e^h) - \sum_j y^j\right) = 0$. Rewrite this expression as

$$\sum_\ell p_\ell \left(\sum_h (x^h_\ell - e^h_\ell) - \sum_j y^j_\ell \right) = 0,$$

and observe the terms in the bracket are non-positive by step 1, so they must be 0 by step 4.

Step 6: y^j is an interior point of $\square$ for all j.

By step 5, $\sum_h x^h = \sum_h e^h + \sum_j y^j \in \mathbb{R}^L_+$. For any $j^* \in J$, of course $y^{j^*} \in Y^{j^*} \subset Y \subset Y + e$. On the other hand, $y^{j^*} = -(\sum_{j \neq j^*} y^j - \sum_h x^h) - e$. Notice $\sum_{j \neq j^*} y^j \in Y$ and $-\sum_h x^h \in \mathbb{R}^L_-$, hence the difference belongs to Y, thus $y^{j^*} \in (-Y - e)$. Observe $(Y + e) \cap (-Y - e)$ is contained in the interior of $\square$, therefore y^{j^*} is an interior point of $\square$.

Step 7: y^j maximizes profit over Y^j for all j.

Suppose conversely there is a production plan $z \in Y^j \setminus \square$ with a strictly higher profit, then by convexity of Y^j for any $\epsilon \in (0,1)$, $\hat{z}(\epsilon) \equiv (1-\epsilon)y^j + \epsilon z \in Y^j$. Further, by step 6, $\hat{z}(\epsilon) \in Y^j \cap \square$ whenever ϵ is small enough. However, this implies $p \cdot \hat{z}(\epsilon) = (1-\epsilon)p \cdot y^j + \epsilon p \cdot z > p \cdot y^j$, which is contradicting with y^j maximizing profit in $Y^j \cap \square$. ∎

Lemma 4.4. *Let $\varphi^h(s)$ be the correspondence defined in the previous proof, then φ^h is continuous over S.*

Proof. Since $\varphi^h(s) = \tilde{B}^h(p, (y^j)_{j \in J}) \cap \square$, it suffices to show $\tilde{B}^h(p, (y^j)_{j \in J})$ viewed as a correspondence from S to $\mathbb{R}^L_+$ is continuous. For the u.h.c part, the proof is quite straight forward, provided that inner product is a continuous function. For the l.h.c part, given the particular form of this modified budget set, we turn to consider the following two cases.

Case 1. Suppose $x \in \tilde{B}^h(p, (y^j)_{j \in J})$, $p_n \to p$ and $y_n = (y^1_n, \ldots, y^J_n) \to y = (y^1, \ldots, y^J)$ with $\sum_j \theta^h_j p \cdot y^j < 0$. Then there is $p \cdot x \leq p \cdot e^h$, and use the same argument as in lemma 3.2 will prove the lower hemi-continuous provided that $e^h \gg 0$.

Case 2. Suppose now $\sum_j \theta^h_j p \cdot y^j \geq 0$, then there is $p \cdot x \leq p \cdot e^h + \sum_j \theta^h_j p \cdot y^j$. Once again, the same method in the proof of lemma 3.2, i.e. defining $t(p, y) = (p \cdot e^h + \sum_j \theta^h_j p \cdot y^j)/(p \cdot x)$ where $p \cdot x > 0$ and letting $x_n = x$ where $p \cdot x = 0$, suffices to complete the proof. ∎

4.3 Pareto Optimality in $\mathscr{E}$

We will establish the Pareto optimality of C.E. in an economy with production. Notice that in an economy with production, the resource constraint for each household takes a very different from as in a pure exchange economy, hence we can not demonstrate the Pareto optimality using those results in pure exchange economy directly. Alternatively, we need to extend the concepts discussed in the case of pure exchange economy to incorporate properly the resource constraint in an economy with production.

At this stage, we do not assume $\mathscr{E} = \big((e^h, u^h, \theta^h)_{h\in H}, (Y^j)_{j\in J}\big)$ satisfies any conditions listed in theorem 4.3, nor the assumptions about the production set in section 4.1. However, we do emphasize that whenever we are talking about an *allocation* in $\mathscr{E}$, the feasibility condition is presumed to be satisfied.

Definition 4.2. *Let $\langle (x^h)_{h\in H}, (y^j)_{j\in J}\rangle$ and $\langle (w^h)_{h\in H}, (z^j)_{j\in J}\rangle$ be two allocations in $\mathscr{E}$. We say the former one is* Pareto superior *to the latter one, if $u^h(x^h) \geq u^h(w^h)$ for all h and $u^h(x^h) > u^h(w^h)$ for at least one h.*

Definition 4.3. *An allocation $\langle (x^h)_{h\in H}, (y^j)_{j\in J}\rangle$ is Pareto optimal in $\mathscr{E}$, if there is no other allocation which is Pareto superior to this one.*

To spell out the Pareto optimality of C.E. in $\mathscr{E}$ in a precise way, i.e. only the allocation in a C.E. has to do with the welfare justification, we define *competitive allocation* as in the case of pure exchange economy.

Definition 4.4. *We say $\langle (x^h)_{h\in H}, (y^j)_{j\in J}\rangle$ is a competitive allocation in $\mathscr{E}$, if there exists a price vector $p \in \mathbb{R}^L_+$ s.t. $\langle p, (x^h)_{h\in H}, (y^j)_{j\in J}\rangle$ is a C.E. in $\mathscr{E}$.*

By, we could lay out the main result in this section regarding to the welfare property of C.E., that is the *first theorem of welfare economics* in $\mathscr{E}$.

Theorem 4.5. *Let $\mathscr{E} = \big((e^h, u^h, \theta^h)_{h\in H}, (Y^j)_{j\in J}\big)$ be a production economy, and assume utility to be weakly monotone. Then, all competitive allocations in $\mathscr{E}$ are Pareto optimal.*

Proof. Let $\langle (x^h)_{h\in H}, (y^j)_{j\in J}\rangle$ be a competitive allocation, and p be the price vector with which the competitive allocation consists of a C.E.

Suppose conversely there is an allocation $\langle (w^h)_{h\in H}, (z^j)_{j\in J}\rangle$ in $\mathscr{E}$ which is Pareto superior to $\langle (x^h)_{h\in H}, (y^j)_{j\in J}\rangle$.

First, since there is one h with $u^h(x^h) < u^h(w^h)$, then $p \cdot w^h > p \cdot e^h + \sum_j \theta_j^h p \cdot y^j$ for this h; otherwise $w^h \in B^h(p, (y^j)_{j \in J})$, which contradicts the optimality of x^h in h's budget set. Further, observe for each j, z^j yields a profit at most as high as $p \cdot y^j$, thus $\sum_j \theta_j^h p \cdot y^j \geq \sum_j \theta_j^h p \cdot z^j$ provided θ_j^h is non-negative. To sum up, we have $p \cdot w^h > p \cdot e^h + \sum_j \theta_j^h p \cdot z^j$.

Second, since for each $h \in H$ there is $u^h(x^h) \leq u^h(w^h)$, there is $p \cdot w^h \geq p \cdot e^h + \sum_j \theta_j^h p \cdot y^j$, otherwise by weak monotonicity there exists a $\tilde{w}^h$ which yields a higher utility than x^h. Moreover, since $p \cdot y^j \geq p \cdot z^j$ for all j, we have $p \cdot w^h \geq p \cdot e^h + \sum_j \theta_j^h p \cdot z^j$ for all h.

Combining these two, it follows that $\sum_h p \cdot w^h > \sum_h p \cdot e^h + \sum_j p \cdot z^j$. However, given $\langle (w^h)_{h \in H}, (z^j)_{j \in J} \rangle$ is an allocation in $\mathscr{E}$, there is $\sum_h w^h \leq \sum_h e^h + \sum_j z^j$, which leads to a contradiction provided that p is non-negative. ∎

4.4 Individualized Economy

Since we put no restriction on the separability on the technology, we could split a firm j into H parts according to the shares of j owned by each household. Now each household h owns entirely a firm described by the production set $\theta_j^h Y^j$. Under this point of view, a new economy is defined with $J \times H$ firms owned by H households, and we call it an *individualized economy* of $\mathscr{E}$, denoted by $\tilde{\mathscr{E}}$. Observe that the budget set of each household in this economy is exactly the same as in $\mathscr{E}$. Moreover, on one hand, whenever y^j maximizes the profit of firm j over Y^j given p, $\theta_j^h y^j$ will maximize the profit of firm (h, j) over $\theta_j^h Y^j$ with the same price vector; on the other hand, once y_j^h maximizes the profit of firm (h, j) for some h given p, then y_j^h / θ_j^h will maximize the profit of firm j in the primitive economy $\mathscr{E}$ with the same price vector. As a result, we have following corollary of theorem 4.3.

Corollary 4.6. *Let $\mathscr{E} = \big((e^h, u^h, \theta^h)_{h \in H}, (Y^j)_{j \in J}\big)$ be an economy satisfying all conditions of theorem 4.3, and $\tilde{\mathscr{E}}$ be the individualized economy of $\mathscr{E}$, then the competitive equilibria of this two economies coincide.*

► Feb.24, 2010

The most prominent advantage of considering individualized economy is that it enables us to discuss *core allocation* in an economy with production. Since in an individualized economy, all firms are individually owned, there is no conceptual difficulty of defining core allocations in this economy. For notational simplicity, let $\tilde{\mathscr{E}} = \big(e^h, u^h, (Y_j^h)_{j\in J}\big)_{h\in H}$ denote the individualized economy. Note, as in discussion about Pareto optimality in the previous section, here we do not assume $\mathscr{E}$ satisfies any conditions listed in theorem 4.3, including the assumptions about production sets in section 4.1.

Definition 4.5. *For a subset S of H, an allocation $\langle (x^h)_{h\in S}, (y_j^h)_{(h,j)\in S\times J}\rangle$ is called an S-allocation in $\tilde{\mathscr{E}}$, if $\sum_{h\in S} x^h \le \sum_{h\in S} e^h + \sum_{h\in S}\sum_{j\in J} y_j^h$.*

Definition 4.6. *An allocation $\langle (x^h)_{h\in H}, (y_j^h)_{(h,j)\in H\times J}\rangle$ is a core allocation in $\tilde{\mathscr{E}}$, if there exists no S-allocation $\langle (w^h)_{h\in S}, (z_j^h)_{(h,j)\in S\times J}\rangle$ where S is a subset of H, such that*

$$\begin{aligned} &u^h(w^h) \ge u^h(x^h), \quad \forall\, h \in S;\\ &u^h(w^h) > u^h(x^h), \quad \textit{for at least one } h \in S. \end{aligned}$$

The following theorem demonstrates that C.E.s in an economy with production entail a welfare property that is stronger than the Pareto optimality.

Theorem 4.7. *Let $\tilde{\mathscr{E}} = \big(e^h, u^h, (Y_j^h)_{j\in J}\big)_{h\in H}$ be an individualized economy with utilities satisfying weak monotonicity. Then each competitive allocation $\langle (x^h)_{h\in H}, (y_j^h)_{(h,j)\in H\times J}\rangle$ in $\tilde{\mathscr{E}}$ is also a core allocation.*

Proof. The proof is almost the same as that of theorem 4.5, while the only difference is that we sum over h in S instead of in H in this proof and get $\sum_{h\in S} p\cdot w^h > \sum_{h\in S} p\cdot e^h + \sum_{h\in S}\sum_j \theta_j^h p\cdot z^j$, which contradicts the feasibility of an S-allocation. ∎

Chapter 5

Miscellaneous about Competitive Equilibria

5.1 Competitive Allocation as Limiting Core

The existence theorem about competitive equilibria in an economy doesn't tell us how these equilibria can be achieved by means of market activity, or in a much weaker sense, how competitive equilibria could be viewed as results from interactive activities of market participants, saying household and firms. From another stand point, as we showed in previous section, under very weak condition, each competitive allocation is in the core of the economy. And since core allocations could be viewed as the somehow reasonable results from interactive activities of market participants, i.e. no coalition is needed for getting a higher utility thus everyone will stay in the market and trade with each other, we could regard competitive equilibria as reasonable consequences in a (private) competitive market economy. However, one problem is there may exist much more core allocations in a economy than competitive allocations, and this fact prevent us from convincing the competitive equilibria being the unique reasonable consequences of market activity. Why should the household prefer a competitive equilibrium to a core allocation?

There were some reasonable arguments advocating competitive equilibria as proper and necessary result of competitive market as early as in 19th century. In his prominent book, Edgeworth (1881) illustrated that the core of an economy would shrink to competitive equilibria as the number of consumer tending to infinity by using his famous box, of course, there are only two types of consumer with the number of each type going to infinity. However this result depended essentially on geometrical interpretation of the economy with only two types of consumers, and it is quite not clear about whether or not the desired result will also hold when there are more than two types of consumers.

This problem is solved astonishingly in an elegant paper by Debreu and Scarf (1963), in which all proof is short and easy to understand. In this article, Debreu and Scarf considered a pure exchange economy $\mathscr{E}(r)$ consisting of m types of consumers and within each type there are r identical households, i.e. with identical preference and endowment. By imposing strictly convex preference[1] for each household, they showed in each core allocation for $\mathscr{E}(r)$, each household in the same type will get the same consumption bundle. Therefore, one only need to consider m consumption bundles when changing r. They also pointed out two facts that the set of core allocations of $\mathscr{E}(r+1)$ will be contained in that of $\mathscr{E}(r)$ and the competitive allocation of the economy with one household in each type will also be a competitive allocation of $\mathscr{E}(r)$[2], thus $\cap_{r=1}^{\infty}\{\text{core allocations in } \mathscr{E}(r)\}$ is not an empty set, which means one can consider an allocation[3] that belongs to the core for all $\mathscr{E}(r)$. With this preparation, Debreu and Scarf laid out following theorem.

Theorem. *If $(x_1, \ldots, x_m)$ is in the core for all r, then it is a competitive allocation.*

Therefore, roughly, we can say competitive equilibria become the unique reasonable consequence of the competitive market activities when there are infinitely many consumers in the market, at least if we restrict our criterion of "reasonable" to core allocations.

Debreu and Scarf (1963) also give two extensions of above theorem, that one is a relaxation on preference being convex but not strictly convex, and the other one is a similar result in a set up of production economy.

No surprising, one may wonder what will happen if no identical households are assumed, or equivalently what's going on if the number of types changes. For this kind of questions, the answer is similar to above theorem, that the core will shrink to competitive equilibria when the total number of household in the economy tends to infinity, provided that the eco-

[1] Preference $\succsim$ is strictly convex in a consumption space X, if for any x, $y \in X$ with $x \succsim y$, there is $\alpha x + (1-\alpha)y \succ$ y for all $\alpha \in (0,1)$.

[2] I.e. households of type t in $\mathscr{E}(r)$ will get the same C.E. consumption bundle as the only household of type t in $\mathscr{E}(1)$.

[3] No matter what r is, we can always treat an allocation in $\mathscr{E}(r)$ as consisting of m consumption bundles.

nomy becomes more "competitive", which can be formalized as

$$\frac{\sup_h ||e^h||}{||\sum_h e^h||} \to 0, \text{ when } H \to \infty,$$

i.e. the ratio of endowment of any household comparing to the total endowment falls to zero as the number of households goes to infinity.

5.2 Uniqueness of Competitive Equilibrium

One problem with competitive equilibria, as Nash equilibria, is that there may exist more than one equilibrium. For example, consider an 2×2 Edgeworth box with $e^1 = (2, 0)$, $e^2 = (0, 2)$, and $u^1(x, y) = \min(x, y)$, $u^1(x, y) = \min(x, y)$. Then it's obvious that every point on the line from $(0, 0)$ to $(2, 2)$ is a competitive allocation.

The most famous condition addressing this problem is called *gross substitution*, which is a sufficient condition to ensure the uniqueness of competitive equilibrium.

Consider a pure exchange economy $\mathscr{E} = (e^h, u^h)_{h \in H}$ where demand function $x^h(p) \equiv x^h(p, e^h) = \operatorname{argmax}_{z \in B^h(p, e^h)} u^h(z)$ is well defined, e.g. imposing strict convexity to the preference, and let $z^h(p) = x^h(p) - e^h$ denote the excess demand function, where both $x^h(p)$ and $z^h(p)$ are vector valued function from $\mathbb{R}^L_+$ to $\mathbb{R}^L_+$. Further define aggregate excess demand function as $z(p) = \sum_h z^h(p)$. Notice, $z(p) = z(\alpha \cdot p)$ for all $p \in \mathbb{R}^L_+$ and all scalar $\alpha > 0$.

Definition 5.1. *$z(p)$ satisfies gross substitute property, if for all $\ell \in L$ and all p, $p' \in \mathbb{R}^L_+ \setminus \{0\}$ s.t. $p'_\ell > p_\ell$ and $p'_k = p_k$ $\forall\, k \neq \ell$, there is $z_k(p') > z_k(p)$, $\forall\, k \neq \ell$.*

Gross substitution characterizes a property of the excess demand function, that once you increase the price of one commodity and keep all other prices unchanged then the excess demand for all commodities except this one goes up.

As we know, a price vector p and a set of allocations $(x^h)_{h \in H}$ consist of a competitive equilibrium in $\mathscr{E}$, if and only if

$$z(p) = \sum_h x^h(p) - \sum_h e^h = \sum_h x^h - \sum_h e^h = 0.$$

With this observation, we could easily prove following uniqueness theorem about competitive equilibrium.

Theorem 5.1. *Let $\mathscr{E} = (e^h, u^h)_{h\in H}$ be a pure exchange economy where demand function is well defined and all possible competitive equilibria consist of strictly positive price vectors* [4]. *If the aggregate excess demand function $z(p)$ of $\mathscr{E}$ satisfies gross substitute property, then there is unique competitive equilibrium*[5] *in $\mathscr{E}$.*

Proof. It suffices to show if p^1 and p^2 are such that $z(p^1) = 0 = z(p^2)$, then they are collinear, i.e. $p^1 = \alpha p^2$ with $\alpha > 0$. Suppose conversely, then, since p^1, $p^2 \gg 0$, we can assume $p^2 \gneq p^1$ with $p^2_\ell = p^1_\ell$ and for some component strict inequality holds. Consider a procedure in which you increase the price for one commodity k other than l from p^1_k to p^2_k. Observe $z_\ell(\cdot)$ will not decrease in this procedure, and moreover, it will increase by a strict positive amount at least in one step since $p^2_k > p^1_k$ for some k. Hence we have $z_\ell(p^1) \gneq z_\ell(p^2)$, which contradicts with $z(p^1) = 0 = z(p^2)$. ∎

An example of gross substitution is the demand function derived from constant elasticity of substitution (CES) utility function. Let $u(x_1, \ldots, x_L) = (\alpha_1 x_1^\rho + \cdots + \alpha_L x_L^\rho)^{1/\rho}$ be a CES utility function where $\alpha_1, \ldots, \alpha_L > 0$ and $\rho \in (-\infty, \infty)$. Suppose initial endowment to be $e = (e_1, \ldots, e_L) \gg 0$, and the corresponding Mashallian demand function is $x(p, e) = (x_1(p, e), \ldots, x_L(p, e))'$. It can be showed that $\partial x_k / \partial p_\ell > 0$, for $k \neq \ell$.

In general, no extension of gross substitution property is made for a production economy, since it seems not reasonable to assume that whenever the price of one commodity goes up, a firm would increase its demand (as input) or reduce its supply (as output) of other commodities.

5.3 Second Welfare Theorem

In fact, second welfare theorem, sometimes also called the second fundamental theorem for welfare economics, said nothing which makes sense, and it's an empty theorem. This theorem asserts that every Pareto optimal point in an economy with convex preferences and convex production sets can be achieved as a competitive allocation by appropriate lump-sum transfers of wealth to each agent.

[4] This is a somehow technical condition which is not emphasized in MWG pp. 613.

[5] Of course, if $\langle p, (x^h)_{h\in H}\rangle$ is a C.E., then for any positive scalar α, $\langle \alpha p, (x^h)_{h\in H}\rangle$ is also a C.E., and we shall refer these two as the same competitive equilibrium in $\mathscr{E}$.

However, from the existence theorem for competitive equilibrium proved in previous sections, it's not too difficult to show that for each Pareto optimal point, the social planner can first redistribute the initial endowments $e = \sum_h e^h$ to each household and each firm according to the Pareto optimal allocation $\langle (x^h)_{h\in H}, (y^j)_{j\in J}$, at the same time from the Pareto optimal allocation the social planner could find a proper price vector p such that each household h maximizes its utility within $B^h(p, (y^j)_{j\in J})$ and each firm j maximizes its profit in Y^j, then the social planner could decompose all the firms in a way like constructing an individualized economy and redistribute $\theta^h_j y^j$ to household h. After all this done, the desired Pareto optimal is a natural consequence of these redistributed endowments, or if you like, lump-sum transfers of wealth, since the social planner just puts the economy into a competitive equilibrium.

More intuitively, we can consider a pure exchange economy with differentiable utility functions, and suppose we have a Pareto optimal allocation $(x^h)_{h\in H}$ which is a interior point of $\mathbb{R}^{LH}_+$. We're going to show $\nabla u^1(x^1) = \cdots = \nabla u^H(x^H)$, where $\nabla f = \left(\frac{\partial f}{\partial x_1}, \cdots \frac{\partial f}{\partial x_L}\right)$ denotes the gradient of $f(\cdot)$. Suppose not, say, $\nabla u^i(x^i) \neq \nabla u^j(x^j)$, then we could find a vector Δe s.t. $e^i + \Delta e$, $e^j - \Delta e \in \mathbb{R}^L_+$, and $\nabla u^i(x^i)\cdot \Delta e$, $\nabla u^j(x^j)\cdot(-\triangle e) > 0$[6]. But this leads to a contradiction since $e^i + \Delta e$, $e^j - \Delta e$ is also attainable within initial resource constraint. Now, let the price vector $p = \nabla u^1(x^1)$, and redistribute the endowment as $(x^h)_{h\in H}$, we get a competitive equilibrium, but in fact there is no trading here since the economy has already been put in a competitive equilibrium.

[6] You may recall a problem in the final exam by Prof. Muench last semester which showed how you could find such a vector.

Chapter 6

Uncertainty

▶ Mar.1, 2010

6.1 Time, Uncertainty and Information Structure

This section is supplemented completely by Yan Liu. A general form of information structure will be introduced, which I wish to be of some help in understanding the setup of models in various fields in economics whenever time and uncertainty play a fundamental role. Nonetheless, all our discussion in the following sections will based on a simplest version of information structure, and no difficulty should be there if you decide to skip this section.

Consider an economy whose activities extend from 0 to ∞. Denote time period $t = 0, \ldots, \infty$. Basically, the uncertainty arises because the economy may stay in an arbitrary *state* $s_t \in S_t$, where S_t, *state space at* t, is the set of all possible states at t, e.g. the economy may be in a drought or not in a particular year t, thus $S_t = \{\text{drought, normal}\}$. We'll always assume there is no uncertainty in the initial period $t = 0$ [1], i.e. S_0 has only one element; on the contrary, there are uncertainty from $t = 1, \ldots, \infty$, i.e. S_t has more than one element for all t from 1 to infinity. In general, no restriction on S_t to be finite set is imposed.

Let $S = \prod_{\tau=0}^{\infty} S_\tau \equiv S_0 \times \cdots \times S_t \times \cdots$ be the set of all possible states in the economy, and let $S^t = \prod_{\tau=0}^{t} S_\tau$ be the set of all possible states up until time t, for all $t = 1, 2, \ldots$. It

[1] We can also consider the case in which uncertainty appears from this initial period, however it seems few models are set up in this way.

is clear that $S^t \subset S^{t+1}\ \forall\, t$. Further, we call $s^t = (s_0, s_1, \ldots, s_t) \in S^t$ a *path* from 0 to t, and $s = (s_0, s_1, \ldots) \in S$ a *path* of this economy.

Of course, our objective is to distinguish different commodities at time t. It is obvious that an apple and an orange at time t should be different commodities, no matter what state it is in the economy. However, in an uncertain circumstance, not only physical property or merely time is of great importance to a complete characterization of a commodity, but the state in which the commodity is at time t, moreover, perhaps all the underlying historic states where it has stayed before time t are critical for a fully understanding of its current relevance to the economy. For example, consider an economy extends for three periods, and $S_t = \{\text{drought, normal}\}$ for $t = 1, 2$. Suppose you have one bottle of water at $t = 2$, and also assume it is in a drought at $t = 2$. In such a case, it's quite possible that your valuation of the water will differ given different states in which the economy was in the previous period, i.e. you may valuate the water much higher if it was also a drought at $t = 1$. In this sense, we would like to distinguish this particular bottle of water with a label of $\langle$drought, drought$\rangle$, and say this bottle of water is a bottle of water contingent to an *event* $\langle$drought, drought$\rangle$[2].

In above example, we make a comprehensive distinction of commodities in a uncertain world by using *event contingent* notion[3]. However, the particular event $\langle$drought, drought$\rangle$ in this example is merely a point of state space $S^2 = S_1 \times S_2$[4], and it could be more convenient if we define an event as a subset of S^2. We could elaborate this idea by following example. Suppose we use precipitation measured by millimeter to indicate a particular state where the economy stays, hence the state spaces for this economy now become $S_1 = S_2 = [0, \infty)$. Moreover, let 50mm be the criterion of drought, so whenever $s_t \leq 50$ the economy is in a drought. With this notation, the previous event $\langle$drought, drought$\rangle$ has a new form $\{(s_1, s_2) | 0 \leq s_1, s_2 \leq 50\}$ which is a subset of $S^2 = S_1 \times S_2$.

With this understanding, we want define a event e^t to be a subset of S^t. However, not an arbitrary subset could be called as an event. Consider a set defined as $e = \{(s_1, s_2) | 0 \leq s_1 \leq 50,\ 0 \leq s_2 \leq 40\}$ in the previous example. e is an subset of S^2, but it is not an event, since we have never defined what is $\{0 \leq s_2 \leq 40\}$.

Therefore, first define *spot event* e_t as a non-empty subset of S_t, and *spot event set* E_t as

[2] Event contingent commodity is a more general concept compared with *state contingent commodity* that will be introduced in the following section, in which a two period setup is laid out.

[3] We use this term following Debreu (1959: Ch.7).

[4] Notice S_0 is s singleton set, hence it doesn't matter to omit S_0 from S^2.

a collection of all spot events at t of which the union contains S_t[5]. Notice, two spot events, as subsets of state space, may have non-empty intersection.

All spot event sets, with underlying state spaces, are given *a priori* as a component of the economy, as the agent set, relevant substance and the action set (like production plans in a production economy) in this economy, and implicitly, time structure in this economy is also fully specified at the same time.

And then, define a *event* e^t as a sequence of spot events taking the form of $\langle e_0, \dots, e_t \rangle$, which is equivalent to the recursive form $\langle e^{t-1}, e_t \rangle$; naturally, the corresponding *event set* is defined as a set of all events at t,i.e. $E^t = E_0 \times \cdots \times E_t$. Obviously, $E^t \subset E^{t+1}$, and a path $s^t \in e^t$ iff $s_0 \in e_0, \dots, s_t \in e_t$.

It is clear that once all event sets are specified in the economy, the spot event sets are also specified.

Uncertainty unfolds according to the time. At the beginning of each period, the economy will switch into a new state. This state may not be observable, but all events containing this state will be observed. With this information, all future events with this history are also determined.

Hence we give following definition of information structure.

Definition 6.1. *A collection of event spaces $(E^t)_{t\geq 0}$ with underlying state spaces $(S_t)_{t\geq 0}$, both of which are given as components of an economy $\mathscr{E}$, are called the information structure of $\mathscr{E}$, and each E^t is called the information set by time t.*

By now, we have not employed any probabilistic concepts to characterize the uncertainty in the economy, and the information structure are described in the term of events. Yet, for a more analytic framework, we would like to have the information structure compatible with a probability model, where more powerful tools can be used.

To achieve this objective, the only modification is to extend spot event set E_t to be a σ-field, of which the elements are subsets of S_t[6]. Let E_t denote this σ-field as well, and call it

[5] This condition is a natural requirement as we don't want to see the economy stays in some state that no event occurs.

[6] Given state space Ω, a σ-field $\mathscr{F}$ is defined as a collection of subsets of Ω which satisfies: i. $\Omega \in \mathscr{F}$; ii. if $A \in \mathscr{F}$, then the complement, $A^c \in \mathscr{F}$; iii. if $A_n \in \mathscr{F}$, $n = 1, \dots, \infty$, then $\cup_n A_n \in \mathscr{F}$. Given state space S_t and spot event set E_t, one can find a *minimal* σ-field $\mathscr{F}_t$ that contains E_t, and we call it the σ-field generated by E_t, which is also denoted $\sigma(E_t)$.

spot event field. Then, define *event field by time t*, E^t, as $E_0 \times \cdots \times E_t$, which is also a σ-field. Finally, define *event field* $E = \cup_{t \geq 0} E^t \equiv \prod_{t=0}^{\infty} E_t$, so a probability $P : E \to [0, 1]$ can be defined. Now, the probabilistic version of our information structure becomes a probability space (S, E, P). In most models, the σ-field E^t is also called *information set* by time t.

6.2 Arrow-Debreu Economy

We're going discuss uncertainty in an economy.

Let $L = \{1, \ldots, 5\}$ be a set of 5 different commodities, $S = \{1, \ldots, 4\}$ be 4 possible states in which our economy could be, and $T = \{1, 2, 3\}$ be 3 different time periods during which activities take place.

We can consider the commodities in a funny way, saying *state contingent* commodities, i.e. a commodity ℓ which is available at time t in state s, denoted as ℓts, is a different commodity from $\ell t's'$ where $t's'$ is a different time-state combination[7]. Since time could be viewed as a special kind of state, it's convenient to omit time index, and use "state contingent" only to denote different commodities. In addition, we assume there are 60 markets here, i.e. *complete market*, each of which is opened for trading a distinct state contingent commodity. Thus oranges in a sunny day could be traded with apples in a rainy day, or even with oranges from the same tree but in a rainy day, at least in the sense of promises of delivery in a specific state.

But actually, it is a very bad model to assume complete markets, since we need too many markets to support all the trading between two different state contingent commodities, especially when the number of possible states in the economy is huge. In real world, it is always the case that there are too many states, yet without enough markets.

Even though, it is of great value for us to consider a complete market setup for uncertainty in an economy, because with the notion of state contingent commodities, we could treat uncertainty exactly in the same way as an deterministic complete market model, in which existence of competitive equilibria is already established, and various concepts of welfare optimality have been investigated.

Formally, consider a two period economy consists of a set of households $H = \{1, \ldots, H\}$

[7] One index could be the same, i.e. even if one of $t = t'$ and $s = s'$ is true, ℓts is still different from $\ell t's'$.

and a set of firms $J = \{1, \dots, J\}$, where a private ownership is assumed, saying $\left(\theta_j^h\right)_{(h,j)\in H\times J}$ is given with $\sum_h \theta_j^h = 1$. There is no uncertainty in first period $t = 0$, and the economy may stay in one state $s \in S$ in the second period $t = 1$, where $S = \{1, \dots, S\}$ is a finite set of possible states[8] in $t = 1$. At $t = 0$, no commodity is available, and at $t = 1$, a set of different kinds of commodities $L = \{1, \dots, L\}$ is available, i.e. delivery will be made and all commodities will be consumed once a specific state reveals. Therefore, consider contingent commodity space $\mathbb{R}_+^{LS}$, where each point $x = \{x_{11}, \dots, x_{L1}, \dots, x_{1S}, \dots, x_{LS}\}$ is a collection of the quantity of every kind of commodities in every state. An endowment vector $e^h \in \mathbb{R}_+^{LS}$ is specified for each household h *a priori*, and h knows this information *ex ante*, but only $e_s^h = \{e_{1s}^h, \dots, e_{Ls}^h\}$ will be given to h if state s reveals.

For each state contingent commodity, there is a market opened at the beginning of $t = 0$, hence a (relative) price system $p = (p_{\ell s})_{(\ell,s)\in L\times S} \in \mathbb{R}_+^{LS} \setminus \{0\}$ is observable by all households and firms. Moreover, denote $p_s = (p_{1s}, \dots, p_{Ls}) \in \mathbb{R}_+^L \setminus \{0\}$, and $p = (p_1, \dots, p_S)$.

Each firm will choose a production plan $y^j \in Y^j \subset \mathbb{R}_+^{LS}$, where production set Y^j depicts the technological restriction for possible production allocations in every state, to maximize the *ex ante* profit[9], $p \cdot y^j$ within this price system p. Denote $y_s^j = (y_{1s}^j, \dots, y_{Ls}^j) \in \mathbb{R}^L$, and $y^j = (y_1^j, \dots, y_S^j)$. One could imagine this firm j writes down this production plan as a promise, and after the state at $t = 1$ reveals, it will deliver what ever written as positive entries in the promise y_s^j and receive whatever written as negative entries.

Each household h will choose a consumption plan $x^h \in \mathbb{R}_+^{LS}$ according to the observed price system, the endowment known *a priori* and the profit share from firms, by maximizing *ex ante* utility [10], given by utility function $u^h(\cdot)$ defined over $\mathbb{R}_+^{LS}$. More specifically, the *ex*

---

[8] Using the notions in the previous section, each state here by itself is a event.

[9] One might wonder why a firm should maximize a so defined profit. In fact, rewrite $p \cdot y^j$ as $\sum_s p_s \cdot y_s^j$, where y_s^j is production plan for state s. We see once j maximizes $p \cdot y^j$, the *ex ante* profit $p_s \cdot y_s^j$ in state s will also be maximized. However, since p_s represents relative prices, $p_s \cdot y_s^j$ is the *ex post* profit in state s under p_s as well, and hence is also maximized.

[10] One may wonder as in the case for firms why household h would maximize such a *ex ante* utility. Since its *ex post* utility should come from its consumption in a particular state s, hence it seem more reasonable for h to maximize its utility separately in each state. However, for such a reasoning, one conceptual problem arisen here is what utility function h should use for such a purpose?

Since $u^h(\cdot)$ is defined on $\mathbb{R}_+^{LS}$, it seems that the utility level in state s may depend on its consumption plan for other state as well, if we use $u^h(\dots, x_s^h, \dots)$ directly to measure its utility.

However, consider a case in which h has two states, one is in good luck and h might win \$500 or more in lottery, while the other one is in bad luck and h could only win less \$20. If possible, let $u(g, b)$ denote *ante* resource constraint of h is given by a budget set as follows,

$$B^h(p,(y^j)_{j\in J}) = \left\{ x \in \mathbb{R}_+^{LS} \,\middle|\, p \cdot x \leq p \cdot e^h + \sum_{j\in J} \theta_j^h \, p \cdot y^j \right\}.$$

And h is going to maximize its utility on this budget set and make the decision about its consumption plan at $t = 1$ according to this price system. One can imagine h writes down its *net demand* $x^h - e^h - \sum_j \theta_j^h y^j$ as a promise, and acts in a similar way as firm j.

In general, we call such a setup a Arrow-Debreu economy. Evidently, we can treat this economy in the same way as a deterministic economy. We could define Arrow-Debreu equilibrium, which is merely competitive equilibrium in this setup, as we did before, and we could establish existence of competitive equilibria using the same theorems. Now, we just laid out the definition for equilibrium as follows.

Definition 6.2. *$\langle p, (x^h)_{h\in H}, (y^j)_{j\in J}\rangle$ is a competitive equilibrium $(C.E.)$ of an Arrow-Debreu economy, if*

- *Firms maximize profit.*
  $y^j \in \text{argmax}_{z\in Y^j}\ p\cdot z,\ \forall\ j \in J.$
- *Households maximize utility.*
  $x^h \in \text{argmax}_{z\in B^h(p,(y^j)_{j\in J})}\ u^h(z),\ \forall\ h \in H.$
- *Market clear.*
  $\sum_h (x^h - e^h) - \sum_j y^j = 0.$

h's utility function with g being the money won in good luck and b in bad luck, and also assume $u(g,b)$ is differentiable. Should h's utility level $u(g,15)$ increase as he is in bad luck already and wins only \$15 ex post while if he's in good luck and wins more and more money? This question seems ridiculous since h can only be in one possible state, either good or bad luck, but can not be in both. Thus it seems reasonable to assume $\partial u(g,\bar{b})/\partial g$=0 for any given $\bar{b}$, saying there's no cross effect between different states.

And in this sense, one should be able to write $u^h(x_1^h,\ldots,x_S^h) = \sum_s u_s^h(x_s^h)$, where $u_s^h(\cdot)$ is a state specific utility function. Once $u^h(\cdot)$ has this form, h could maximize $u(x_1^h,\ldots,x_S^h)$ with respect to x_s only, and no worried about effects from consumption plans in other states is needed.

Debreu (1960) proves that once utility function has such a *state independent* property, then it could indeed be written in additive form. It is also worth to mention that expected utility by definition posses such a property.

Back to our original problem, whenever state independent property is assumed, then maximizing $u^h(\cdot)$ on the *ex ante* budget set is equivalent to maximize $u^h(\cdot)$ w.r.t x_s^h only on a degenerate budget set $\{x \in \mathbb{R}_+^L : p_s\cdot x_s \leq p_s \cdot e_s^h + \sum_{j\in J}\theta_j^h\ p_s\cdot y_s^j\}$ for every state s. Since p_s represents relative price in s, the last assertion implies x_s^h also maximizes *ex post* utility in this particular state.

6.3 Assets, Rational Expectation and Radner Equilibrium

As mentioned above, the disadvantage of a setup like Arrow-Debreu economy is that too many markets need to be assumed existing *ex ante* to fulfill the trading demand for the economy, and this is highly unrealistic in real economy. Fortunately, there is another approach to formulate uncertainty in an abstract economy.

For simplicity, we consider an exchange economy. The basic setup takes the same form as Arrow-Debreu economy, except there is no production sector and no firm shares for household.

However, instead trading promises in at $t = 0$ in $L \times S$ markets, households trade K assets $A^1, \ldots, A^K$ in K markets, where $A^k = (A^k_1, \ldots, A^k_S) \in R^S_+$ and whenever h has one unit A^k then h will receive a return of A^k_s dollars "money" at $t = 1$ if s reveals. Of course, here "money" should be represented by some real commodity otherwise it makes no sense for households to purchase any physical commodities with this "money", so we set $1 \in L$ for each state $s \in S$ to be the *numeraire*, and hence there are S numeraires. *Short position* is allowed, and also assume all assets are perfect divisible, so the portfolio $\theta^h = (\theta^h_1, \ldots, \theta^h_K)$ [11] held by household h is a vector in $\mathbb{R}^K$, hence $\sum_k \theta^h_k A^k_s$ is the total "money" h receives in s with this portfolio θ^h. Further, let $\pi = (\pi_1, \ldots, \pi_K)$ denote the prices for each asset observed in the K markets, and assume no initial assets for every household, therefore $\pi \cdot \theta^h \leq 0$, reading as all portfolio should be *self financing*.

Instead of evaluating any assets, households only evaluate an *ex ante* determined consumption plan as in an Arrow-Debreu economy. Therefore, no direct way exists to tell us how a household h would choose his portfolio θ^h. However, we will see how to overcome this conceptual difficulty after we investigate the procedure in which households choose their consumption plan.

Assume each household h has its own expectation at $t = 0$ *ex ante* about the *spot prices* $p_s = (p_{1s}, \ldots, p_{Ls}) \in \mathbb{R}^L_+ \setminus \{0\}$ prevailing at $t = 1$ for every state s, and as before commodity 1 is set to be the numeraire. Assume h already has a portfolio θ^h, so the total "money" resource is $p_s \cdot e^h_s + p_{1s} \sum_k \theta^h_k A^k_s$ in state s. According to these resource constraints, h chooses a consumption plan $x^h = (x^h_1, \ldots, x^h_S)$ s.t. $p_s \cdot x^h_s \leq p_s \cdot e^h_s + p_{1s} \sum_k \theta^h_k A^k_s$ for all s. Now observe that households may get a higher utility by adjusting their portfolios,

[11] Whenever $\theta^h_k < 0$, it is called a short position and h need to pay $\theta^h_k A^k_s$ dollars to the buyer if s reveals at $t = 1$.

so after forming their expectation about the spot prices, they maximize utility by choosing portfolios and consumption plans at the same time.

Summing up, the budget set for household h will take following form,

$$B^h(p,\pi) = \left\{ (\theta^h, x^h) \in \mathbb{R}^K \times \mathbb{R}_+^{LS} \,\middle|\, \pi \cdot \theta^h \leq 0 \text{ and } p_s \cdot (x_s^h - e_s^h) \leq p_{1s} \sum_k \theta_k^h A_s^k,\ \forall\, s \in S \right\},$$

note the profile of spot prices $p = (p_1, \dots, p_S)$ is of h's own expectation.

Every household makes the portfolio decision and chooses consumption plan for the next period, then after a particular state s is revealed at the beginning of second period, L markets open for the state contingent commodities $1s$ to Ls, in which all delivery are accomplished. But a essential problem follows: if the prices for these L commodities don't coincide with households' expectations, how could market clear be satisfied?

The breakthrough in this setup turns out to be a fundamental notion that explains, or in fact defines how peoples' expectations will be realized in the second period. This notion is *rational expectation*. By assuming all households have rational expectation, it means not only all of them have same expectations about spot prices in the future, but also these expected prices will indeed clear the markets whenever they open in a particular state. This latter characterization for rational expectation is also called *self fulfill*. Hence, once we impose rational expectation assumption to the household, an equilibrium concept analogous to competitive equilibrium could be laid out without any difficulty. Formally, such a equilibrium concept is called Radner equilibrium.

Definition 6.3. *$\langle \pi, p, (\theta^h, x^h)_{h\in H}\rangle$ constitutes a Radner equilibrium in a pure exchange economy $\mathscr{E} = (e^h, u^h)_{h\in H}$ under uncertainty $S = (1, \dots, S)$ with assets $A^1, \dots, A^K \in \mathbb{R}_+^S$, if*

- *Household maximizing utility,*
  for all $h \in H$, $u^h(x^h) \geq u^h(\tilde{x}^h)$, $\forall\, (\tilde{\theta}^h, \tilde{x}^h) \in B^h(p,\pi)$.

- *Asset market clear,*
  $\sum_h \theta^h = 0$.

- *Commodity market clear,*
  $\sum_h x^h = \sum_h e^h$.

6.4 Complete Market and Asset Structure

It turns out that asset structure, i.e. complete or incomplete, is a fundamental factor for the implications of Radner equilibrium. To formalize our discussion, let $A = (A^1, \dots, A^K)$ be the return matrix of these K assets where A^k is treated as a column vector.

Definition 6.4. *Let S be the number of states in an economy. Then, the asset structure of $A^1, \dots, A^K$ is said to be complete, if* $\operatorname{rank}(A) = S$*; and incomplete, if* $\operatorname{rank}(A) < S$.

Using a different notation, the asset structure is complete iff $\operatorname{span}(A^1, \dots, A^K) = \mathbb{R}^S$. Note also, the maximal number of rank A is S, since A is an $S \times K$ matrix. Also note, with this notation, the commodity budget constraint can be written as

$$\begin{bmatrix} p_1 \cdot (x_1^h - e_1^h) \\ \vdots \\ p_S \cdot (x_S^h - e_S^h) \end{bmatrix} \leq \begin{bmatrix} p_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & p_{1S} \end{bmatrix} A\theta^h,$$

where θ^h is arranged as a column vector.

Now we consider a special kind of assets. Let $A^1, \dots, A^S$ be S unit vectors[12] in $\mathbb{R}^S$, and it is often called Arrow-Debreu security for each one of such kind of assets. Following theorem asserts that once assuming such an asset structure (which is obviously complete), then Radner equilibrium coincides with Arrow-Debreu equilibrium. And in this sense, complete asset structure and complete market are the two sides of a same coin.

Theorem 6.1. *Assume $A^1, \dots, A^S$ to be S assets in a pure exchange economy $\mathscr{E}$ under uncertainty with weakly monotonic utility, where each A^s is a unit vector in $\mathbb{R}^S$. Then, Arrow-Debreu equilibrium $\langle p, (x^h)_{h\in H}\rangle$ with $p \gg 0$ is the same as Radner equilibrium $\langle \pi, p, (\theta^h, x^h)_{h\in H}\rangle$ with $\pi, p \gg 0$, in the sense that their equilibrium allocations are the same.*

Proof. We divide this proof into two parts.

(i).We show how to construct π and $(\theta^h)_{h\in H}$ such that $\langle \pi, p, (\theta^h, x^h)_{h\in H}\rangle$ constitutes a Radner equilibrium where p and $(x^h)_{h\in H}$ come from an Arrow-Debreu equilibrium.

Let Λ be an diagonal matrix with p_{1s} as its (s, s) element, then Λ is invertible since $p_{1s} > 0$. Define $\pi = \mathbf{1}\Lambda = (p_{11}, \dots, p_{1S})$, where $\mathbf{1} = (1, \dots, 1)$. Let $w_h = (p_1 \cdot (x_1^h -$

[12] I.e. A^s is a vector of which the s'th component is 1 and all other components are 0.

$e_1^h), \ldots, p_S \cdot (x_S^h - e_S^h))$ be a column vector. Since utility is weakly monotonic, households will exhaust their resources in an Arrow-Debreu equilibrium, so $\mathbf{1}w^h = p \cdot (x^h - e^h) = 0$. And by market clear in Arrow-Debreu equilibrium, $\sum_h w_h = 0$. Define $\theta^h = \Lambda^{-1} w^h$, hence commodity budget feasibility are satisfied by definition. And we have $\sum_h \theta^h = \Lambda^{-1} \sum_h w^h = 0$, thus assets market clear is satisfied. Further, $\pi \cdot \theta^h = \mathbf{1}\Lambda\Lambda^{-1} w^h = \mathbf{1}w^h = 0$, thus portfolio θ^h meets with the budget feasibility.

We assert (θ^h, x^h) maximizes $u^h(\cdot)$ on the budget set for Radner equilibrium. Let $(\tilde{\theta}^h, \tilde{x}^h) \in B^h(p, \pi)$ be a point in the budget set for Radner equilbrium, then $\mathbf{1}\Lambda \cdot \tilde{\theta}^h = \pi \cdot \tilde{\theta}^h \leq 0$. Observe that $\tilde{w}^h \leq \Lambda \tilde{\theta}^h$, where $\tilde{w}^h = (p_1 \cdot (\tilde{x}_1^h - e_1^h), \ldots, p_S \cdot (\tilde{x}_S^h - e_S^h))$, hence $p \cdot (\tilde{x}^h - e^h) = \mathbf{1}\tilde{w}^h \leq \mathbf{1}\Lambda\tilde{\theta}^h \leq 0$. Hence $\tilde{x}^h$ belongs to the budget set for the Arrow-Debreu equilibrium, and so there is $u^h(x^h) \geq u^h(\tilde{x}^h)$, which justifies our assertion.

(ii).We show how to construct a price vector $\hat{p}$ such that $\langle \hat{p}, x^h \rangle$ constitutes an Arrow-Debreu equilibrium where $\langle \pi, p, (\theta^h, x^h)_{h \in H} \rangle$ is a Radner equilibrium.

Without loss of generality, we assume $p_{1s} = 1$, hence Λ is an identity matrix. Define $\hat{p} = (\pi_1 p_1, \ldots, \pi_S p_S)$ where p_s is viewed as row vector. Using the same notation as in (i), we have $\pi \cdot \theta^h \leq 0$ and $w^h \leq \Lambda \theta^h = \theta^h$. Hence $\hat{p} \cdot (x^h - e^h) = \sum_s \pi_s p_s \cdot (x_s^h - e_s^h) = \pi \cdot w^h \leq \pi \cdot \theta^h \leq 0$, so x^h meets with the budget feasibility for an Arrow-Debreu equilibrium with a price vector $\hat{p}$. Since market clear is the same between these two equilibrium, only maximality of x^h remains to be shown.

Let $\tilde{x}^h$ be a point in the budget set for Arrow-Debreu equilibrium, so $\pi \cdot \tilde{w}^h = \sum_s \pi_s p_s \cdot (\tilde{x}_s^h - e_s^h) = \hat{p} \cdot (\tilde{x}^h - e^h) \leq 0$. Define $\tilde{\theta}^h = \tilde{w}^h$, so we have $\pi \cdot \tilde{\theta}^h = \pi \cdot \tilde{w}^h \leq 0$. Therefore $(\tilde{\theta}^h, \tilde{x}^h)$ is point in the budget set for the Radner equilibrium, hence $u^h(x^h) \geq u^h(\tilde{x}^h)$. ∎

Chapter 7

Matching

This chapter tends to be a supplementary review of some results of matching, including Conway's theorem, the lattice structure of the set of stable matchings, and the uniqueness of stable matching. After all, it is highly recommended to read the original paper of Gale and Shapley (1962).

7.1 Basic notations and results

All through this chapter, we will assume there are finite men and women of equal number. All people have a *strict* and *complete* ranking over the group of opposite gender. A *matching*, denoted by μ, is a set of pairs in which each person is assigned (only) one partner, and a *stable matching* is defined according to Gale and Shapley (1962). For each man m and woman w, $\mu(m)$ and $\mu(w)$ represent m's and w's partner respectively.

Furthermore, denote *men-proposing procedure* by MPP, likewise WPP for *women-proposing procedure*. Correspondingly, denote the resulting matchings from MPP and WPP by μ_{MPP} and μ_{WPP}. For each man m, we define $\text{Poss}(m)$ as the set of women to whom m gets married in all stable matchings, likewise, we could define $\text{Poss}(w)$ for each woman w.

The following two propositions are due to Gale and Shapley (1962).

Theorem 7.1. *Both μ_{MPP} and μ_{WPP} are stable.*

Theorem 7.2. *When MPP is followed, each man m gets his top choice in* $\mathrm{Poss}(m)$. *Analogously, each woman w gets her top choice in* $\mathrm{Poss}(w)$ *when WPP is followed.*

Remark 7.1. The first theorem implies that $\mathrm{Poss}(m)$ and $\mathrm{Poss}(w)$ are not empty for all m and w.

The next result is a simple implication of the theorem 7.2.

Theorem 7.3. *If $\mu_{\mathrm{MPP}} = \mu_{\mathrm{WPP}}$, then the stable matching is unique.*

Proof. Suppose the converse is true, that there is a stable matching μ differing from μ_{MPP}. Then, there exists a couple $m \leftrightarrow w$ in μ such that m has a different wife w' in μ_{MPP}. Note that w' is m's top choice in $\mathrm{Poss}(m)$, thus m prefers w' to w. Likewise, since m is the husband of w' in μ_{WPP}, it follows that w' must prefer m to her husband in μ, say m'. Now, look at μ again. Among two couples $m \leftrightarrow w$ and $m' \leftrightarrow w'$, we have found that m and w' both prefer each other to their own spouses, hence μ can not be stable, which completes our proof. ∎

7.2 Conway's Theorem

Let S be the set of all stable matchings. For any two stable matchings μ and μ', define an operation $\mu \vee_M \mu'$ as follows: For each man m, let m choose among his wives in μ and μ' the most preferred one. As a result, this operation forms a new set of pairs denoted by the same notation μ and μ'. Note this definition doesn't exclude the possibility of two men choosing the same woman. Analogously, define an operation $\mu \wedge_M \mu'$ by letting each man choose his least preferred women among his wives in μ and μ'. Likewise, we could define the same kind of operations according to women's preferences, which we shall denote by $\mu \vee_W \mu'$ and $\mu \wedge_W \mu'$.

With these notations, we have the following Conway's theorem.

Theorem 7.4. *Let $\lambda = \mu \vee_M \mu'$ where $\mu,\ \mu' \in S$, then we have*

- *λ is a matching;*
- *λ is a stable matching;*

- *each woman in λ gets her least preferred man from μ and μ'.*

Analogous results hold for $\wedge_M$, $\vee_W$, and $\wedge_W$.

Proof. First, it's suffice to show that two men will not choose the same woman. Suppose conversely, there are two men m_1 and m_2 both choose the same woman w among their partners in μ and μ'. Without loss of generality, suppose w is m_1's wife in μ and m_2's wife in μ'. In μ, we know m_2 prefers w to his wife in the same matching, so w must prefer m_1 to m_2 otherwise μ is unstable. However this leads to a contradiction to the stability of μ', since now m_1 prefers w to his wife in this matching, while w prefers m_1 to her husband m_2.

Second, suppose λ is unstable, then without loss of generality, we could assume there are two men m_1 and m_2 such that m_1 prefers $\lambda(m_2)$ to $\lambda(m_1)$ while $\lambda(m_2)$ prefers m_1 to m_2 as well. If both $m_1 \leftrightarrow \lambda(m_1)$ and $m_2 \leftrightarrow \lambda(m_2)$ are couples in either μ or μ', then the stability in either matching is violated. Together with symmetry, it implies that we only need to consider the case that $\lambda(m_1) = \mu(m_1)$ and $\lambda(m_2) = \mu'(m_2)$. In μ', we have $\lambda(m_2)$ prefers m_1 to m_2, on the one hand; and since $\lambda(m_1)$ is the preferred one among $\mu(m_1)$ and $\mu'(m_1)$ by m_1, we have m_1 prefers $\lambda(m_2)$ to $\mu'(m_1)$, on the other hand. This leads to a contradiction to the stability of μ'. Hence λ must be stable.

Third, suppose conversely that there is a woman w who gets her most preferred man m from μ and μ'. By symmetry, we can assume $m = \mu(w)$. Then in μ', we have $m' \leftrightarrow w$ and $m \leftrightarrow \mu'(m)$, where w prefers m, and m prefers w also by the definition of λ. This contradiction completes our proof. ∎

Remark 7.2. It can be easily verified that $\vee_M = \wedge_W$ and $\wedge_M = \vee_W$, where the equalities hold in the sense that the operations on both sides result in the same matching.

7.3 Lattice Structure of S

A *finite lattice* is a finite set X over which a partial order $\succsim$ is defined such that for any two elements $x,\ y \in X$ there exist w and z in X which satisfy

$$z \succsim x, y \text{ and if there is } z' \in X \text{ s.t. } z' \succsim x, y, \text{ then } z' \succsim z;$$
$$x, y \succsim w \text{ and if there is } w' \in X \text{ s.t. } x, y \succsim w', \text{ then } w \succsim w'.$$

In addition, z is called the *minimal upper-bound* of $\{x, y\}$, and w is called the *maximal lower-bound* of $\{x, y\}$. Given a finite lattice, it is obvious that there exists a *maximal element* $\bar{x} \in X$ such that $\bar{x} \succsim x$ for all $x \in X$; likewise, there exists a *minimal element* $\underline{x} \in X$. Therefore, if $\bar{x} = \underline{x}$, we conclude that X is a singleton.

Turn to S, the set of all stable matchings. Evidently, given $\mu,\ \mu' \in S$, $\mu \vee_M \mu'$ and $\mu \wedge_M \mu'$ are the minimal upper-bound and maximal lower-bound of μ and μ' respectively, according to the partial ordering defined by men's preferences.[1] Moreover, it is easy to verify that μ_{MPP} and μ_{WPP} are the maximal and minimal element of S respectively according to men's preference. Hence S is a finite lattice. Similarly, we can use women's preference to define a partial ordering under which S is also a finite lattice.

Employing the fact that S is a finite lattice under men's preference, it follows that if $\mu_{\text{MPP}} = \mu_{\text{WPP}}$, then there is a unique stable matching. This gives an alternative proof for theorem 7.3.

[1]More precisely, for two stable matchings μ and μ', $\mu \succsim \mu'$ is defined as if each man in μ is no worse of than in μ according his preference.

Bibliography

Arrow, Kenneth J. and Gerard Debreu, "Existence of an Equilibrium for a Competitive Economy," *Econometrica*, July 1954, *22* (3), 265–290. 16, 17, 34

Debreu, Gerald, "Topological Methods in Cardinal Utility Theory," in "Mathematical Methods in the Social Sciences," Standford University Press, 1960, chapter 1. 52

Debreu, Gerard, *Theory of Value*, New York: John Wiley and Sons, 1959. 48

_ **and Herbert Scarf**, "A Limit Theorem on the Core of an Economy," *International Economic Review*, 1963, *4* (3), 235–246. 42

Edgeworth, F. Y., *Mathematical Psychics*, London: Kegan Paul, 1881. 41

Eilenberg, Samuel and Deane Montgomery, "Fixed Point Theorems for Multi-Valued Transformations," *American Journal of Mathematics*, 1946, *68* (2), 214–222. 14

Gale, David and Lloyd S. Shapley, "College Admissions and the Stability of Marriage," *American Mathematical Monthly*, 1962, *69* (1), 9–15. 57

von Neumann, John and Oskar Morgenstern, *Theory of Games and Economic Behavior*, 2nd ed., Princeton: Princeton University Press, 1947. 6