Making Sovereign Debt Safe with a Financial Stability Fund*

Yan Liu† **Ramon Marimon**‡ **Adrien Wicht**§

March 22, 2023

Abstract

We develop an optimal design of a Financial Stability Fund that coexists with the international debt market. The sovereign can borrow defaultable bonds on the private international market, while having with the Fund a long-term contingent contract subject to limited enforcement constraints. The Fund contract does not have *ex ante* conditionality, but requires an accurate country-specific risk-assessment (DSA), accounting for the Fund contract. The Fund periodically announces the level of liabilities the country can sustain to achieve the constrained-efficient allocation. The Fund is only required *minimal absorption* of the sovereign debt, but it must provide insurance (Arrow-securities) to the country. Furthermore, with the Fund *all sovereign debt is safe independently of the seniority structure*; however, seniority of the Fund, with respect to the private lenders, may require a greater *minimal absorption* than a *pari passu* regime. We calibrate our model to the Italian economy and show it would have had a more efficient path of debt accumulation with the Fund.

Keywords: Recursive contracts, limited enforcement, debt stabilisation, debt overhang, safe assets, seniority structure

JEL Classification: E43, E44, E47, E62, F34, F36, F37

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*We would like to thank Aitor Erce, Alessandro Dovis, Liyan Shi, Pedro Teles and the participants of the 2020 ADEMU Workshop, the 2021 EEA meeting, the 2022 ESM Research Seminar, the 2022 SED annual meeting, the 2022 IEF conference, the 2022 Sovereign Debt Conference at Minneapolis Fed, and other seminars where this work has been presented, as well as to the referees on a previous version, for helpful comments. We acknowledge the financial support of the Max Weber Chair programme of the EUI and of the European Stability Mechanism. The views expressed in this study are the authors' and do not necessarily reflect those of the European Stability Mechanism. All remaining errors are our own.

†Wuhan University; yanliu.ems@whu.edu.cn

‡Universitat Pompeu Fabra - BSE, CREi, EUI, CEPR and NBER; ramon.marimon@upf.edu

§European University Institute; adrien.wicht@eui.eu

1 Introduction

In the last few years, the public debt-to-GDP ratio has reached historic levels in the world. For instance, in the European Union (EU), the average indebtedness of Member States amounted to 86% of GDP in 2022, whereas it was 66% in 2000. Some countries such as Greece, Italy, Portugal and Spain have already reached a debt ratio above 100%.[1] This is the result of three consecutive crises — the global financial crisis of 2007–2009, the European sovereign debt crisis of 2010–2012 and the COVID-19 crisis. In response to these crises, important institutional and policy changes took place, making the Euro area and the EU more resilient but, for the time being, more indebted.[2] As a result of these changes, at the end of 2021, Euro area institutions were playing a leading role in their sovereign debt market, holding more than 30% of the sovereign debt of all Euro area countries.[3] Nevertheless, the question of how to efficiently stabilise the sovereign debt — for example, with complementary official lending programmes — remains open.

To address this question, we design a Financial Stability Fund (Fund) as a *constrained-efficient mechanism*, in line with Ábrahám et al. (2021).[4] While the latter assumes that the Fund absorbs all the sovereign debt of a country and focuses on the borrower's perspective, we emphasize the lender's side of the contract and derive the optimal relationship between the private competitive lenders and the Fund. More precisely, we assume that sovereign countries can raise debt in the private international market and in the Fund.[5] While private international lenders solely offer credit (i.e. long-term non-contingent defaultable bonds), the Fund provides both credit and insurance (i.e. Arrow securities) in the form of long-term

[1]According to AMECO, General Government Gross Debt in 2022: Euro area 94%, Italy 145%, Portugal 115%, Spain 114% and Greece 171%.

[2]In particular, starting the European Banking Union, founding the European Stability Mechanism, implementing asset purchasing programmes by the ECB, some including purchases of Euro area sovereign debt, and the COVID-19 Next Generation EU (NGEU) programme of the EU making, *de facto*, the European Commission the world's largest official lender, with unprecedented emissions of EU debt.

[3]Particularly, the sovereign debt holdings by Euro area institutions represents for Cyprus, Italy, Portugal and Spain more than 40% of their GDP and for Greece more than 120%.

[4]The main difference with respect to Ábrahám et al. (2021) is threefold. First, we do not consider an exclusive contract between the Fund and the contracting countries. Second, we use growth shocks to better analysze the interest rate-growth differential (i.e. $r - g$). Third, we abstract from moral hazard as we focus on the lending side of the contract.

[5]The adjective 'private' is used to distinguish lenders on the international market relative to the Fund.

state-contingent securities.[6] The Fund's intervention is constrained to prevent default and to satisfy a strict *debt sustainability analysis* (DSA), which is an evaluation of the present value of the sovereign's future surpluses (net savings). The Fund takes into account the country's indebtedness (i.e. commitments) with private lenders, which brings the issue of whether the Fund possesses seniority. In line with the official lending practice, we consider two regimes: *pari passu* (i.e. no seniority) and *seniority* of Fund's liabilities over private liabilities.

The Fund does not impose *ex ante* conditions, provided there can be feasible contracts with the country, given its existing sovereign debt.[7] This requires upfront a detailed risk-assessment of the country and a calibration of the economy which allow the Fund to compute the optimal borrowing policy the sovereign should adopt. This policy defines the *total* debt holdings and insurance necessary to reach the *constrained efficient allocation*. Then, in any given period, the Fund plays a dual role with respect to the country with a long-term contract: first, it announces the total liabilities that the country can sustain for next period, provided they maintain the contract with the Fund; second, after the country has contracted some, or all, of its debt liabilities with private lenders, the Fund implements its contract, for the period, with its insurance and, if needed, its additional lending. In the decentralized version of our economies, in a rational expectations *Recursive Competitive Equilibrium* (RCE), the announced state-contingent liabilities coincide with the realized ones, and the *constrained efficient allocation* is implemented.[8]

The characterization of this RCE implementation is remarkable. First, the Fund stabilizes the entire indebtedness of the sovereign. In other words, the entire sovereign debt becomes safe, without default risk. Second, we assume that there is sufficient private demand for safe assets and, therefore, there is only need for a *minimal intervention policy* (MIP) of the Fund in the sovereign debt market. Such intervention consists of an insurance component with an additional guarantee on long-term debt holdings by private lenders when the DSA binds. Third, all sovereign debt is safe independently of the seniority structure. However, seniority

[6]We consider that the private lenders cannot credibly provide insurance due to strategic complementarity in their actions. As a large player, the Fund is unaffected by such coordination issue.

[7]There may be very high levels of debt that may require restructuring to make the Fund contract feasible or the country may prefer to implement some *ex ante* reforms to improve its risk-profile; that is, the Fund can, and should, have a *menu of Fund contracts* depending on different risk profiles.

[8]Our characterization of the Fund is a Nash RCE. The Fund does not play the role of a Ramsey planner, since it lacks the authority to fully control the market transactions between the private lenders and the sovereign borrower, neither directly through planned allocations nor indirectly via policy instruments.

of the Fund may require a greater debt absorption by the Fund than a *pari passu* regime. Fourth, the Fund's DSA internalizes a pecuniary externality that competitive private lenders usually do not: the fact that, in an economy with safe debt, marginal lending can be excessive. Fifth, the Fund — as capacity announcer and provider of insurance and, when needed, debt — implements a unique non-autarkic allocation which features no default, therefore, no debt-dilution and no excess lending. Sixth, being a *constrained efficient allocation* the implied fiscal policy is highly counter cyclical. Seventh, the choice of maturity is irrelevant in steady state unless the Fund's DSA binds — in which case it is optimal for the sovereign to choose one-period debt. In sum, the literature on sovereign debt has mostly focused on the borrower's default decision, we contribute by characterising the lenders' optimal policy and its impact on the sovereign debt market.

From this characterization, the third and the fourth findings are particularly novel and require some explanation. Regarding the latter, the depreciation of the value of the debt can take different forms: when debt is nominal, with inflation; when debt is real and defaultable, with default and dilution, and when the debt is real and perceived safe, with excessive lending. The former two have been extensibly studied, the latter one not. In our economies with a Fund, the DSA monitors whether additional borrowings entail expected losses which, in the context of a union of sovereign states, would result in undesired permanent transfers (i.e. debt mutualization). The DSA constraint being binding results in a negative spread, a price signal that lenders should not purchase new debt, but also that, if they can, they should sell their holdings of long-term debt in exchange for riskless assets with a better return. Expectations of these *sudden stop* turbulences can harm the value of long-term bonds. That is, the Fund's MIP can be seen as a prudential policy: if the DSA binds, the Fund is willing to absorb "whatever it takes" of the existing stock of long-term debt, while keeping its commitment to provide insurance, in order to repel the turmoil.

Regarding seniority, the intuition behind our result can be explained in a few words. When the DSA binds, a negative spread arises in the security market since it is the price signal that risk-neutral lenders should stop lending to the sovereign. In particular, the Fund restricts its offerings to the provision of insurance and, if needed, debt, as we have just mentioned. With seniority of the Fund, in principle *partial* default (i.e. only default to private lenders) is possible. To prevent this, the Fund might need to adapt its MIP and may have to hold more debt. Otherwise, there is a risk of *partial* default, which we show materializes with probability one if the DSA does not bind next period. The non-contingency of the *partial* default decision is a consequence of the Arrow component of the Fund contract. It implies that the private lenders deter from further lending and the price signal is, in this case, a positive spread.

As we said, our analysis enables a comparison with existing lending institutions such as the European Stability Mechanism (ESM) and the International Monetary Fund (IMF). We show that the Fund without seniority might need to absorb less debt in our environment, while the ESM and the IMF usually require seniority in their lending programs.[9] Moreover, while it is true that official lending institutions conduct DSAs as a necessary condition to guarantee credits, it is not the case that their resulting debt contracts provide insurance against future DSAs, as the Fund does. In other words, international lending institutions base their lending policy on one of several scenarios — e.g. the 'most likely,' the 'politically preferred,' or the 'worst case' scenario. In contrast, the Fund contract risk-shares among these different scenarios or paths. That is, it provides additional transfers in the worst scenario in exchange for higher payments in the best scenario.[10]

We conduct a quantitative analysis in which we calibrate the outside option of the Fund — an incomplete market economy with defaults — to Italy for the period 1992Q1–2019Q4. Unlike Greece, Portugal and Spain, Italy did not participate to an ESM programme of any sort. It therefore offers the possibility of a counterfactual analyses. Second, it is the third largest country of the Euro Area and it faces a public indebtedness above 100% of GDP with one of the largest spreads in the Euro Area, elements that make the Fund intervention more challenging. The specificity of Italy and its debt management has already been recently studied by Bocola and Dovis (2019). Our contribution is here twofold. First, we study the impact of the introduction of a Fund on Italy's debt sustainability and welfare. Second, we introduce stochastic growth and gauge the relationship between r and g.

[9]The IMF together with the World Bank have a *de facto* seniority, but it is not a formal contractual feature (see Schlegl et al., 2019). In opposition, the ESM has a *de jure* seniority with respect to the market. The only exception to this is Spain. The Spanish program was initially agreed with the EFSF with a standard *pari-passu* clause and managed to extend this feature into the ESM loan.

[10]Recently, the IMF DSA analysis takes the form of a *Stochastic Debts Sustainability Analysis*, (SDSA), where risk paths are 'statistically calibrated.' There are two differences with our analysis. First, we calibrate the parameters of a stochastic dynamic model to the macro-history of the country, in order to generate an exogenous stochastic structure, which provides a risk assessment without the Fund's contract. Second, we compute the constrained-efficient contract design, given our calibration. Third, as it is also done with standard DSA or SDSA, we obtain our 'counterfactual' DSA accounting with the Fund contract.

The main results of our quantitative inquiry are also twofold. First, with the Fund, the Italian debt would have been free of default risk; i.e. its entire debt position would have been safe. This is due to the Fund state-contingent credit line being designed to support a countercyclical fiscal policy with respect to exogenous shocks, but also contingent to the states that endogenous enforcement constraints become binding: reassessing the value of primary surpluses to avoid default, and risk-sharing across states when the DSA would be binding in some state. Providing this 'binding states' insurance is at the source of important welfare gains. Importantly, we show that the sovereign benefits from a greater debt absorption capacity compared to the standard incomplete market economy with defaults. Particularly, receiving state-contingent transfers from the Fund, the sovereign can accumulate debt in states in which defaults would usually happen. Quantitatively, we find in the steady-growth economy with the Fund, the total debt-to-GDP ratio is 221%, while with the MIP the Fund's Italian debt holding is nil. Importantly, in our calibration the DSA constraint is never binding at the steady-state, therefore there are no negative spreads or sudden stops, and the maturity of the debt is irrelevant.

Second, we argue that by accessing the Fund, Italy would have had a more stable evolution of its indebtedness. Using the decomposition of Cochrane (2020, 2022), we show that, in the last two decades, Italy largely increased its public indebtedness despite large primary surpluses. This is due to a strongly positive interest rate-growth differential ($r - g$) dominating the debt accumulation process. The positive differential is a combination of a relatively low, and unstable, growth of the Italian economy with an important risk premium on the Italian sovereign debt. We show that, by accessing the Fund, the Italian government would have reduced these perverse effects and therefore would have ended up with a lower indebtedness. The model predicts that the Italian indebtedness by the end of 2019 would have been around 80% of GDP rather than 135% if Italy could have joined the Fund in 2000. The Italian government's perseverance in maintaining positive primary surpluses, in spite of growth reversals, can be seen as a *commitment to debt sustainability*, in line with the European Union's fiscal policy. Indeed, the accumulation of large primary surpluses dampened the increase in Italian indebtedness, but this was a highly inefficient path to have been followed compared with the path that could have been followed with the Fund.

Relation to the literature Our work is related to the sovereign debt literature pioneered by Eaton and Gersovitz (1981) and subsequently extended by Aguiar and Gopinath (2006) and Arellano (2008).[11] As in Ábrahám et al. (2021), our benchmark economy with defaultable debt builds on Chatterjee and Eyigungor (2012) who introduce long-term bonds. Within this literature, our work is closely related to Hatchondo et al. (2017), who consider the case of adding a non-defaultable bond into the otherwise standard defaultable bond economy, and show that there are welfare gains by swapping defaultable bonds into non-defaultable bonds. Our work also relates more closely to Roch and Uhlig (2018) who model a bailout agency with a *minimal intervention policy* but focus on self-fulfilling debt crises.

Besides this, our study addresses the literature on optimal contracts with limited enforcement constraints such as Kehoe and Levine (2001), Kocherlakota (1996) and, in particular, Kehoe and Perri (2002) and Restrepo-Echavarria (2019) who already applied the Lagrangian-recursive approach developed by Marcet and Marimon (2019). Unlike Aguiar et al. (2019) and Aguiar and Amador (2020), our planner's problem accounts for all creditors in both the objective function and the constraint set and integrates two-sided limited enforcement constraints. Our decentralization relies on the approach of Alvarez and Jermann (2000), while our focus is close to Thomas and Worrall (1994) who already studied international lending contracts, with one-sided limited commitment. Importantly, in the economy with the Fund, the competitive equilibrium implements the *unique* — non autarkic — *constrained-efficient allocation*. Hence, the presence of the Fund eliminates potential multiplicity in equilibria such as the one in Alvarez and Jermann (2000) and Aguiar and Amador (2020).[12] Roch and Uhlig (2018) present a similar result in the environment of Cole and Kehoe (2000).

A more recent literature merges these last two strands of literature and it is the most closely related one to our work. In particular, Dovis (2019) decentralises optimal contracts through partial default and an active debt maturity management, and Müller et al. (2019) through *ex post* state-conditionality given by default and renegotiation procedures. Our approach is not to 'rationalise' *ex post* observed behaviour, but to account for existing constraints. In view of this, we adopt a Nash specification in which the Fund takes the decision in the private bond market as given. We then characterise the constraint efficient allocation and assess it quantitatively in relation to a calibrated version of the benchmark defaultable debt economy. Within this approach, and in contrast to most of the literature, our specific focus is on the role of lender's DSA, as a lender's limited-enforcement constraint.

[11] See also Aguiar and Amador (2014) and Aguiar et al. (2016)

[12] See also Gu et al. (2013), Kirpalani (2017) and Bloise and Vailakis (2022) for multiple equilibria in economies with endogenous borrowing limits.

Finally, as a theoretical foundation for the design of a — effectively running — fiscal fund, able to stabilise sovereign debt and expand the supply of safe assets, our work is related to a large literature regarding the IMF and other international institutions lending practices, and to the debate on the need to develop the Fiscal Union within the European Economic and Monetary Union (EMU) and expand its supply of eurobonds (as it has been done with the Next Generation EU (NGEU) program) as safe assets.[13]

Organisation of the paper We lay down the economic environment and present the Fund contract in Section 3. We expose the decentralized economy in Section 4, which includes the sovereign's, the private lenders' and the Fund's problems. Section 5 develops the Fund's intervention with seniority. After this, we calibrate our model to Italy in Section 6 and present the underlying results in Section 7. Finally, we conclude in Section 8.

2 Environment

We assume an infinite-horizon small open economy with a single homogenous consumption good in discrete time. There is a sovereign borrower acting as a representative agent and taking decisions on behalf of the small open economy, a Fund acting as official lender and a continuum of competitive private lenders.

2.1 The Sovereign Borrower

The sovereign's preference is represented by $\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c_t, n_t)$, where $\beta \in (0,1)$ is the discount factor, n_t is the labor, $1 - n_t$ the leisure and c_t the consumption at time t. The sovereign is relatively impatient as $\beta < 1/(1+r)$. We adopt a specific form of utility function so as to obtain a (stochastic) balanced growth path and to simplify the detrended formulation of the problem: $U(c,n) = u(c) + h(1-n) = \log(c) + \xi \frac{(1-n)^{1-\zeta}}{1-\zeta}$.

The sovereign has access to a labor technology $y = \theta f(n)$ subject to decreasing returns to scale, where $f_n(n) > 0$, $f_{nn}(n) < 0$. Moreover, $\theta \in \Theta$ represents a trend shock to the productivity. It is the only source of uncertainty in the economy. The law of motion of the shock is given by $\theta_t = \gamma_t \theta_{t-1}$, where $\gamma_t \in \Gamma$ represents the growth rate at time t. We denote the history of θ up to time t by θ^t. The exact form of the shock is detailed in Section 6.[14]

Finally, the sovereign has access to a long-term state-contingent contract with the Fund

[13]See Marimon and Wicht (2021) for a discussion on how our Fund proposal relates to this literature and it can be implemented within EMU.

[14]We present in the main text the model with the stochastic trend and keep track of θ in the state space. The detrended version is presented in Appendix B. There we only keep track of γ in the state space.

— a credit-insurance line that we specify below — and long-term debt contracts with a continuum of competitive private lenders. However, it cannot commit to honor the terms of any contract. Given that contracts are all long term, this gives rise not only to *default risk* — i.e. non repayment — but also to *dilution risk* — i.e. devaluation of legacy debt.

In the first part of our analysis, we assume that the Fund contract has no seniority with respect to the private debt contracts. That is, every default is a *full* default as the sovereign reneges its entire debt position. Under such default, the sovereign receives a penalty in the form of a reduced output, $\theta^d \leq \theta$, and loses access to both the private bond market and the Fund. Later, it can reintegrate the private bond market with some probability, λ, but cannot obtain the assistance of the Fund anymore. In the second part of our analysis, we consider the case in which the Fund possesses seniority with respect to the private bond market.[15]

2.2 The Private Lenders

There is a continuum of competitive private lenders which have access to international financial markets. They are risk neutral and discount the future at $\frac{1}{1+r}$ where r is the risk-free rate. Private lenders' contracts are a continuum of simple long-term debt contracts, which we assume have a common maturity and coupon (δ, κ); i.e. given a private lending portfolio of value $b_{l,t}$, $(1-\delta)b_{l,t}$ matures in period t while $\delta\kappa b_{l,t}$ is the coupon payment private lenders must receive from the sovereign for the non-matured debt.[16] As long as there are no spreads — positive or negative — on the debt, private lenders are willing to provide all the debt contracts the sovereign asks for.

2.3 The Financial Stability Fund

Similar to the private lenders, the Fund has access to international financial markets, is risk neutral, discounts the future at $\frac{1}{1+r}$ and breaks even in expectation.

However, the Fund provides a state-contingent contract while private lenders offer non-contingent debt contracts. Particularly, the Fund contract is a state-contingent asset, $a_{l,t}$, with the aforementioned maturity and coupon (δ, κ) which can be decomposed into a debt, $\bar{a}_{l,t}$, and an insurance components, $\hat{a}_{l,t}(\theta^t)$. We consider that the private lenders cannot

[15] We do not consider the case in which the Fund is junior relative to the private lenders as official multilateral lending institutions generally enjoy a preferred creditor status (see Schlegl et al., 2019).

[16] We denote by $b_{l,t}$ the bond held by the private lenders, while we denote by b_t the bond issued by the sovereign at time t. By market clearing, $b_t = -b_{l,t}$ for all t. Furthermore, $b_t > 0$ denotes an asset and $b_t < 0$ denotes a debt from the point of view of the sovereign. We adopt the same notation for all other securities.

credibly provide insurance due to *strategic complementarity* in their actions. More precisely, if a private lender believes that future private lenders are unwilling to provide insurance, it will be itself unwilling to provide insurance as the sovereign will eventually default.[17] As a large player, the Fund is unaffected by such coordination issue.

Moreover, while private lenders are competitive, the design of the Fund contract is based on a risk assessment of the country which, as it is common practice in *debt sustainability analysis* (DSA), also accounts for the effect of the same Fund contract in enhancing the sustainability of the country's sovereign debt. Importantly, liabilities with the Fund cannot be arbitrary. There is a limit on the extent of losses the Fund can make given by $\theta_{t-1}Z \leq 0$. Particularly, if $Z = 0$, the Fund does not tolerate permanent losses on the contract.[18]

Finally, the Fund's withdrawal in the case of a *full* default is permanent, whereas the private bond market's exclusion is temporary.

2.4 Timing of Actions

The timing of actions within the period is:

1. Given $(\theta_{t-1}Z, b_{l,t})$, after the realization of the growth shock θ_t, the Fund announces what is the (state-contingent) sustainable debt capacity of the sovereign country for next period: $\{\omega_l'(\theta^{t+1})\}_{\theta^{t+1}|\theta^t}$, which can be decomposed into a debt component $\bar{\omega}_l'(\theta^t) = b_{l,t} + \bar{a}_{l,t}$, to be allocated between the private lenders and the Fund, and insurance components: $\omega_l'(\theta^{t+1}) - \bar{\omega}_l'(\theta^t) = \hat{a}_{l,t}(\theta^{t+1})$, which must be part of the Fund contract.

2. The sovereign decides whether to default or not and, in the latter case, the sovereign then determines its borrowing with the private bond market before going to the Fund. Given this timing, the sovereign acts as a monopsony and, therefore, the prices competitive lenders face are a function of its total debt liabilities for next period $\bar{\omega}_l'(\theta^t)$.[19]

3. Conditional on no default, the Fund and the sovereign implement the corresponding debt and insurance part of their contract.

[17] Especially, Mateos-Planas and Seccia (2014) show that lenders do not provide Arrow securities for states in which the borrower defaults. See also Kirpalani (2017).

[18] Expected losses must be mutualized if the Fund is only backed with the union's — or other union countries' — primary surpluses. Alternatively $\theta_{t-1}Z < 0$ can also mean that in a particular state θ_t, there can be bounded solidarity transfers. For instance, the NGEU recovery plan mentioned in footnote 2.

[19] This timing rules out self-fulfilling debt crises (Ayres et al., 2018).

3 The Financial Stability Fund

In this section, we specify the Fund contract in a Nash specification where the actions in the private bond market are taken as given.

3.1 Debt and Sustainability

The private lenders' and Fund's contracts establish that at time t and state-history θ^t the country must transfer $\tau_f(\theta^t)$ for its state-contingent liabilities with the Fund and $\tau_p(\theta^t)$ for its non-contingent debt liabilities with the private lenders. We denote $\tau(\theta^t) \equiv \tau_f(\theta^t) + \tau_p(\theta^t)$ as the total transfer the country pays. That is, given a consumption and employment plan $\{c(\theta^t), n(\theta^t)\}_{t=0}^{\infty}$, in period-state (t, θ^t) feasibility implies that

$$\tau(\theta^t) = \theta_t f(n(\theta^t)) - c(\theta^t); \tag{1}$$

that is, $\tau(\theta^t)$ is the *primary surplus* in state-history θ^t. Therefore, if the country's debt with private lenders is $-b_{l,t}$ and its asset position with the Fund is $-a_{l,t}$ for a total amount of $-\omega_{l,t} = -(b_{l,t} + a_{l,t})$, *debt sustainability* requires that the expected present value of future transfers discounted with the risk free rate r should cover the outstanding amount of debt:

$$\mathbb{E}_t \sum_{j=t}^{\infty} \left(\frac{1}{1+r}\right)^{j-t} \tau(\theta^j) \geq \omega_{l,t}.$$

In particular, there is a decomposition of total transfers such that:

$$\mathbb{E}_t \sum_{j=t}^{\infty} \left(\frac{1}{1+r}\right)^{j-t} \tau_p(\theta^j) \geq b_{l,t} \quad \text{and} \quad \mathbb{E}_t \sum_{j=t}^{\infty} \left(\frac{1}{1+r}\right)^{j-t} \tau_f(\theta^j) \geq a_{l,t}. \tag{2}$$

Without loss of generality, we assume that $a_{l,0} = 0$, therefore the initial state is given by $(\theta_0, b_{l,0})$. In contrast with private lenders which only issue non-contingent debt contracts, the Fund provides a state-contingent contract, i.e. $a_{l,t} = \bar{a}_{l,t} + \hat{a}_{l,t}(\theta^t)$. More precisely, it defines contingent transfers for $(t+1, \theta^{t+1})$ at (t, θ^t); i.e. $\tau_f'(\theta^{t+1}) = \tau_f(\theta^t) + \hat{\tau}_f'(\theta^{t+1})$, with $\sum_{\theta^{t+1}|\theta^t} \hat{\tau}_f'(\theta^{t+1}) = 0$ and

$$\tau_f(\theta^t) = \sum_{\theta^{t+1}|\theta^t} \tau_f'(\theta^{t+1}). \tag{3}$$

We later specify the form that these transfers have in a decentralized economy.

However, for the debt to be sustainable two other factors must be taken into account. First, a *sovereign country can default* on its liabilities. Therefore, if in state θ_t the value of the outside default option is $V^{af}(\theta_t)$, to prevent *full* default the Fund contract must satisfy:

$$\mathbb{E}\left[\sum_{j=t}^{\infty} \beta^{j-t} U(c(\theta^j), n(\theta^j)) \middle| \theta^t \right] \geq V^{af}(\theta_t). \tag{4}$$

Second, the Fund contract must account that the *liabilities with the Fund cannot be arbitrary*. Therefore, since the Fund takes into account the private debt liabilities $b_{l,t}$, and both debt liabilities are treated *at par* (a feature we analyze in detail in Section 5) the Fund contract must satisfy:[20]

$$\mathbb{E}\Big[\sum_{j=t}^{\infty}\left(\tfrac{1}{1+r}\right)^{j-t}\tau(\theta^{j})\Big|\theta^{t}\Big] \geq \theta_{t-1}Z + b_{l,t}. \tag{5}$$

The above constraint depends on $Z \leq 0$ and b_l. The former variable indicates the level of redistribution of the Fund. In order to prevent that the Fund provides permanent transfers to the sovereign we will assume that $Z = 0$, i.e. that in no state the Fund contract has expected losses. Similarly, b_l indicates the level of outstanding private debt the sovereign needs to repay. Larger b_l tightens the constraint. Thus, (5) shows an aspect that makes the Fund contract different from an uncontingent defaultable debt contract: in states where the sovereign's future surpluses might not cover additional amount of debt — say, when (5) is binding at θ^t — there is a lending 'sudden stop' to avoid losses that would go beyond the contract's terms. In other words, the Fund contract anticipates these states, defines the appropriate state-contingencies and limits the amount of lending. Thus, the Fund internalizes a pecuniary externality that competitive private lenders usually do not: the fact that marginal lending can be excessive. We therefore interpret (5) as a DSA as it corresponds to an evaluation of the present value of the sovereign's future surpluses.

The design of the Fund contract has two distinct elements. First, it establishes the levels of debt which are sustainable next period, $\{\omega_l'(\theta^{t+1})\}_{\theta^{t+1}|\theta^t}$, according to a DSA. Second, it defines the long-term contract between the Fund and the sovereign, which here takes the form of financial transfers and more explicitly, next section, the form of a state-contingent asset. Both elements are linked, in that the first takes into account the second. In other words,

[20]To obtain equation (5), observe that, conditional on θ^t,

$$\mathbb{E}\Big[\sum_{j=0}^{\infty}\left(\tfrac{1}{1+r}\right)^{j}\tau(\theta^{t+j})\Big|\theta^{t}\Big] = \mathbb{E}\Big[\sum_{j=0}^{\infty}\left(\tfrac{1}{1+r}\right)^{j}\left(\tau_f(\theta^{t+j}) + \tau_p(\theta^{t+j})\right)\Big|\theta^{t}\Big].$$

Using the valuation formula in (2), the previous equation simplifies into

$$\mathbb{E}\Big[\sum_{j=0}^{\infty}\left(\tfrac{1}{1+r}\right)^{j}\tau(\theta^{t+j})\Big|\theta^{t}\Big] \geq \mathbb{E}\Big[\sum_{j=0}^{\infty}\left(\tfrac{1}{1+r}\right)^{j}\tau_f(\theta^{t+j})\Big|\theta^{t}\Big] + b_l(\theta^t).$$

The present value constraint on Fund's lending is $\mathbb{E}\big[\sum_{j=0}^{\infty}\left(\tfrac{1}{1+r}\right)^{j}\tau_f(\theta^{t+j})\big|\theta^{t}\big] \geq \theta_{t-1}Z$, thus the overall participation constraint of the Fund reduces to (5). Note that we cannot consider $\mathbb{E}\big[\sum_{j=0}^{\infty}\left(\tfrac{1}{1+r}\right)^{j}\tau_f(\theta^{t+j})\big|\theta^{t}\big] \geq \theta_{t-1}Z$ and $\mathbb{E}\big[\sum_{j=0}^{\infty}\left(\tfrac{1}{1+r}\right)^{j}\tau_p(\theta^{t+j})\big|\theta^{t}\big] \geq b_l(\theta^t)$ as two separate constraints because the Fund takes the actions in the private bond market as given.

the first can be thought as the total credit line that is sustainable for next period, when it is complemented with insurance. The second, the part of debt, together with the insurance component, to which the Fund is committed for next period and takes the form of a state-contingent security for the sovereign, $\{a_{l,t+1}(\theta^{t+1}, b_{l,t+1})\}_{\theta^{t+1}|\theta^t}$, where $a_{l,t+1}(\theta^{t+1}, b_{l,t+1}) = \omega_{l,t+1}(\theta^{t+1}) - b_{l,t+1}$.

Note that the Fund contract is state contingent with respect to the productivity shocks θ_{t+1} but also with respect to $V^{af}(\theta_{t+1})$ and $\theta_{t-1}Z + b_{l,t+1}$ being binding.

3.2 The Fund Contract Problem

We now turn to the specific design of the Fund announcement and contract. Once the corresponding country's risk assessment regarding $\{\theta_t\}_{t=0}^{\infty}$ has been done, the Fund solves a planner's problem with two agents — the sovereign and the Fund itself — taking into account the participation of a continuum of private lenders in absorbing credit needs. This defines an allocation, of consumption and employment, which the Fund takes as the benchmark policy the sovereign will follow, and the corresponding transfers of the sovereign to the lenders.

We say that $\{c(\theta^t), n(\theta^t)\}_{t=0}^{\infty}$ is a Fund's *constrained-efficient allocation* in *sequential form*, given $b_{l,0}$, if there exist sequences of transfers $\{\tau_p(\theta^t), \tau_f'(\theta^{t+1})\}_{t=0}^{\infty}$, with associate $\{b_{l,t}\}_{t=0}^{\infty}$ satisfying (2), such that:

$$\max_{\{c(\theta^t),n(\theta^t)\}_{t=0}^{\infty}} \mathbb{E}\Big[\mu_{b,0} \textstyle\sum_{t=0}^{\infty} \beta^t U(c(\theta^t), n(\theta^t)) + \mu_{l,0} \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t \tau(\theta^t) \Big| \theta_0\Big] \tag{6}$$
$$\text{s.t. } (5), (4), (3) \text{ and } (1), \text{ for all } (t, \theta^t), t \geq 0.$$

The *constrained-efficient allocation* prescribes that, in period t, the sovereign consumes $c(\theta^t)$ and provides labor $n(\theta^t)$. Furthermore, the Fund's break-even assumption determines the initial weights $(\mu_{b,0}, \mu_{l,0})$. Following, if without private debt there is an interior solution to the Fund's contracting problem, then an optimal solution exists and there are feasible paths of private debt, starting at $b_{l,0}$, subject to an upper bound on how large the initial debt $b_{l,0}$ can be. We come back to this later.

Using the recursive contracts approach of Marcet and Marimon (2019), we can formulate the Fund's problem (6) in *recursive form*. Defining $s \equiv \{\theta^-, \gamma\}$ and $\eta \equiv \beta(1+r) < 1$,

$$\begin{aligned} FV(s, x, b_l) = \mathcal{SP} \min_{\{\nu_b,\nu_l\}} \max_{\{c,n\}} \ & x\big[(1+\nu_b)U(c,n) - \nu_b V^{af}(\theta)\big] \\ & + \big[(1+\nu_l)\tau - \nu_l(\theta^- Z + b_l)\big] + \tfrac{1+\nu_l}{1+r} \mathbb{E}\big[FV(s', x', b_l') \big| \theta\big] \\ \text{s.t. } & \tau = \theta f(n) - c, \end{aligned} \tag{7}$$

$$x' = \frac{1+\nu_b}{1+\nu_l}\eta x \quad \text{with } x_0 \text{ given.} \tag{8}$$

Appendix A presents all the details of such exposition. We denote by x' the prospective Pareto weight of the sovereign relative to the Fund where $\nu_b \geq 0$ and $\nu_l \geq 0$ are the normalized multipliers attached to the sovereign's and the Fund's participation constraints, respectively. The value function of the contracting problem satisfies:

$$FV(s, x, b_l) = xV^b(\theta, x, b_l) + V^l(s, x, b_l), \text{ with}$$
$$V^b(\theta, x, b_l) = U(c, n) + \beta\mathbb{E}[V^b(\theta', x', b_l')|\theta] \quad \text{and} \quad V^l(s, x, b_l) = \tau + \frac{1}{1+r}\mathbb{E}[V^l(s', x', b_l')|\theta].$$

It should be underlined that we take a specific planner's perspective in solving for the Fund contract. The Fund is designing a constrained efficient contract with the sovereign borrower while taking as given the lending policies of the private lenders in the market, and at the same time, the private lenders are aware of the lending decisions of the Fund in the contract. However, the Fund does not play the role of a Ramsey planner in our framework, since it lacks the authority to fully control the market transactions between the private lenders and the sovereign borrower, neither directly through planned allocations nor indirectly via policy instruments. In other words, as we will see more explicitly when, in the next section, we characterize the decentralized economy, the equilibrium between the Fund, the sovereign and the continuum of private lenders has a Nash-competitive equilibrium characterization, not a Ramsey policy implementation.

Unlike Aguiar et al. (2019), our contract accounts for both the private lenders and the Fund on equal footing. There is no dilution of legacy creditors as the entire debt is taken into consideration in the Fund's insurance component. While the Fund *directly* specifies contingent transfers $\tau_f'(\theta^{t+1})$ taking as given $\tau_p(\theta^t)$, effectively the contract accounts for the total surplus, $\tau(\theta^t)$, since only in this way it is capable of consistently stabilising the borrower's entire debt position. As we will see in the next section when we define the *minimal intervention policy* (MIP) of the Fund, this also means that in some circumstances the Fund stands ready to absorb the debt position of the borrower in the form of private bonds, and effectively there is complete credit (risk) transfer from the private lenders to the Fund.

Our present formulation is close to the current rules of international lending institutions such as the IMF or the ESM. The Fund takes into account all the sovereign's debt liabilities — within and outside the Fund — that satisfy the DSA in every possible state. The difference with current practices is that the DSA is usually only conducted at the beginning of the contract, or at certain time intervals, while in our characterisation of the Fund contract, DSA, i.e. (5), is contingent to all states that the contract specifies, including those where participation constraints are binding. This means that our DSA has a different definition of 'sustainability' than existing official multilateral lending institutions. Particularly, sovereign liabilities have to remain sustainable *ex ante* and *ex post* in all considered paths.

Another difference is that in this framework, the Fund provides state-contingent transfers, key component that averts default on equilibrium path as we will see. We argue that private lenders are unable to credibly provide such insurance due to strategic complementarity in their actions — especially when (4) binds. In addition, the Fund has no seniority over privately owned debt. This is in general not the case when official multilateral lenders intervene (cf. footnote 9). In Section 5, we consider an alternative formulation where the Fund liabilities has seniority over privately held sovereign debt. We show that the seniority structure of the Fund might affect the Fund's MIP.

In sum, the Fund can be seen as an official lender which provides financial information and surveillance — i.e. optimal allocation and DSA — as well as financial resources — i.e. contingent transfers — to sovereign market participants.

3.3 The Sovereign's Outside Option

The autarky value of the standard incomplete market model with default represents the sovereign's outside option (Eaton and Gersovitz, 1981; Aguiar and Gopinath, 2006; Arellano, 2008). Since the Fund has no seniority with respect to the privately held sovereign debt, the sovereign reneges its entire debt position if it decides to default. This is what we call a *full* default. The Bellman equation in such situation reads

$$V^{af}(\theta) = \max_{n} \left\{ U(\theta^d f(n), n) \right\} + \beta \mathbb{E}\left[(1-\lambda) V^{af}(\theta') + \lambda J(\theta', 0) \middle| \theta \right], \tag{9}$$

where $\theta^d \leq \theta$ contains the penalty for defaulting. Furthermore, V^{af} corresponds to the value under financial autarky and J to the value of reintegrating the private bond market without the Fund. More precisely, $J(\theta, b) = \max\{V^o(\theta, b), V^{af}(\theta)\}$, with

$$\begin{aligned} V^o(\theta, b) = & \max_{\{c,n,b'\}} U(c, n) + \beta \mathbb{E}\left[J(\theta', b') \middle| \theta \right] \\ & \text{s.t. } c + \tau_p(b') \leq \theta f(n). \end{aligned} \tag{10}$$

Given equation (2), the sequence of private transfers $\{\tau_p(\theta^t)\}_{t=0}^{\infty}$ directly relates to a sequence of private debt $\{b(\theta^t)\}_{t=0}^{\infty}$. Hence, for a given b, by picking b', the borrower directly chooses a certain level of transfer τ_p. We therefore slightly abuse notation and write τ_p as a function of b'. Market clearing implies that in any state, $b + b_l = 0$.

3.4 Properties of the Fund Contract

This subsection demonstrates the main properties of the Fund contract. Other properties such as the inverse Euler equation and the steady state are presented in Appendix C. Proofs are in Appendix D.

We start with the existence of the Fund contract and, for this, we need the following interiority assumption (Marcet and Marimon, 2019).

Assumption 1 (Interiority)**.** *There is an $\epsilon > 0$, such that, for all $\theta^t \in \Theta^t$ with associate $\{b_{l,t}\}_{t=0}^{\infty}$ satisfying (2), there is a sequence $\{\ddot{c}(\theta^t), \ddot{n}(\theta^t)\}$ satisfying for all $t \geq 0$,*

$$\mathbb{E}\Big[\sum_{j=t}^{\infty} \beta^{j-t} U(\ddot{c}(\theta^j), \ddot{n}(\theta^j))\Big|\theta^t\Big] \geq V^{af}(\theta_t) + \epsilon,$$
$$\mathbb{E}\Big[\sum_{j=t}^{\infty} \big(\tfrac{1}{1+r}\big)^{j-t}\big(\theta_j f(\ddot{n}(\theta^j)) - \ddot{c}(\theta^j)\big)\Big|\theta^t\Big] \geq \theta_{t-1} Z + b_{l,t} + \epsilon.$$

This assumption ensures the uniform boundedness of the Lagrange multipliers. For equations (4) and (5), it requires that, in spite of the enforcement constraints, there are strictly positive rents to be shared among the contracting parties. In our environment, since rents to be shared are positively correlated with productivity shocks, this assumption is easily satisfied given that default is costly. Otherwise, there may not exist a constrained-efficient risk-sharing agreement.

Proposition 1 (Existence)**.** *In the specified environment,*[21] *if Assumption 1 is satisfied, for every θ there is a $\underline{b}_l(\theta) > 0$ such that if $b_{l,0}(\theta) \leq \underline{b}_l(\theta)$, then there exists a unique Fund's allocation with initial condition $(\theta, b_{l,0}(\theta))$. Furthermore, there is a $\underline{t}(\theta, \underline{b}_l(\theta))$ such that for $t > \underline{t}(\theta, \underline{b}_l(\theta))$ the detrended Fund contracts are at the steady state.*

Proposition 1 is made of three parts. First, a Fund contract exists if — among other requirements — the initial level of private indebtedness is not too high, as to Assumption 1 to be satisfied. However, if an economy is in an initial state $(\theta, b_{l,0}(\theta))$ but $b_{l,0}(\theta) > \underline{b}_l(\theta)$ then the private debt will need to be restructured — i.e. to a $\ddot{b}_{l,0}(\theta) \leq \underline{b}_l(\theta)$ — for a Fund contract to exist. In other words, there is a strict risk-assessment of the sovereign and, provided that the existing level of private liabilities is sustainable, if there is a Fund contract

[21] We define a specific functional form for $U(c, n)$, but it is enough to assume that U is continuous, strictly monotone and concave in $\mathbb{R}^+ \times [0, 1]$.

then no other *ex ante* conditionality is needed. Second, the Fund contract allocation in terms of consumption and employment is unique and, third, it is characterised by an ergodic distribution which we detail in Appendix C.

Corollary 1 (Full Default)**.** *In a Fund contract, the sovereign does not enter in full default.*

The sovereign's participation constraint (4) implies no (*full*) default on equilibrium path. The Fund always provides state-contingent transfers to the sovereign. This sustains the chosen sequence of private liabilities, $\{\tau_p(\theta^t)\}_{t=0}^{\infty}$, and ensures that the sovereign finds optimal not to default. As a result, there is no dilution of legacy creditors.

This shows the importance of the state contingency of the Fund's transfer. Without this feature, the Fund would not be capable of accounting for the possibility of default in each state $\theta' \in \Theta$ specifically.

4 The Decentralized Economy

The previous section derived the Fund contract from the perspective of a mixed centralized-private economy. It had the advantage that it allowed a full characterization of the Fund contract, but the disadvantage of having the sovereign in the shadow, with its actions being decided by the Fund contract. We now consider the decentralized version of the economy in which the Fund and the private lenders trade securities with the sovereign.

4.1 Market Structure and Private Sector

As before, the financial market is composed of private lenders and the Fund. The sovereign has therefore two funding opportunities. On the one hand, it can borrow long-term defaultable bonds, b', on the private bond market at a unit price of $q_p(\theta, \bar{\omega}')$, where $\bar{\omega}$ is defined momentarily. On the other hand, it can trade Θ state contingent securities $a'(\theta')$ at a unit price of $q_f(\theta', \omega'(\theta')|\theta)$. At the start of a period, the sovereign holds a in the Fund and b in the private bond market which together sum to an entire position of $\omega = a + b$. A fraction $1 - \delta$ of each financial asset matures today and the remaining fraction δ is rolled-over and pays a coupon κ. Given this, the transfer to the Fund and the private lenders are, respectively

$$\tau_f(\theta) = \sum_{\theta'|\theta} q_f(\theta', \omega'(\theta')|\theta)(a'(\theta') - \delta a(\theta)) - (1 - \delta + \delta\kappa)a(\theta), \tag{11}$$

$$\tau_p(\theta) = q_p(\theta, \bar{\omega}')(b'(\theta) - \delta b(\theta)) - (1 - \delta + \delta\kappa)b(\theta). \tag{12}$$

The assets provided by the Fund are state contingent, while private bonds are not. More precisely, the portfolio $a'(\theta')$ can be decomposed into a common bond $\bar{a}'$ that is independent of the next period state, traded at the implicit bond price $q_f(\theta,\bar{\omega}') \equiv \sum_{\theta'|\theta} q_f(\theta',\omega'(\theta')|\theta)$,[22] and an insurance portfolio of Θ Arrow securities $\hat{a}'(\theta')$. Thus we have that $a'(\theta') = \bar{a}' + \hat{a}(\theta')$ with $\bar{a}' = \frac{\sum_{\theta'|\theta} q_f(\theta',\omega'(\theta')|\theta)a'(\theta')}{q_f(\theta,\bar{\omega}')}$ and $\sum_{\theta'|\theta} q_f(\theta',\omega'(\theta')|\theta)\hat{a}'(\theta') = 0$ which represents the market clearing condition of Arrow securities.[23] We denote the total debt position by $\bar{\omega} = \bar{a} + b$.

The economy is decentralised as a competitive equilibrium with endogenous borrowing and lending constraints following Alvarez and Jermann (2000) and Krueger et al. (2008). Under the above market structure, the sovereign's problem reads

$$W^b(\theta,a,b) = \max_{\{c,n,b',\{a'(\theta',b')\}_{\theta'\in\Theta}\}} U(c,n) + \beta\mathbb{E}\big[W^b(\theta',a'(\theta',b'),b')\big|\theta\big] \tag{13}$$

$$\text{s.t. } c + \sum_{\theta'|\theta} q_f(\theta',\omega'(\theta')|\theta)(a'(\theta',b') - \delta a) + q_p(\theta,\bar{\omega}')(b' - \delta b) \tag{14}$$

$$\leq \theta f(n) + (1-\delta+\delta\kappa)(a+b), \text{ and}$$

$$\omega'(\theta') = a'(\theta',b') + b' \geq \mathcal{A}_b(\theta'). \tag{15}$$

Equation (15) is the equivalent to the participation constraint (4), which prevents defaults. The endogenous borrowing limit $\mathcal{A}_b(\theta')$ is such that

$$W^b(\theta',\ddot{a}'(\theta',\ddot{b}'),\ddot{b}') = V^{af}(\theta') \text{ for all } \ddot{a}'(\theta',\ddot{b}') + \ddot{b}' = \mathcal{A}_b(\theta'). \tag{16}$$

In words, the Fund limits the sovereign's indebtedness such that the sovereign's expected lifetime utility from repaying its debts is at least as high as that of defaulting. The borrowing limit is therefore a no-default borrowing constraint (Zhang, 1997). Particularly, it is tight enough in the sense of Alvarez and Jermann (2000) to prevent default but allows as much risk sharing as possible. We explain the dependence of $a'(\theta',b')$ on b' when we derive the decentralized Fund's problem.

Private lenders solve a static problem. However, we express it in recursive form to later formulate the DSA of the Fund. We have

$$W^p(\theta,a_l,\bar{a}_p,b_l) = \max_{\{c_p,b_l',\bar{a}_p'\}} c_p + \frac{1}{1+r}\mathbb{E}\big[W^p(\theta',a_l',\bar{a}_p',b_l')\big|\theta\big] \tag{17}$$

$$\text{s.t. } c_p + q_p(\theta,\bar{\omega}')(b_l' - \delta b_l) + q_f(\theta,\bar{a}_p')(\bar{a}_p' - \delta\bar{a}_p) \leq (1-\delta+\delta\kappa)(b_p + \bar{a}_p).$$

[22] As showed below, by no arbitrage condition, $\sum_{\theta'|\theta} q_f(\theta',\omega'(\theta')|\theta)$ depends only on $\bar{\omega}'$, thus we write $q_f(\theta,\bar{\omega}')$ directly.

[23] Note that $\tau_f'(\theta') = \tau_f(\theta) + \hat{\tau}_f(\theta')$, where $\tau_f(\theta) = q_f(\theta,\bar{\omega}')(\bar{a}'(\theta) - \delta a(\theta)) - (1-\delta+\delta\kappa)a(\theta)$ and $\hat{\tau}_f(\theta') = q_f(\theta',\omega'(\theta')|\theta)\hat{a}'(\theta')$.

An important object which emanates from this problem is the private lending policy, $b_l' = B_l(\theta, a_l, b_l)$ which is taken as given by the Fund.

The private lenders also have access to the bonds issued by the Fund. This enables that the bond price in the Fund and in the private bond market coincide through arbitrage. We will consider equilibria where, without loss of generality, $\bar{a}_p = 0$, therefore we simplify notation by eliminating $\bar{a}_p$ if not necessary. Notably, we write $W^p(\theta, a_l, 0, b_l) \equiv W^p(\theta, a_l, b_l)$. Besides this, trade of private bonds satisfies the following transversality condition:

$$\lim_{n\to\infty} \mathbb{E}\Big\{ \Big[\prod_{j=0}^{n} Q_p\big(\theta^{t+j}, \bar{\omega}\big(\theta^{t+j}\big)\big) \Big] b_l\big(\theta^{t+j}\big) \Big| \theta^t \Big\} = 0, \quad \text{with} \tag{18}$$

$$Q_p\big(\theta^{t+j}, \bar{\omega}\big(\theta^{t+j}\big)\big) = \frac{q_p\big(\theta^{t+j}, \bar{\omega}\big(\theta^{t+j}\big)\big)}{1 - \delta + \delta\kappa + \delta q_p\big(\theta^{t+j+1}, \bar{\omega}\big(\theta^{t+j+1}\big)\big)}. \tag{19}$$

The implicit interest rate in the private bond market is $r_p(\theta, \bar{\omega}') \equiv \frac{1}{Q_p(\theta, \bar{\omega}')} - 1$. As we will see, it is possible that $r_p(\theta, \bar{\omega}') < r$ generating a wedge between the lenders' discount factor and the pricing kernel. That is why the valuation equation (2) holds with inequality.

4.2 The Decentralised Fund Contract

We can further decentralise the Fund contract. We show that, given the realization of the state, the Fund formulates a capacity announcement stating the level of indebtedness that remains sustainable in all future states. The maximization problem of the Fund is given by

$$W^f(s, a_l, b_l) = \max_{\{c_f, \{a_l'(\theta', b_l')\}_{\theta' \in \Theta}\}} \; c_f + \tfrac{1}{1+r} \mathbb{E}\big[W^f(s', a_l'(\theta', b_l'), b_l') \big| \theta\big] \tag{20}$$

$$\text{s.t. } c_f + \textstyle\sum_{\theta'|\theta} q_f(\theta', \omega'(\theta')|\theta)(a_l'(\theta', b_l') - \delta a_l) \le (1 - \delta + \delta\kappa) a_l, \tag{21}$$

$$\omega_l'(\theta') = a_l'(\theta', b_l') + b_l' \ge \mathcal{A}_f(\theta', b_l'), \tag{22}$$

$$\text{with } b_l' = B_l(\theta, a_l, b_l) \text{ given,}$$

where $B_l(\theta, a, b)$ is the lending policy of the private lenders and $s \equiv \{\theta^-, \gamma\}$. Again, we remove $\bar{a}_p$ to simplify notation as $\bar{a}_p = 0$ in equilibrium.

Note that in (22), $\omega_l'(\theta')$ and $a_l'(\theta', b_l')$ are simultaneously determined for a given b_l'.[24] That is, the Fund, as a security trader choosing $a_l'(\theta', b_l')$, determines $\omega_l'(\theta')$ by (22); alternatively, the Fund, as capacity announcer, could have chosen $\omega_l'(\theta')$ and use (22) to determine $a_l'(\theta', b_l')$.

[24]Even if it will not happen in equilibrium, the Fund must have a policy for the case that the interaction between the sovereign and private lenders ends with an over-lending which makes the continuation of the contract unfeasible. Then, its policy is to do as it does at the beginning of the Fund contract: discontinue the contract unless there is a debt restructuring that makes its intervention possible and credible.

The variable $\mathcal{A}_f(\theta', b_l)$ represents an endogenous limit defined as

$$W^f(s', \mathcal{A}_f(\theta', b_l') - b_l', b_l') = \theta^- Z. \tag{23}$$

This condition restricts the extent of losses. Particularly, it ensures that the present discounted value of the Fund's assets are at least equal to $\theta^- Z \leq 0$. Specifically, when $Z = 0$, $\mathcal{A}_f(\theta', b_l')$ ensures that the sovereign's liabilities can be absorbed by the Fund without incurring permanent losses. Adding equations (23) to the value of the lender (17) and applying the transversality condition (18), we obtain

$$W^f(s', \mathcal{A}_f(\theta', b_l') - b_l', b_l') + W^p(\theta', \mathcal{A}_f(\theta', b_l') - b_l', b_l') = \theta^- Z + b_l'.$$

This gives the decentralised counterpart of the Fund's participation constraint in (5),

$$W^l(s', a_l'(\theta', b_l'), b_l') \equiv W^f(s', a_l'(\theta', b_l'), b_l') + W^p(\theta', a_l'(\theta', b_l'), b_l') \geq \theta^- Z + b_l', \tag{24}$$

We interpret condition (24) as a proper DSA since it links the value of the current lending with its prospective stream of cashflow. This DSA takes into account the sovereign's entire debt position — within and outside the Fund — in every possible state. Moreover, owing to the trade of Arrow securities, it is contingent on all the states that the contract specifies, including those states where participation constraints are binding.

Note that, with $\bar{a}_p(\theta) = 0$, the market clearing condition in the Fund is given by $a(\theta, b) + a_l(\theta, b) = 0$ for all (θ, b). In addition, the initial asset holdings of the sovereign in the Fund, $a(\theta_0, b_0) = -a_l(\theta_0, b_0) = 0$, are given.

4.3 Properties of the Competitive Equilibrium

We first define a (recursive) competitive equilibrium in this environment and then characterize the price dynamic and the optimal holdings of assets.

Definition 1 (Recursive Competitive Equilibrium (RCE))**.** *Given value functions for the outside value options of the sovereign, $V^{af}(\theta')$, and of the lenders, $\theta^- Z + b_l$, a* Recursive Competitive Equilibrium *(RCE) consists of: prices $q_f(\theta', \omega'(\theta')|\theta)$ and $q_p(\theta, \bar{\omega}')$; value functions $W^b(\theta, a, b)$, $W^f(s, a_l, b_l)$, and $W^p(\theta, a_l, b_l)$; endogenous limits, $\mathcal{A}_b(\theta')$ and $\mathcal{A}_f(\theta', b_l')$, and policy functions $c(\theta, a, b)$, $c_f(\theta, a_l, b_l)$, $c_p(\theta, a_l, b_l)$, $n(\theta, a, b)$, $a'(\theta', b') = A(\theta', \theta, a, b, b')$, $a_l'(\theta', b_l') = A_l(\theta', \theta, a_l, b_l, b_l')$, $b' = B(\theta, a, b)$ and $b_l' = B_l(\theta, a_l, b_l)$, which are solutions to the problems of the sovereign, private lenders and the Fund, and all markets clear. Particularly, the announcement $\omega_l'(\theta')$ is equal to its equilibrium value, i.e. $\omega_l'(\theta') = a_l'(\theta', b_l') + b_l' = -\omega'(\theta')$.*

The last part of the definition makes it clear that the RCE has a Nash specification. The Fund's capacity announcement $\{\omega_l'(\theta')\}_{\theta'|\theta}$ is not a constraint in either (13) or (17), while, as we said, it is part of (20).

We now characterize the price dynamic and the optimal holdings of assets in the decentralised environment. Using the fact that the borrowing constraints of the sovereign and the Fund do not bind at the same time, the price is determined by the agent whose constraint is not binding (Krueger et al., 2008).[25] Defining $\eta \equiv \beta(1+r)$, it follows that

$$q_f(\theta', \omega'(\theta')|\theta) = \frac{\pi(\theta'|\theta)}{1+r}\left[(1-\delta+\delta\kappa) + \delta \sum_{\theta''|\theta'} q_f(\theta'', \omega''(\theta'')|\theta')\right] \max\left\{\frac{u_c(c(\theta', a', b'))}{u_c(c(\theta, a, b))}\eta, 1\right\}. \tag{25}$$

Given the above price schedule, the intertemporal discount factor is defined by

$$Q_f\big(\theta', \omega'(\theta')|\theta\big) \equiv \frac{q_f\big(\theta', \omega'(\theta')|\theta\big)}{1-\delta+\delta\kappa+\delta\sum_{\theta''|\theta'} q_f\big(\theta'', \omega''(\theta'')|\theta'\big)}. \tag{26}$$

The implicit interest rate in the Fund is then defined by $r_f(\theta, \bar{\omega}') \equiv \frac{1}{Q_f(\theta, \bar{\omega}')} - 1$ with $Q_f(\theta, \bar{\omega}') \equiv \sum_{\theta'|\theta} Q_f\big(\theta', \omega'(\theta')|\theta\big)$.

Provided that the private lenders have access to the Fund's securities, no arbitrage is possible between the Fund and the private bond market for the borrower. Hence, the bond prices in the Fund and the private bond market are alike.

Proposition 2 (Bond Price)**.** *In an RCE, for all* $(\theta, \omega'(\theta'))$,

$$\sum_{\theta'|\theta} q_f\left(\theta', \omega'(\theta')|\theta\right) = q_p(\theta, \bar{\omega}').$$

Moreover, whenever (24) *binds,* $\sum_{\theta'|\theta} q_f\left(\theta', \omega'(\theta')|\theta\right) > \frac{1-\delta+\delta\kappa}{1+r-\delta}$.

Given the definition of the price in (25), if the Fund's DSA is binding, the price of a bond reads $q_f(\theta, \bar{\omega}') > \frac{1}{1+r}\sum_{\theta'|\theta} \pi(\theta'|\theta)\big[(1-\delta+\delta\kappa) + \delta q_f(\theta', \bar{\omega}'')\big]$, or equivalently $Q_f(\theta, \bar{\omega}') > \frac{1}{1+r}$ implying that $r_f(\theta, \bar{\omega}') < r$. In words, when the Fund's DSA binds, a *negative spread* appears. The Fund's binding DSA has therefore two opposite effects. On the one hand, accumulating debt, $\bar{a}_l' > 0$, is cheaper owing to the fact that $q_f(\theta, \bar{\omega}')$ is above the risk-free price. On the other hand, buying insurance, $\hat{a}_l'(\theta') < 0$, becomes more expensive. This effect is even stronger provided that the trade of Arrow securities has to be such that $\sum_{\theta'|\theta} q_f(\theta', \omega'(\theta')|\theta)\hat{a}_l'(\theta') = 0$.

[25] If both constraints would bind at the same time, Assumption 1 would be violated.

The negative spread is a strong signal that the Fund refrains from further lending and causes private lenders to stop lending to the sovereign as the rate of return settles below r. Indeed, the private lenders are willing to *borrow* from the Fund in terms of a portfolio of securities which constitutes risk free asset a_p, and investing the funds to earn a risk free rate r. Nevertheless, the binding DSA of the Fund also prevents such trading activities. As a result, a binding DSA in (24) not only restricts the provision of the Fund's insurance to the sovereign, it also sustains a no-trade equilibrium in the private bond market: there is a private lending 'sudden stop'.

Thus, the Fund's DSA internalizes a pecuniary externality that competitive private lenders usually do not: the fact that marginal lending can be excessive. As the sovereign's entire debt position is perceived as safe — i.e. without default risk, private lenders may not realize that the present value of future transfers does not cover additional lending. The amount of debt held by the private lenders is therefore crucial in the DSA. As already highlighted, the right-hand side of (24) depends on b_l' and, due to the lending sudden stop, $b_l' \leq \delta b_l$. Furthermore, from (2), when (24) binds, we have $\theta^- Z + \delta b_l \geq \omega_l'(\theta')$. Given that $\omega_l'(\theta') = \bar{\omega}_l' + \hat{a}_l'(\theta')$, the maximal amount of debt the Fund may have to absorb is $\theta^- Z + \delta b_l - \min\left\{\hat{a}_l'(\theta') : (24) \text{ binds } \wedge \ \pi(\theta'|\theta) > 0\right\}$.

The Fund's intervention when the DSA binds has two components. On the one hand, the Fund provides $\breve{a}_l \equiv \theta^- Z - \min\left\{\hat{a}_l'(\theta') : (24) \text{ binds } \wedge \ \pi(\theta'|\theta) > 0\right\}$ which corresponds to the solidarity component $Z \leq 0$ and the complement to the maximal amount of insurance the sovereign may receive with positive probability. On the other hand, from the perspective of the private lenders, the Fund has to guarantee a maximal absorption of δb_l. In other words, the Fund must stand ready to guarantee just enough lending for the sovereign to honour its long-term liabilities. This is because the private lending sudden stop endangers the ability of the sovereign to maintain the value of its long-term debt, either directly — under the counterfactual interpretation that each period the sovereign buys and sells the long-term debt — since it may not be able to borrow from the private lenders to cover it; or, indirectly since private lenders may want to sell their holdings of over-priced, low-return, long-term debt in exchange for safe assets. The Fund's guarantee is therefore a form of prudential policy which is active as long as debt is long term (i.e. $\delta > 0$). Nevertheless, for any $\delta > 0$, such guarantee may never materialize in steady state.

Proposition 3 (Private Debt)**.** *In a RCE, in the states in which* (24) *binds,* $b_l' \leq \delta b_l$*. Conversely, in the states in which* (24) *does not bind, the division of* $\bar{\omega}_l'$ *between* b_l' *and* $\bar{a}_l'$ *is* *indeterminate.*

However, when the Fund's DSA in (24) does not bind, the sovereign can equally access the private bond market and the Fund. In this case, given Proposition 2, debt is as expensive in the Fund as in the private bond market and the sovereign can accumulate debt in both locations. Therefore, the sovereign is indifferent between holding debt in the private bond market or in the Fund. It is then without loss of generality that we can set $\bar{a}'_l = 0$ whenever (24) does not bind. As we have said, our underlying assumption is that as long as there are no spreads (positive or negative) on the debt's interest rates, private lenders are willing to buy all the debt being offered by the sovereign. We can therefore define the Fund's *minimal intervention policy* (MIP) in the following terms.[26]

Definition 2 (The Fund's Minimal Intervention Policy (MIP))**.** *For a given state* (θ, b_l)*, we say that the the Fund implements a Minimal Intervention Policy (MIP) if* $\bar{a}'_l = \underline{a}(\theta, b_l)$ *where, if* (24) *binds* $\underline{a}(\theta, b_l) \in [\check{a}_l, \check{a}_l + \delta b_l]$ *with* $\check{a}_l \equiv \theta^- Z - \min\left\{\hat{a}'_l(\theta') : (24) \text{ binds } \wedge \ \pi(\theta'|\theta) > 0\right\}$ *and* $\underline{a}(\theta, b_l) = 0$ *otherwise.*

We have taken the maturity structure of the debt as given, as we do in Section 6 when we calibrate our model to the Italian economy. However,

Corollary 2 (Debt maturity)**.** *If the sovereign takes into account the Fund contract in deciding its maturity structure, it prefers to choosing an alternative* $\hat{\delta} = 0$ *whenever* (24) *binds in steady state with* $\delta > 0$*. Otherwise, the choice of* δ *is irrelevant.*

This result follows from the fact that with long-term debt, in particular with the MIP, the Fund's guarantee towards private lenders can be up to δb_l. Hence, the closer is δ to 0, the lower is the amount of private debt the Fund may need to absorb every period when (24) binds. In other words, the choice of maturity in the presence of the Fund solely matters when (24) binds for some $\theta' \in \Theta$ with $\delta > 0$ in steady state.

[26] Unlike Ábrahám et al. (2021), we abstract from moral hazard. However, this does not impact the conclusion we draw from the interaction between the Fund and the private lenders. With moral hazard, the decentralised Fund contract features — in addition to what we have here — taxes on portfolio holdings. Those taxes are budget neutral and are set to ensure that the allocation remains incentive compatible. As a result, the moral hazard problem has no direct impact on the bond price and only an indirect impact on debt holdings through taxation. Thus, it does not affect the interplay between the Fund and the private bond market, provided the taxes are levied on all sovereign debt.

This recasts the findings of Aguiar et al. (2019). While it is true that there might be welfare losses with long maturities relative to short ones, the Fund contract remains constrained efficient for all $\delta \in [0,1]$. This is because, in the Fund contract, there is no sovereign debt dilution, since there is no default. Furthermore, given that Corollary 1 holds, the value of long term debt is preserved even when the Fund — to prevent excess lending — stops any additional lending.

Proposition 4 (Second Welfare Theorem (SWT)). *Given initial conditions* $\{\theta_0, b_0, x_0\}$*, the Fund's constrained-efficient allocation can be decentralised as a RCE with endogenous borrowing and lending limits.*

In solving the Fund contract, the Fund takes the strategy of the private lenders and the sovereign as given. In particular, it solves for consumption and leisure and the corresponding transfers, which, given the lending strategy of the private agents, split between the private lenders and the Fund, with the latter also providing insurance. The proof of the Second Welfare Theorem (SWT) requires to map the structure of the Fund's contract, accounting for its lending from competitive private lenders, into the more decentalized market structure of the RCE. A key step is to map the state of the Fund (θ, x) to the state of the RCE (θ, a_l, b_l), given the lending strategy of the lenders; that is, to map from (θ, x) to (θ, ω_l) and, giving b_l, a_l is determined. This map is given by the identification of the consumption policies and the Fund's consumption first-order condition

$$u_c(c(\theta, a, b)) = u_c(c(\theta, x, b_l)) = \frac{1 + \nu_l(\theta, x, b_l)}{1 + \nu_b(\theta, x, b_l)} \frac{1}{x},$$

and, since by (8) the right hand side is equal to η/x', the law of motion of the co-state variable x maps into the borrower's Euler equation. Using this, we define the Fund contract as a long-term state-contingent asset and derive the corresponding asset prices. Then, we map policies and value functions and show that they satisfy the RCE conditions of Definition 1. Furthermore, by Proposition 1, the constrained-efficient allocation is unique (when Assumption 1 is satisfied), therefore the RCE of Proposition 4 can take different forms (e.g. different asset structures), but the corresponding RCE allocation is also unique.

The SWT is satisfied in many environments, this is not the case for the First Welfare Theorem (FWT) since multiplicity of equilibria usually prevails; in particular, inefficient equilibria, such as autarky. Autarky is dismissed under the *High Implied Interest Rates* condition, which guarantees that the value of the equilibrium allocation (the present value of sovereign consumption plus transfers) is finite. Alternatively, bounded prices provide the same guarantee and, in general, equilibrium boundedness follows from standard monotonicity of preferences and an interiority, *free disposal*, assumption. We follow this tradition and introduce the equivalent to *free disposal* (already satisfied in our economies) when there are endogenous limit constraints. The following interiority assumption — which, in our environment, is implied by our Assumption 1 — also dismisses autarky as a RCE.

Assumption 2 (Decentralized interiority)**.** *There is an $\epsilon > 0$, such that, for all equilibrium states (θ, a, b) the sovereign, lenders and Fund problems — (13), (17) and, (20) — have a solution when the right hand sides of constraints (15) and (22) are replaced by $\mathcal{A}_b(\theta') + \epsilon$ and $\mathcal{A}_f(\theta'), b_l') + \epsilon$, respectively.*

Proposition 5 (First Welfare Theorem (FWT))**.** *Given initial conditions $\{\theta_0, b_0, a_0\}$, a RCE with endogenous borrowing and lending limits, satisfying Assumption 2 implements the constrained efficient allocation of the Fund.*

In the decentralized economy it is even more explicit that the Fund takes the strategy of the private lenders and the sovereign as given, as well as asset prices contingent on the sovereign's liabilities. The proof of the FWT requires the (inverse) map from the market structure — given by problems (13), (17) and (20), and the corresponding equilibrium conditions — to the structure of the Fund contract problem. The starting point is the first-order condition of the sovereign's problem (13): $u_c(c(\theta, a, b)) = \varkappa(\theta, a, b)$, where $\varkappa(\theta, a, b)$ is the Lagrange multiplier of the budget constraint (14). From the sovereign's and Fund's Euler equations we obtain the following intertemporal relation between these multipliers:

$$\varkappa(\theta, a, b) = \eta \frac{1 + \mathring{\nu}_b(\theta', a', b')}{1 + \mathring{\nu}_l(\theta', a', b')} \varkappa'(\theta', a', b'),$$

where $\mathring{\nu}_b(\theta', a', b')$ and $\mathring{\nu}_l(\theta', a', b')$ are normalized Lagrange multipliers of the endogenous limit constraints (15) and (22). As it can be seen, this intertemporal relation mirrors the law of motion of the co-state variable (8), which is at the core of Fund's problem. With the (inverse) map of value and policy functions it follows that the RCE allocation is a solution to the Fund's problem. Furthermore since, again by Proposition 1, the solution is unique the RCE allocation (when Assumption 2 is satisfied) must also be unique. We conclude this section with a characterization of the RCEs in our economies with a Fund.

Corollary 3 (No autarky, default, debt dilution or excess lending)**.** *Under Assumption 2, if the Fund can and implements a Minimal Intervention Policy, there exists a unique*

RCE allocation that features no autarky, no default — therefore no dilution of legacy private lenders — and no excess lending in the private sovereign debt market.

Corollary 3 follows from Propositions 1 and 5, together with Definition 2. The latter strengthens the uniqueness with a policy of the Fund that breaks the indeterminacy between the debt allocation between private investors and the Fund and, furthermore, guarantees that there is no turmoil in the sovereign debt market (which, *ex ante* could trigger non-anchored expectations) when there is a *sudden stop* because the lending limit binds.

This corollary states that the presence of the Fund rules out potential multiplicity in equilibrium. First, it prevents the private lenders to coordinate to an autarkic equilibrium, when a non-autarkic equilibrium is feasible (which it is by Assumption 2 and the standard convexity assumptions of our environment). Second, there is no default and, therefore, the Fund implements a saving equilibrium as defined by Aguiar and Amador (2020) in which the claim of legacy private creditors is never diluted when there is long-term debt. Third, there is no excess lending, which may occur in an economy where debt is being perceived as being safe, thanks to the MIP when there is long-term debt.

5 The Seniority Structure of the Fund

So far, we assume that the Fund has no seniority with respect to the privately held sovereign debt. We therefore consider that a default always implicates both the Fund and the private lenders. We now relax this assumption allowing for a *partial* default in which the sovereign defaults only on its private liabilities while remaining in the Fund.[27]

5.1 The Sovereign and the Private Lenders under Seniority

Compared to the case without seniority, the sovereign possesses two outside options. On the one hand, it can default on both the private lenders and the Fund. This represents the case of *full* default considered previously. On the other hand, the sovereign can repudiate its private debt while remaining in the Fund. We refer to this situation as a *partial* default because the sovereign solely defaults on the private lenders. That is, if the sovereign has an outstanding debt of $\bar{\omega} = \bar{a} + b$, it defaults on b and repays $\bar{a}$. We assume that the default penalty and the re-access probability are the same in *partial* and *full* defaults.[28]

[27] We do not consider the case in which the Fund is junior with respect to the private lenders as this is not the practice in official lending (cf. footnotes 9 and 15).

[28] This is without loss of generality. We could also have assumed a different output penalty and re-access probability under *partial* and *full* defaults.

There is a clear tradeoff when deciding whether to enter *partial* default. On the one hand, in *partial* default, the sovereign is less productive — i.e. $\theta^d \leq \theta$ — for some time. On the other hand, the sovereign repudiates is private liabilities — i.e. $b = 0$ — and continues to receive support from the Fund. That is, unlike in *full* default, it can still trade bonds and insurance with the Fund. Given this, the state space in the decentralized economy is now (θ, a, b, d_p) where $d_p = 1$ if the sovereign is in *partial* default and $d_p = 0$ otherwise. Hence, in a given state (θ', a', b'), the sovereign does not enter in *partial* default if

$$W^b(\theta', a', b', 0) \geq W^b(\theta', a', 0, 1), \tag{27}$$

where the value upon *partial* default reads

$$\begin{aligned} W^b(\theta, a, 0, 1) = & \max_{\{c,n,\{a'(\theta',0,d'_p)\}_{\theta',d'_p}\}} \quad U(c,n) + \beta \mathbb{E}\big[(1-\lambda)W^b(\theta', a'(\theta',0,1),0,1) \\ & \qquad\qquad\qquad\qquad + \lambda W^b(\theta', a'(\theta',0,0),0,0) \big| \theta\big] \\ & \text{s.t.} \quad c + \sum_{\theta'|\theta,d'_p} q_f(\theta', a'(\theta',0,d'_p),0|\theta)(a'(\theta',0,d'_p) - \delta a) \leq \theta^d f(n) + (1-\delta+\delta\kappa)a, \\ & \qquad a'(\theta',0,d'_p) = \bar{a}'(0) + \hat{a}'(\theta',d'_p) \geq \mathcal{A}_b(\theta',d'_p). \end{aligned}$$

We then define $V^{ap}(\theta, a) \equiv W^b(\theta, a, 0, 1)$.[29] In the case of *partial* default, the endogenous borrowing limit is defined as $W^b(\theta', \mathcal{A}_b(\theta', 1), 0, 1) = V^{af}(\theta')$, while in the case of repayment $W^b(\theta', \ddot{a}'(\theta', \ddot{b}', 0), \ddot{b}', 0) = V^{af}(\theta')$ for all $\ddot{a}'(\theta', \ddot{b}', 0) + \ddot{b}' = \mathcal{A}_b(\theta', 0)$.

Compared to the case without seniority, a' is now a function of the *partial* default status next period, d'_p. As the bond component $\bar{a}'$ is not contingent, it is the Arrow component, $\hat{a}'$, that depends on d'_p. This is because the sovereign is less productive in *partial* default — i.e. $\theta^d \leq \theta$ — and repudiates its liabilities towards private lenders — i.e. $b = 0$. The Fund's insurance component must therefore discriminate whether the sovereign is in default as the sovereign's risk profile changes.

The private lenders' problem remains static as in Section 4. Thus, the private bond price is given by

$$q_p(\theta, \bar{a}', b') = \frac{\mathbb{E}\{(1 - D(\theta', a', b'))[1 - \delta + \delta\kappa + \delta q_p(\theta', \bar{a}'', b'')]|\theta\}}{1 + r}, \tag{28}$$

where $D(\theta, a, b) = D_p(\theta, a, b) + D_f(\theta, a, b)$ with $D_p(\theta, a, b) = 1$ if $V^{ap}(\theta, a) > W^b(\theta, a, b, 0)$ and $V^{ap}(\theta, a) \geq V^{af}(\theta)$ and $D_p(\theta, a, b) = 0$ otherwise, while $D_f(\theta, a, b) = 1$ if $V^{af}(\theta, a) >$

[29] The value under repayment is a simple extension of (13) with the additional state variable $d_p = 0$.

$W^b(\theta, a, b, 0)$ and $V^{af}(\theta, a) > V^{ap}(\theta)$ and $D_f(\theta, a, b) = 0$ otherwise. The value under *full* default might coincide with the value under *partial* default. Hence, if the sovereign is indifferent between the two types of default, we assume it selects the *partial* default.

However, the price may not depend on the total level of debt $\bar{\omega}'$ anymore but on $\bar{a}'$ and b' separately. As we will see, the split of $\bar{\omega}'$ between $\bar{a}'$ and b' becomes relevant as in *partial* default the sovereign defaults on b' but repays $\bar{a}'$.

5.2 The Fund under Seniority

The Fund still aims at making the sovereign's debt safe. Thus, even though it possesses seniority, its capacity announcement continues to relate to the sovereign's entire indebtedness as in the case without seniority. However, in addition to (θ, a, b), the announcement now includes the default status d_p. That is, depending on the *partial* default decision, the sovereign does not receive the same amount of resource from the Fund. Again, this is because a *partial* default affects the sovereign's risk profile.

As we have seen previously, the sovereign's participation constraint continues to relate to the case of *full* default. The rationale is that, in the contract with seniority, the sovereign defaults on the Fund only in the case of a *full* default. A *partial* default solely affects private lenders. This means that if the value under *partial* default is greater than the value of *full* default in some states, *partial* defaults can occur. In other words, the sovereign's participation constraint alone is not sufficient to prevent *partial* defaults.

The Funds's participation constraint may change in the case of seniority. Particularly, the transfer to the private lenders is now given by $\tau_{p,t} = q_p(\theta_t, \bar{a}_{t+1}, b_{t+1})b_{t+1} - (1 - \delta + \delta\kappa + \delta q_p(\theta_t, \bar{a}_{t+1}, b_{t+1}))b_t(1 - D_t)$. Hence, depending on whether there are *partial* defaults, given (2), b_t might not be the same as in the Fund's participation constraint without seniority. The difference comes from the private bond market exclusion and the haircut following a default. Furthermore, given that $\theta^d \leq \theta$, a *partial* default impacts the sovereign's output which also affects the transfer to the Fund, τ_f. Hence, only without *partial* default does the participation constraint with and without seniority coincide.

5.3 The Fund's Minimal Intervention Policy under Seniority

To evaluate the importance of the seniority assumption, we need to check whether the sovereign is willing to follow the Fund's capacity announcement when we impose seniority. For this purpose, we define the Fund's MIP under seniority such that a *partial* default is never optimal.

First, observe that a *partial* default can occur only when the sovereign holds private debt. In other words, if $\bar{a}'_l = \bar{\omega}'_l$ and $b' = 0$, there is no *partial* default. In opposition, if *partial* default is optimal next period, the sovereign would like to set $b' = \bar{\omega}'$ and $\bar{a}' = 0$. Moreover, if $\theta^d = \theta$ for all θ, there is no penalty upon *partial* default meaning that it is not possible to sustain debt in the private bond market. This follows from the standard result in Bulow and Rogoff (1989). In the same logic, if $\theta^d < \theta$ for at least one θ, then the sovereign can hold some level of private debt without being willing to enter *partial* default. Thus, the MIP is the minimal level of debt the Fund should absorb such that (27) holds for all θ' for which $\pi(\theta'|\theta) > 0$.

Definition 3 (The Fund's Minimal Intervention Policy (MIP) under Seniority)**.** *For a given state θ, we say that the the Fund implements a Minimal Intervention Policy (MIP) under seniority if $\bar{a}'_l = \max\{\underline{a}(\theta, b_l), \underline{\underline{a}}(\theta)\}$ where $\underline{a}(\theta, b_l)$ is given in Definition 2 and $\underline{\underline{a}}(\theta) = \max\big\{\{-\bar{a}' > 0 : (27) \text{ binds } \wedge\ \pi(\theta'|\theta) > 0\} \cup \{0\}\big\}$.*

The sovereign does not have any incentive to enter *partial* default if Definition 3 is satisfied. Thus we come up with the following proposition.

Proposition 6 (Partial Default)**.** *In equilibrium, if $\bar{a}'_l \geq \underline{\underline{a}}(\theta)$, the sovereign never enters in partial default. Conversely, if $0 < \bar{a}'_l < \underline{\underline{a}}(\theta)$, the sovereign is willing to enter in partial default such that $0 < \mathbb{E}[D_p(\theta', \bar{a}'_l, b'_l)|\theta] \leq 1$ if the DSA binds in at least one θ' for which $\pi(\theta'|\theta) > 0$ and $\mathbb{E}[D_p(\theta', \bar{a}'_l, b'_l)|\theta] = 1$ otherwise.*

The first part of the proposition directly follows from Definition 3: the MIP under seniority is such that there is no *partial* default. Note that depending on the severity of the output penalty and the duration of the private bond market exclusion, (27) may hold in all states with $\underline{\underline{a}}(\theta) = 0$. In other words, under the Fund's seniority, the Fund's MIP can be identical to the one without seniority given in Definition 2. This is the case in our calibration. The second part of the proposition states that if the DSA never binds next period, the *partial* default decision is not state contingent. The non-binding DSA means that there is no private lending sudden stop and the sovereign can freely accumulate debt in the private bond market. In addition, as already highlighted, the Fund adapts the insurance it provides to the sovereign according to the *partial* default status. Arrow securities therefore aim at equating wealth not only across productivity states but also across repayment states. Thus, for a given level of debt in the Fund, the repayment decision is not state contingent when the DSA does not bind. This has a direct impact on the price charged by private lenders.

Corollary 4 (Private Lenders and Partial Default)**.** *In equilibrium, for a given debt capacity announcement* $\bar{\omega}'_l$*, if the DSA does not bind next period, the private lenders set* $q_p(\theta, \bar{a}', b') = 0$ *for all* $b' < -(\bar{\omega}'_l - \underline{\underline{a}}(\theta))$*.*

If the DSA binds with strictly positive probability next period, as we have seen in Section 4, a negative spread arises and the private lenders prefer borrowing from the Fund rather than lending to the sovereign. In other words, private lenders would like that the Fund absorbs all the sovereign debt. In this case, there is no possibility for the sovereign to deviate — say by accumulating more private debt — from the Fund's policy. In opposition, if the DSA does not bind next period, the private lenders anticipate that as soon as the Fund's intervention violates the definition of the MIP, the sovereign defaults on its private liabilities with probability one in the next period. In this case, the private lenders are not trying to borrow from the Fund but to escape from a *partial* default. Accordingly, given (28), they set $q_p(\theta, \bar{a}', b') = 0$ for all $b' < -(\bar{\omega}'_l - \underline{\underline{a}}(\theta))$. Again, the sovereign cannot deviate from the Fund's policy.

All in all, depending on the output penalty upon default and the re-access probability, the Fund's MIP might differ in the case with and without seniority. In the former, the Fund may need to absorb relatively more debt. However, in equilibrium, the sovereign cannot profitably deviate from the Fund's policy. Thus, the entire debt position remains safe as no default — either *partial* or *full* — arise on equilibrium path.[30]

6 Calibration

We calibrate the parameters of the model economy by fitting the sovereign debt model (9)–(10), i.e. the one without the Fund, to quarterly data of Italy over the period 1992Q1 to 2019Q4.[31] Table 1 summarizes the value of each parameters.

We calibrate the productivity growth rate shock γ_t with a Markov regime switching AR(1) process to the sample productivity series of Italy. We choose a specification of 2 regimes, with the worst regime capturing the crisis period (i.e the Great Financial crisis) observed in the data. Specifically, we estimate the following model for the (net) growth rate $\gamma_t - 1$ with

[30]Wicht (2023) shows that in an environment without state-contingent contracts, seniority matters for official multilateral lending institutions such as the IMF and the World Bank.

[31]The calibration starts in 1992 due to data availability and ends in 2019 owing to the pandemic. Appendix E contains detailed explanations on data sources, measurement, and additonal information on shock process estimation.

Table 1: Parameter Values

Parameter	Value	Definition	Targeted Moment
A. Direct measures from data			
α	0.5295	labor share	labor share
λ	0.032	return probability	average exclusion period
r	0.0132	risk-free rate	annual real short-term rate
δ	0.9297	bond maturity	bond maturity
κ	0.0543	bond coupon rate	bond coupon rate
B. Based on model solution			
β	0.96	discount factor	$\overline{b'/y}$
ψ	0.746	productivity penalty	$\rho(\text{spread}, y)$
ζ	0.29	labor elasticity	$\overline{n}$, $\sigma(c)/\sigma(y)$ and $\rho(n, y)$
ξ	1.265	labor utility weight	
C. By assumption			
Z	0	Fund's outside option	

the expectation maximization (EM) algorithm of Hamilton (1990):

$$\gamma_t - 1 = (1 - \rho(\varsigma_t))\mu(\varsigma_t) + \rho(\varsigma_t)(\gamma_{t-1} - 1) + \sigma(\varsigma_t)\epsilon_t, \tag{29}$$

where ς_t denotes the regime at t, and $\rho(\varsigma_t)$, $\mu(\varsigma_t)$, and $\sigma(\varsigma_t)$ denote regime specific parameters. As shown in Appendix E, such a regime switching process can capture the sudden drop in productivity dynamics around crisis periods. In the computation, we further discretize the shock process using the method of Liu (2017) with 15 grid points for each regime. Aguiar and Gopinath (2006) show that given a CRRA utility in consumption $\frac{c^{1-\sigma}}{1-\sigma}$, one requires that $\lim_{t\to\infty} \mathbb{E}_0 \beta^t(\theta_{t-1}^{1-\sigma} - 1)/(1-\sigma) = 0$, so that the discount utility can be well defined with stochastic trend. For the case of log utility, this amounts to $\lim_{t\to\infty} \mathbb{E}_0 \beta^t \log \theta_{t-1} = 0$, which holds automatically in our setup. We subsequently detrend an 'allocation' variable x_t by θ_{t-1}: $\tilde{x}_t = x_t/\theta_{t-1}$.

The preference parameters for labor supply are set to $\zeta = 0.29$ and $\xi = 1.265$. These are used to match the average fraction of working hours and its correlation with GDP, together with the volatility of consumption relative to GDP. The risk free interest rate is fixed to $r = 1.32\%$, the average real short-term interest rates of the Euro area. We further set $\delta = 0.9297$ and $\kappa = 0.0543$ to match the average Italian bond maturity and coupon rate (coupon payment to debt ratio), respectively. Finally, we fix $\beta = 0.96$ to match the average indebtedness relative to annual output. The production function is Cobb-Douglas

$f(n) = n^\alpha$, and we set $\alpha = 0.5295$ to match the average labor share in Italy.

The default penalty is asymmetric as in Arellano (2008). To ensure that we can properly detrend the penalty, we consider

$$\theta_t^d = \theta_{t-1}\psi\mathbb{E}\gamma_t \text{ if } \theta_t \geq \theta_{t-1}\psi\mathbb{E}\gamma_t \quad \text{and} \quad \theta_t^d = \theta_t \text{ if } \theta_t < \theta_{t-1}\psi\mathbb{E}\gamma_t.$$

One sets $\psi = 0.746$ to match the correlation of spread with respect to output. Furthermore, we fix $\lambda = 0.032$ which corresponds to an average default duration between 7 and 8 years. This is consistent with the average default length Italy recorded during its defaults on external debt in the 1930s and the 1940s (Reinhart and Rogoff, 2011). Note that under such parameter values, the MIP is the same with and without seniority.

7 Quantitative Analysis

In this section, we first assess the fit of the model to the data. We then compare the economy with and without the Fund through various exercises.

7.1 Model Fit and Comparison

The fit of the model with respect to the data is depicted in Table 2. As we calibrate the model to Italy, the relevant benchmark is the economy without the Fund. To compute the moments we run 5,000 simulations of the model with 600 periods each, and we discard the first 200. For the volatilities and correlation statistics, we filter the simulated data — except the spread — through the HP filter with a smoothness parameter of 1600.

Table 2: Data and Models

Targeted Moments				Non-Targeted Moments			
Variable	Data	Without Fund	With Fund	Variable	Data	Without Fund	With Fund
			A. First Moments				
$b'/y_{annual}\%$	117.64	116.20	221.00	$ps/y\%$	2.09	6.49	9.54
$n\%$	38.64	38.23	39.93	spread%	2.50	0.43	0.00
			B. Second Moments				
$\sigma(c)/\sigma(y)$	1.27	1.25	0.28	$\sigma(\text{spread})$	0.96	0.11	0.00
$\rho(n, y)$	0.68	0.63	0.99	$\sigma(n)/\sigma(y)$	0.75	1.42	0.62
$\rho(\text{spread}, y)$	-0.16	-0.25	0.00	$\rho(c, y)$	0.53	0.04	0.95
				$\sigma(ps/y)/\sigma(y)$	1.09	2.32	0.72
				$\rho(ps/y, y)$	0.29	0.71	0.98

As one can see, the model replicates well the average indebtedness of Italy owing to the long-term debt structure (Chatterjee and Eyigungor, 2012). We are also matching the share of hours worked and its correlation with output given the specification of the shocks. The same holds true for the volatility of consumption. In addition, the model replicates well the correlation of the spread with output. However, it cannot match the average spread observed in the data.[32] In terms of other non-targeted moments, the model also exaggerates the volatility and the correlation of the primary surplus.

In addition, Table 2 compares the economy with and without the Fund. The difference between the two is important. First, the Fund enables a greater accumulation of debt in total. Particularly, the Fund almost doubles the debt capacity of the economy. Nevertheless, with the MIP, the Fund's debt holdings is nil given that the Fund's DSA never binds in steady state as we will see. Second, there is no spread with the Fund, while the spread attains 0.43% without the Fund. Hence, the Fund achieves the goal of making sovereign debt safe — i.e. without default risk. Third, consumption is much less volatile in the presence of the Fund. This means that there is a greater risk sharing across states. This comes from the highly pro-cyclical surplus. In other words, in periods of distress, the Fund provides resources to sustain consumption. Such mechanism is less marked in the economy without the Fund owing to the risk premium attached on the debt and the lack of state contingency.

7.2 Policy Functions and Financial Variables

To gain better understanding of the working of the Fund, we first present the numerical solutions of the policy functions of the Fund under our calibration. Figure 1 depicts the the different policy functions for zero private debt as a function of $(\gamma, \tilde{x})$, while Figure 2 depicts the main financial variables.[33] All figures relate to the detrended version of the model presented in Appendix B. We focus on three main values of the growth rate: the smallest one, γ_{min}, the median one, γ_{med}, and the highest one, γ_{max}. We denote the annual output by $\tilde{y}$.

[32] Models of sovereign defaults in the spirit of Aguiar and Gopinath (2006) and Arellano (2008) have difficulty to match the average spreads. Chatterjee and Eyigungor (2012) manage to match an average spread of 8% by means of long-term debt and quadratic output penalty but do not use growth shocks. Bocola and Dovis (2019) also match the average spread using multiple maturities but target an average spread of 0.61%.

[33] In appendix G, Figure G.2 and G.5 present the main policy functions and financial variables as a function of $(\gamma, \tilde{\omega})$ and $(\gamma, \tilde{b})$, respectively.

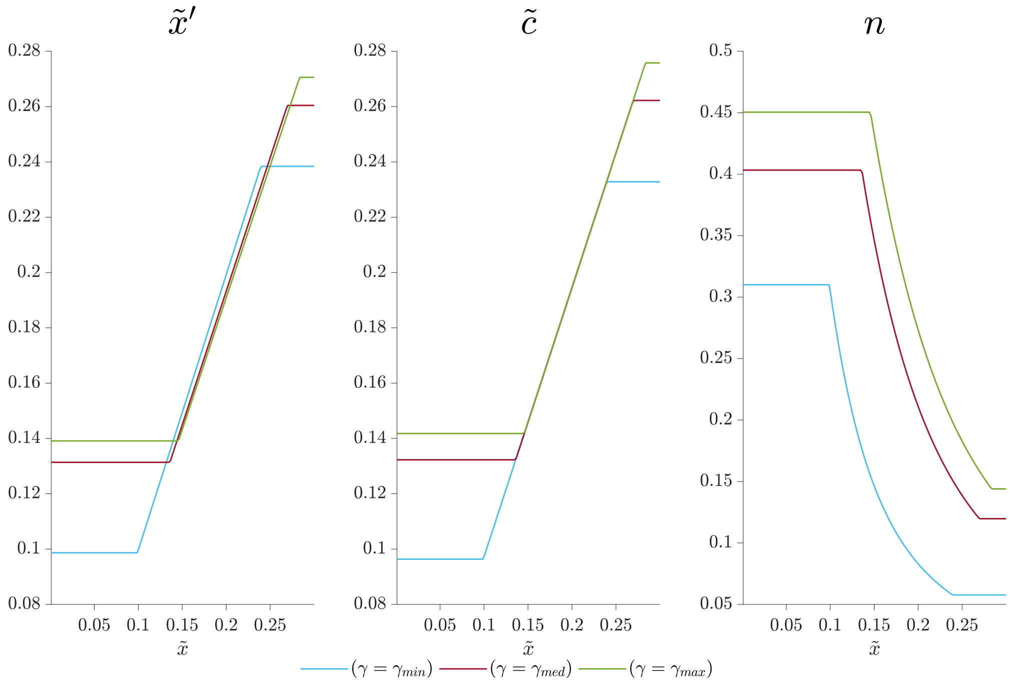

Figure 1: Optimal Policies with Zero Private Debt as Function of $(\gamma, \tilde{x})$

Figure 1 presents the optimal policies with respect to the future relative Pareto weight, consumption and labor as function of $(\gamma, \tilde{x})$. As explained in Section 3 and Appendix A, the recursive formulation of the Fund relies on the relative Pareto weight $\tilde{x}$ which keeps track of the binding constraints. With a logarithmic utility, one has that $\tilde{c} = \tilde{x}' \frac{\gamma}{\eta}$. Both $\tilde{c}$ and $\tilde{x}'$ are increasing, while n is decreasing in the current relative Pareto weight $\tilde{x}$. In each panel, the horizontal line on the left hand side is determined by the sovereign's binding participation constraint, while the horizontal line on the right hand side is determined by the Fund's binding participation constraint. The line rejoining both horizontal lines is determined by the first best allocation and has a slope of $\eta < 1$.

We now turn to the financial variables depicted in Figure 2.[34] The first row of the figure represents the prospective debt holdings of the sovereign. Consistent with the definition of MIP, when the Fund's DSA does not bind, the credit line of the Fund is nil. Conversely, when the Fund's DSA binds, there is a private lending sudden stop. With zero initial private debt this translates into a complete stop of private lending activities. In this case, the debt accumulation is largely reduced.

The second row of Figure 2 depicts the current asset holdings and the interest spreads.

[34]In appendix G, Figure G.4 presents the same financial variables but for different levels of private debt.

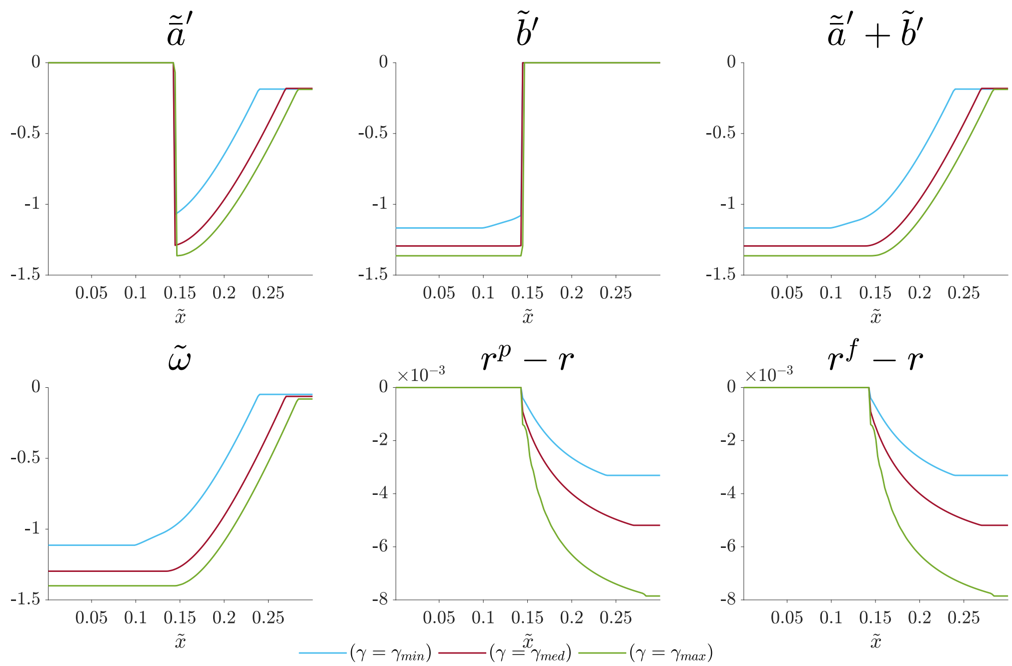

Figure 2: Financial Variables with Zero Private Debt as Function of $(\gamma, \tilde{x})$

One sees that when the Fund's DSA is binding, ω is very close to zero because of Definition 2 and the fact that $Z = 0$ and $\tilde{b} = 0$. As $\omega = \bar{\omega} + \hat{a}(\gamma)$, this tells us that if the Fund's DSA is binding today then the value of the sovereign's debt is in great part offset by the value of the realized Arrow security. Hence, when the Fund's DSA binds, the sovereign is limited in the trade of both Arrow securities and bonds.

Regarding interest rates, the Fund's and private bonds market's spreads are nil when the Fund's DSA is not binding consistent with Corollary 1.[35] In contrast, spreads are negative when the Fund's DSA is binding consistent with Proposition 2. As one can see, the negative spread remains relatively modest in terms of magnitude.

7.3 Steady State Analysis

As detailed in Appendices A and C, the relative Pareto weight, $\tilde{x}$, is key to the dynamics of the model economy as it represents a sufficient statistic of the contract's binding constraints. We first explain the dynamic of the relative Pareto weight before simulating the economy with and without the Fund in steady state.

Figure 3 displays the law of motion of the relative Pareto weight. The dark grey region represents the ergodic set given in Definition C.1. The light grey region represents the basin

[35] The default set of the economy with and without the Fund is presented in Figure G.6 in Appendix G.

of attraction of the ergodic set. As one can see, the convergence path to the steady state depends on the level of privately held debt. Especially, the larger is the level of private debt, the closer the economy gets to the ergodic set. This is different than in Ábrahám et al. (2021) where the convergence path solely depends on $\tilde{x}$ ad γ.

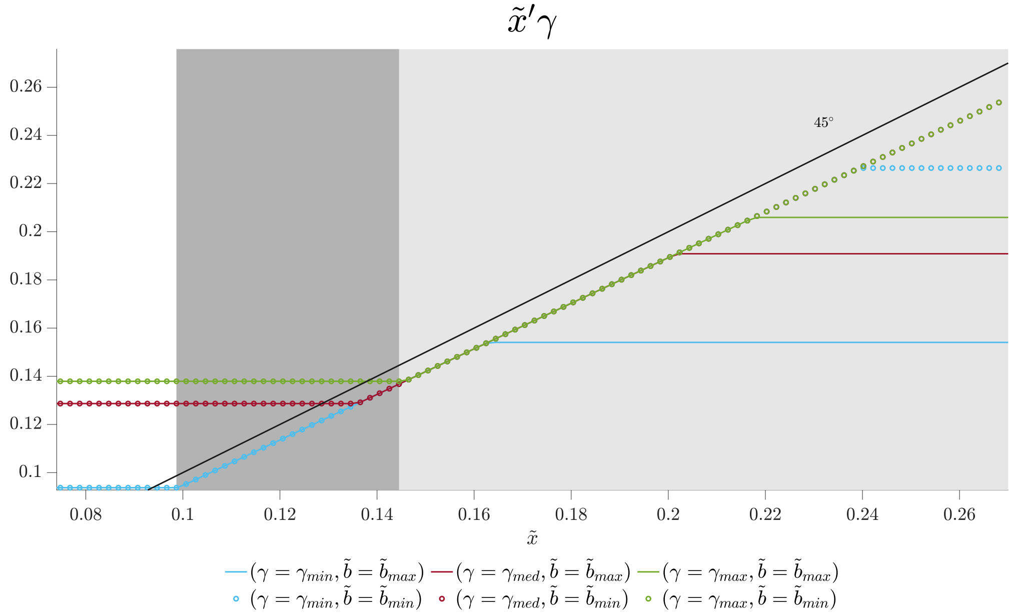

Figure 3: Evolution of the Relative Pareto Weight in Steady State as a Function of $(\gamma, \tilde{b}, \tilde{x})$

Most importantly, we see that the Fund's DSA does not bind in steady state. This has three main consequences. First, the private lending sudden stop exposed in Proposition 3 does not arise in the long run. Second, consistent with Corollary 2, the average bond maturity is irrelevant. Finally, in line with Definition 2, the Fund's holding of sovereign debt is nil — i.e. $\bar{a}' = 0$. In other words, the Fund solely provides insurance.

We simulate the economy within the ergodic set of relative Pareto weights. For this purpose, we generate one history of shocks for 400 periods in steady state starting with the lowest Pareto weight in the ergodic set. To avoid that the initial conditions blur the results, the first 200 periods are discarded. To gauge the impact of the Fund's intervention in this exercise, we simulate both the economy with and without the Fund in parallel.

Figure 4 depicts the simulation result with the grey region representing the periods in which the economy without the Fund is in default. With the Fund's intervention, the economy has a more stable consumption path over time. The sovereign avoids the major fluctuations of consumption that characterise the standard incomplete market economy with defaults. Moreover, the sovereign is able to accumulate private debt at the risk-free rate in

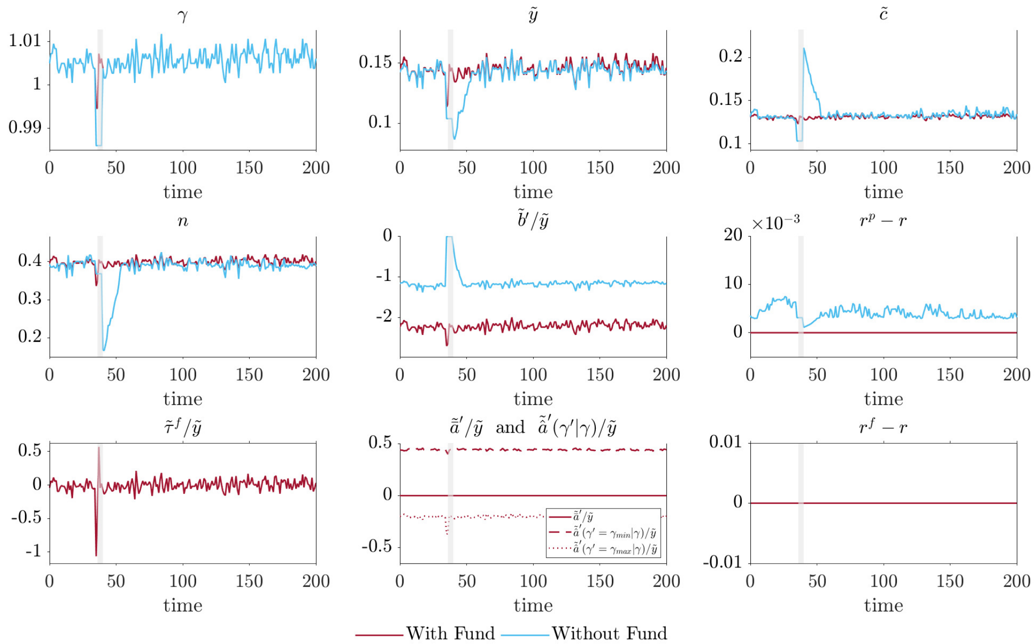

Figure 4: Simulation of a Typical Path

regions where it would normally default without the Fund. This is entirely due to the fact that the entire debt position is hedged by Arrow securities. To get a sense of the insurance component, we display the Arrow securities purchased today for the highest and the lowest states tomorrow. Two points deserve to be noted. First, the portfolio of Arrow securities is procyclical as it closely follows the shock process. Second, the positions taken in Arrow securities are substantial. If one focuses on $\tilde{a}'(\gamma'|\gamma)$ for $\gamma' = \gamma_{min}$, we see that it amounts on average to roughly 50% of annual GDP. Instead of looking at the Arrow securities one can observe the Fund's primary surplus, $\tilde{\tau}_f$, which also moves procyclically and largely oscillates around zero since $Z = 0$.

Figure 5 depicts the impulse response functions resulting from a stark negative growth shock on selected key variables.[36] The responses are computed as the mean of 5,000 independent shock histories starting with the lowest growth shock as well as initial debt holdings and relative Pareto weights drawn from the ergodic set. In the very first periods following the negative shock's realization, the Fund provides additional insurance to the sovereign. This sustains the existing level of debt and prevents a large decrease in consumption and a large

[36]Figures G.7 and G.8 in Appendix G present the impulse response functions to a negative and to a positive shock of all relevant variables in the model, respectively.

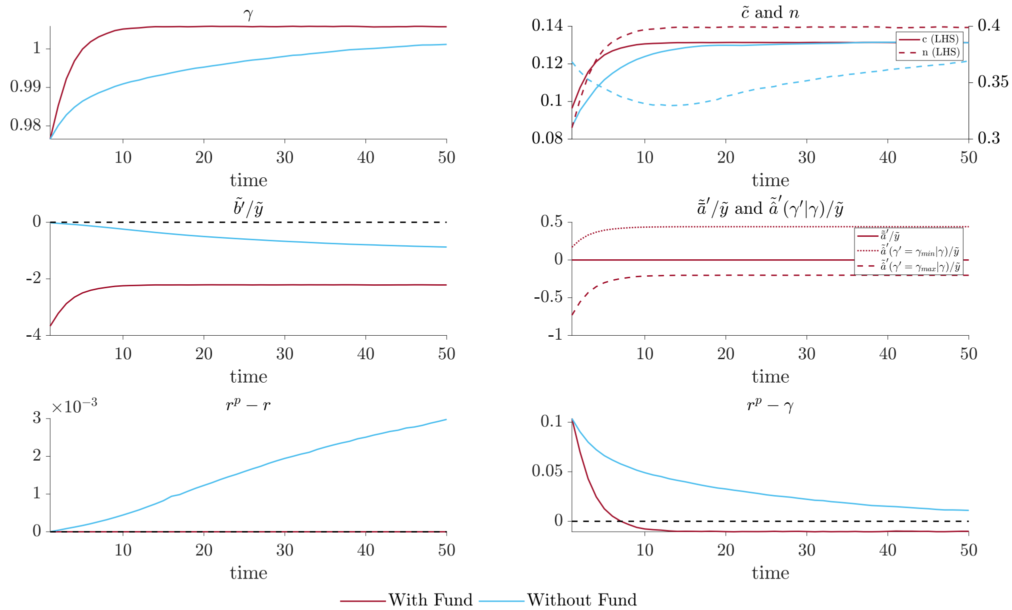

Figure 5: Impulse Response Functions to a Negative γ Shock

increase in labor supply. Hence, without the Fund's intervention, the sovereign repudiates its debt and is obliged to provide more labor to avoid a massive reduction in consumption. Thus, the immediate impact of a sudden low growth shock is more severe in the absence of the Fund. In the long run, the sovereign without the Fund is likely to repudiate debt again and therefore reaches a lower level of steady state indebtedness. Besides this, the economy with the Fund avoids the positive spread in the private bond market. It can therefore reach more quickly a low level of $r^p - \gamma$ easing debt management.

7.4 Welfare Analysis

Sharp difference in the dynamics of the economy with and without Fund translates into superior welfare implications of the Fund. The first column of Table 3 represents the welfare gains of the Fund's intervention in consumption equivalent terms at zero initial debt holdings. Recall that the sovereign which has access to the Fund can hold debt in the Fund or in the private bond market. Thus, to adequately compare the two economies, we compare them for the same *total* debt holdings. That is, the welfare comparisons are computed at the points where $\omega = 0$ for the economy in the Fund and at $b = 0$ for the economy outside the Fund. The welfare computation is presented in Appendix F.

Welfare gains are significant with the Fund's intervention. With zero initial debt, the consumption-equivalent welfare gains are on average 14%. Moreover, the largest welfare gains are recorded in low growth states. Thus, the Fund's intervention is mostly valued when the

Table 3: Welfare Comparison at Zero Initial Debt

State	Welfare Gains (%)	Maximal Debt Absoption (% of GDP)	
		With Fund	Without Fund
$\gamma = \gamma_{min}$	15.27	596	224
$\gamma = \gamma_{med}$	14.01	244	125
$\gamma = \gamma_{max}$	13.82	204	104
Average	14.05		

sovereign is in a difficult economic situation. As mentioned above, welfare gains are the consequence of two main features of the Fund's intervention. First, the Fund provides state-contingent transfers and therefore enhances consumption smoothing. Second, it enables a greater accumulation of debt in general. As one can see in the last two columns of Table 3, the the maximal debt absorption of the economy is almost always twice larger with the Fund than without.

To be more precise on the source of the aforementioned welfare gains, in Appendix F, we provide a decomposition of the welfare gains. We show that they are mostly due (i.e. above 90%) to the greater debt capacity and the insurance component; among these two factors debt capacity represents the largest share of total gains (i.e. circa 85%).

7.5 Debt Dynamic Decomposition

We further decompose the evolution of the debt according to Cochrane (2020, 2022): sovereign debt at the end of the year, v_{t+1}, is equal to its value at the beginning of the year, v_t, plus the net cost of keeping debt, $r_t^p - \gamma_t$, and the year's primary deficit (excluding interest payment), $-s_t$, so that $v_{t+1} = v_t + r_t^p - \gamma_t - s_t$, assuming no discounting for simplification. In our environment, the primary surplus without interest payment corresponds to $\tilde{b}_{t+1} - \tilde{b}_t$ for the economy without the Fund and $\tilde{\tilde{\omega}}_{t+1} - \tilde{\omega}_t$ for the economy with the Fund.

Figure 6 depicts the decomposition for Italy as well as the model economy with and without the Fund in logarithmic scale. We generate the two panels for the model economy by feeding the smoothed growth path of Italy over 2000Q1–2019Q4 into the model and start with the same level of debt of Italy in 2000Q1.[37] We then obtain the path of debt and interest rate through the optimal policy functions. The blue line represents the evolution of

[37] We consider a smoothed version of the growth path to avoid defaults in the economy without the Fund.

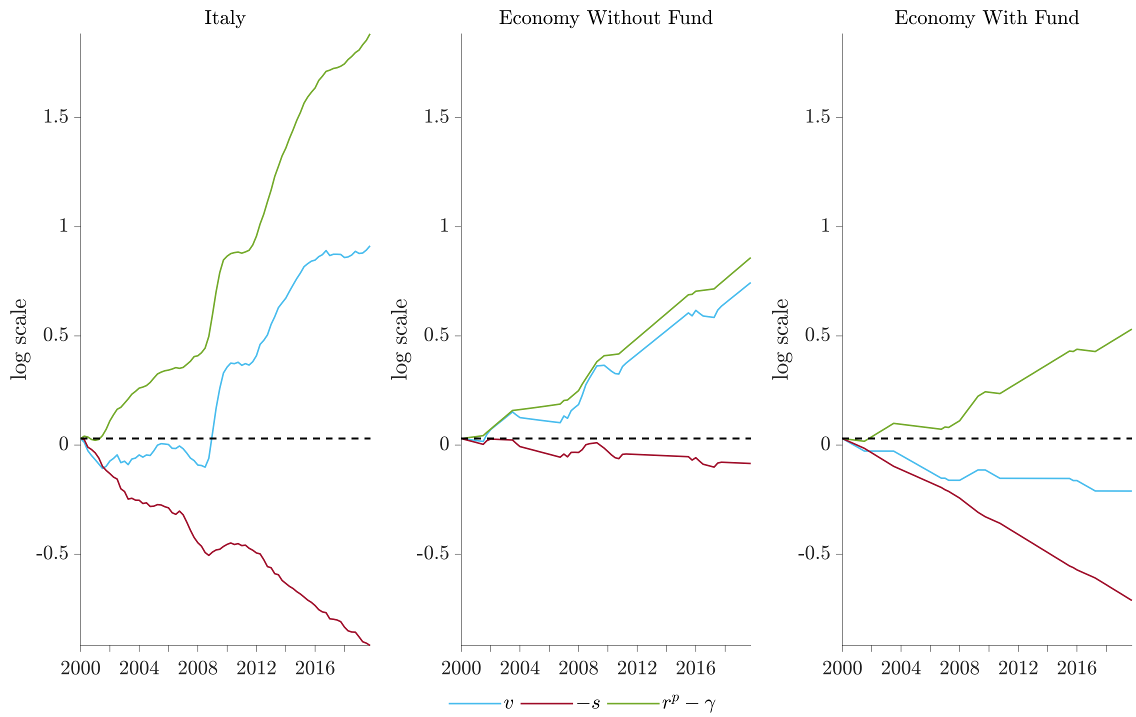

Figure 6: Cochrane Decomposition

the value of debt which is the combination of the green line (i.e. $r^p - \gamma$) and the red line (i.e. $-s$). In view of this, had the accumulation of debt been costless (i.e. $r^p - \gamma = 0$), then the blue line would coincide with the red line.

We observe that the evolution of Italy's debt is the result of two conflicting forces: a remarkable history of increasing accumulated primary surpluses and two decades of growth decline resulting in accumulated costs $r^p - \gamma$. The model without the Fund replicates well the dynamic of the Italian public indebtedness. It nonetheless minimises the positive impact of primary surpluses and the negative impact of the interest rate-growth differential.

Turning to the economy with the Fund, we see that the evolution of debt is flatter than in the economy without. This comes from two components. On the one hand, the rate at which the sovereign issues debt is at most risk free. This therefore largely reduces the $r^p - \gamma$ cost compared to the economy without the Fund. On the other hand, the Fund provides insurance through Arrow securities. This eases debt management by making fiscal policy countercyclical as shown previously. As a result, the debt path is more smooth. Particularly, the model predicts that the Italian indebtedness by the end of 2019 would have been around 80% of GDP rather than 135% if Italy could have joined the Fund in 2000.[38] This shows that

[38]We obtain this figure by computing the model implied debt-to-GDP ratio at the end of the sample

the path followed by the Italian economy in the last two decades was highly inefficient. Even though the accumulation of large primary surpluses dampened the increase in the Italian indebtedness, it prevented a proper countercyclical policy which could have corrected the interest rate-growth differential.

With the intervention of the Fund, debt can be located both in the private bond market and the Fund itself. We can further decompose the value of the debt as $v_t = v_t^P + v_t^F$, where v_t^P and v_t^F are the value of the debt held in the private bond market and the Fund, respectively. Figure G.9 in Appendix G presents the above decomposition. We note two elements. On the one hand, given the MIP, most of the value of the debt emanates from the private bond market. On the other hand, the Fund dampens the dynamic of debt over time. In other words, it counterbalances the large level of indebtedness in the private bond market through insurance.[39]

8 Conclusion

A starting point of this research has been the recognition that in a monetary union, such as the Euro area and as the result of the 21st Century crises, not only sovereign debt is very high, but also that a large fraction of the union-countries' sovereign debt is being held in Euro area institutions. This has helped 'stressed countries', reducing sovereign debt spreads — for example, in the Euro crisis in 2012. However, a simple maturity transformation or a long-term holding of sovereign debts may not be the most efficient debt management policy for the union. In fact, Ábrahám et al. (2021) has already shown that there can be high efficiency and welfare gains from having a *Financial Stability Fund*, with the proviso that the Fund absorbs all the sovereign debt of a country. We remove this proviso and show that the gains are still very high. Particularly, we show that Fund's intervention needs only to be minimal. Such *minimal intervention policy* (MIP) consists of an insurance component with an additional guarantee on long-term debt holdings by private lenders when the DSA binds.

In sum, there are many interesting features to our results but we want to emphasize the two key elements that give the Fund a leading role in 'making sovereign debt safe' even with a MIP. The two elements also require innovation with respect of existing official lender's

period using the decomposition of Cochrane (2020, 2022).

[39]The Fund contract with Italy has a positive asset position in the Fund ($v_t^F < 0$). This is part of the insurance component allowing, for example, an increase in debt in the private market during crises, without triggering surge accumulations of Italian debt, as it happened in 2008 - 2012 (see the first panel of Figure 6), in spite of the Italian government maintaining positive primary surpluses.

practices. First, the existence of a proper country risk-assessment, accounting for the effect of the constrained-efficient Fund contract. Second, the role of the Fund state-contingent contract in defining a thick contingent (and contention) wall between the level of liabilities which is sustainable and the level which is not. And, linking the two, its role in coordinating lenders' and sovereign's beliefs with its debt capacity announcements. As we said, most of the sovereign literature has focused on default problems, but in a mature union, outright default or exit may be rare events.[40] However, with the uncertainty and challenges that even advanced economies face, debt sustainability can remain a persistent concern for years to come and, even if sovereign debt is perceived to be safe, excessive lending can be a problem that private lenders may not internalize. We hope our work will not only contribute to the existing literature but also to face these challenges.

More specifically, we show in our calibration to the Italian economy and subsequent simulations and computations, how important welfare gains can be achieved by improving existing official lending practices offering long-term state-contingent Fund contracts, even when there is debt accumulation or $r - g$ uncertainty, as most countries nowadays face. Furthermore, in line with current official lenders policies, we show that these results — which also imply a massive increase of safe assets — can be achieved with a MIP, of the Fund, in terms its holdings of sovereign debt, much lower than the current Euro area holdings of sovereign debt by Euro area institutions.

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[40]The recent Brexit shows that exit can happen or, alternatively, that the union was still immature.

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Online Appendix
(Not For Publication)

A The Fund Contract in Recursive Form

Using the recursive contracts approach, defining $s \equiv \{\theta^-, \gamma\}$, we say that $c(\theta, x, b_l)$, $n(\theta, x, b_l)$, $\nu_b(\theta, x, b_l)$ and $\nu_l(\theta, x, b_l)$ are a saddle-point solution to the Fund's contacting problem in *recursive form*, given $b_{l,0}$, if there exist a Fund's value function $FV(s, x, b_l)$, transfer policies $\tau_p(\theta, x, b_l)$ and $\tau_f'(\theta', x, b_l)$, with associate private lending policy $b_l' = B_l(\theta, x, b)$ satisfying (2), i.e., $\mathbb{E}_{t+1} \sum_{j=t+1}^{\infty} \left(\frac{1}{1+r}\right)^{j-t-1} \tau_p(\theta^j) \geq b_{l,t+1}$, such that:

$$FV(s, x, b_l) = \mathcal{SP} \min_{\{\nu_b, \nu_l\}} \max_{\{c,n\}} x\big[(1+\nu_b)U(c,n) - \nu_b V^{af}(\theta)\big] \tag{A.1}$$

$$+ \big[(1+\nu_l)\tau - \nu_l(\theta^- Z + b_l)\big] + \tfrac{1+\nu_l}{1+r}\mathbb{E}\big[FV(s', x', b_l')\big|\theta\big]$$

$$\text{s.t. } \tau = \theta f(n) - c,$$

$$x' = \frac{1+\nu_b}{1+\nu_l}\eta x, \tag{A.2}$$

$$\text{with } x_0 \text{ given.}$$

The value function of the contracting problem satisfies:

$$FV(s, x, b_l) = xV^b(\theta, x, b_l) + V^l(s, x, b_l), \text{ with} \tag{A.3}$$

$$V^b(\theta, x, b_l) = U(c,n) + \beta\mathbb{E}[V^b(\theta', x', b_l')|\theta], \tag{A.4}$$

$$V^l(s, x, b_l) = \tau + \frac{1}{1+r}\mathbb{E}[V^l(s', x', b_l')|\theta]. \tag{A.5}$$

We denote by x' the prospective Pareto weight of the sovereign relative to the Fund where $\eta \equiv \beta(1+r) < 1$ and $\nu_b \geq 0$ and $\nu_l \geq 0$ are the normalized multipliers attached to the sovereign's and the Fund's participation constraints, respectively.[41] That is, the constraint qualification constraints are

$$\nu_b\Big[V^b(\theta, x, b_l) - V^{af}(\theta)\Big] = 0, \tag{A.6}$$

$$\nu_l\Big[V^l(s, x, b_l) - (\theta^- Z + b_l)\Big] = 0. \tag{A.7}$$

The contracting problem in recursive form takes into account the existence a private lending policy, $b_l' = B_l(\theta, x, b)$.[42] The sequence of private transfers $\{\tau_p(\theta^t)\}_{t=0}^{\infty}$ directly relates to a

[41] The normalization of the Pareto weights is the same as the one in Ábrahám et al. (2021).

[42] In this Nash specification of the Fund contract, the effect of τ_f on $B(\theta, x, b)$ is not taken into account.

sequence of private debt $\{b_l(\theta^t)\}_{t=0}^{\infty}$. Hence, for a given b_l, by picking b_l', the private lenders directly choose a certain level of transfer τ_p. The exact relationship between τ_p and b_l' is detailed in Section 4.

We obtain the optimal consumption and leisure policies, $c(\theta, x, b_l)$ and $n(\theta, x, b_l)$ by taking the first-order conditions of problem (A.1),[43]

$$u_c(c) = \frac{1+\nu_l}{1+\nu_b}\frac{1}{x}, \tag{A.8}$$

$$\theta f_n(n) = \frac{h_n(1-n)}{u_c(c)}. \tag{A.9}$$

This results in a total transfer policy $\tau(\theta, x, b_l)$ which corresponds to the lending capacity the Fund computes and announces every period. The lending capacity enables the economy to reach the constrained efficient allocation.

The relative Pareto weight, x, evolves according to the binding participation constraints. Particularly, it increases when the sovereign's constraint binds (i.e. $\nu_b > 0$) and decreases when the Fund's constraint binds (i.e. $\nu_l > 0$). In the former case, the sovereign's consumption increases not to generate default incentives, while in the latter case, the sovereign's consumption decreases to avoid expected losses from the lenders' perspective.

B Detrended Model

As in Aguiar and Gopinath (2006), we consider a growth shock to the productivity of the following form $\theta_t = \gamma_t \theta_{t-1}$, where γ_t represents the growth rate and θ_t the trend at time t. We detrend the variables for allocations (except for labor n where we normalize the time endowment to 1) of the model by dividing them by θ_{t-1}. We normalize $\theta_{-1} = 1$, then the initial states satisfy $\theta_0 = \gamma_0$. We then denote by $\tilde{c}_t$ the detrended form of c_t such that $\tilde{c}_t = \frac{c_t}{\theta_{t-1}}$ represents the deviation from the trend. It follows that $U(c_t, n_t) = \ln(\theta_{t-1}) + U(\tilde{c}_t, n_t)$, and clearly, $\ln(\theta_{t-1})$ does not affect optimal choice. By the homogeneity of the sovereign's recursive problem, we have the detrended formulation as

$$\widetilde{W}^b(\gamma, \tilde{a}, \tilde{b}) = \max_{\left\{\tilde{c}, n, \tilde{b}', \{\tilde{a}'(\gamma')\}_{\gamma' \in \Gamma}\right\}} U(\tilde{c}, n) + \beta \mathbb{E}\left[\widetilde{W}^b(\gamma', \tilde{a}'(\gamma', \tilde{b}'), \tilde{b}') \Big| \gamma\right] \tag{B.1}$$

$$\text{s.t. } \tilde{c} + \sum_{\gamma'|\gamma} q_f(\gamma', \tilde{\omega}'(\gamma')|\gamma)(\gamma \tilde{a}'(\gamma', \tilde{b}') - \delta \tilde{a}) + q_p(\gamma, \tilde{\bar{\omega}}')(\gamma \tilde{b}' - \delta \tilde{b})$$

$$\leq \gamma f(n) + (1 - \delta + \delta\kappa)(\tilde{a} + \tilde{b}), \text{ and}$$

[43] The first-order condition with respect to consumption tells us that the sovereign can infer x_t from $u'(c_{t-1})$ given $(x_0, b_{l,0})$.

$$\tilde{\omega}'(\gamma') = \tilde{a}'(\gamma', \tilde{b}') + \tilde{b}' \geq \tilde{\mathcal{A}}_b(\gamma').$$

The sovereign's outside option in detrended form takes the following form

$$\widetilde{V}^{af}(\gamma) = \max_n \left\{U(\gamma^d f(n), n)\right\} + \beta \mathbb{E}\left[(1-\lambda)\widetilde{V}^{af}(\gamma') + \lambda \tilde{J}(\gamma', 0) \middle| \gamma\right],$$

The detrended Fund's problem in sequential form is given by

$$\max_{\{\tilde{c}(\gamma^t), n(\gamma^t)\}_{t=0}^{\infty}} \mathbb{E}\left[\mu_{b,0} \sum_{t=0}^{\infty} \beta^t U(\tilde{c}(\gamma^t), n(\gamma^t)) + \mu_{l,0} \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right)^t \left(\prod_{i=0}^{t-1} \gamma_i\right) \tilde{\tau}(\gamma^t) \middle| \theta_{-1}\right] \tag{B.2}$$

$$\text{s.t.} \quad \mathbb{E}\left[\sum_{j=t}^{\infty} \beta^{j-t} U(\tilde{c}(\gamma^j), n(\gamma^j)) \middle| \gamma^t\right] \geq \widetilde{V}^{af}(\gamma_t), \tag{B.3}$$

$$\mathbb{E}\left[\sum_{j=t}^{\infty} \left(\frac{1}{1+r}\right)^{j-t} \left(\prod_{i=t}^{j-1} \gamma_i\right) \tilde{\tau}(\gamma^j) \middle| \gamma^t\right] \geq Z - \tilde{b}(\gamma^t), \tag{B.4}$$

$$\tilde{\tau}(\gamma^t) = \gamma_t f(n(\gamma^t)) - \tilde{c}(\gamma^t), \quad \forall \gamma^t, t \geq 0,$$

$$\text{with } \mu_{b,0},\ \mu_{l,0},\ \{\tilde{b}(\gamma^t)\}_{t=0}^{\infty} \text{ given.}$$

And in recursive form

$$\widetilde{FV}(\gamma, (\tilde{x}, 1), \tilde{b}_l) = \mathcal{SP} \min_{\{\nu_b, \nu_l\}} \max_{\{\tilde{c}, n\}} \tilde{x}\left[(1+\nu_b) U(\tilde{c}, n) - \nu_b \widetilde{V}^{af}(\gamma)\right] \tag{B.5}$$

$$+ \left[(1+\nu_l)\tilde{\tau} - \nu_l (Z + \tilde{b}_l)\right] + \frac{1+\nu_l}{1+r} \gamma \mathbb{E}\left[\widetilde{FV}(\gamma', (\tilde{x}', 1), \tilde{b}_l') \middle| \gamma\right]$$

$$\text{s.t.} \quad \tilde{\tau} = \gamma f(n) - \tilde{c},$$

$$\tilde{x}' = \frac{1+\nu_b}{1+\nu_l} \frac{\eta}{\gamma} \tilde{x}, \tag{B.6}$$

Note that we have also enlarged the co-state $\tilde{x}$ to $(\tilde{x}, 1)$, which contains the same information, but now $\widetilde{FV}$ is homogeneous of degree one in $(x, 1)$, which is convenient in the proof of existence of a Fund contract. Nevertheless, we will only make this explicit extension when it is necessary. The value function takes the form of

$$\widetilde{FV}(\gamma, (\tilde{x}, 1), \tilde{b}_l) = \tilde{x} \widetilde{V}^b(\gamma, (\tilde{x}, 1), \tilde{b}_l) + \widetilde{V}^l(\gamma, (\tilde{x}, 1), \tilde{b}_l), \text{ with} \tag{B.7}$$

$$\widetilde{V}^b(\gamma, (\tilde{x}, 1), \tilde{b}_l) = U(\tilde{c}, n) + \beta \mathbb{E}[\widetilde{V}^b(\gamma', (\tilde{x}', 1), \tilde{b}_l') | \gamma], \text{ and}$$

$$\widetilde{V}^l(\gamma, (\tilde{x}, 1), \tilde{b}_l) = \tilde{\tau} + \frac{1}{1+r} \gamma \mathbb{E}[\widetilde{V}^l(\gamma', (\tilde{x}', 1), \tilde{b}_l') | \gamma].$$

Taking the first-order conditions with respect to $\tilde{c}$ and n leads to

$$u_c(\tilde{c}) = \frac{1+\nu_l}{1+\nu_b} \frac{1}{\tilde{x}} \quad \text{and} \quad \gamma f_n(n) = \frac{h_n(1-n)}{u_c(\tilde{c})},$$

The consumption is therefore equal to $\tilde{c} = \tilde{x}'\frac{\gamma}{\eta} \equiv \tilde{z}'\gamma$. From this, we see that whenever the growth rate of the economy settles below one, the relative Pareto weight increases. However, the consumption does not react to changes in γ. In fact, the consumption is affected only when one of the participation constraints binds.

Finally, for completeness, the decentralised Fund problem in detrended form is given by

$$\widetilde{W}^f(\gamma, \tilde{a}_l, \tilde{b}_l) = \max_{\{\tilde{c}_f, \{\tilde{a}_l'(\gamma', \tilde{b}_l')\}_{\gamma' \in \Gamma}\}} \tilde{c}_f + \frac{1}{1+r}\gamma \mathbb{E}\left[\widetilde{W}^f(\gamma', \tilde{a}_l'(\gamma', \tilde{b}_l'), \tilde{b}_l') \middle| \gamma\right] \tag{B.8}$$

$$\text{s.t. } \tilde{c}_f + \sum_{\gamma'|\gamma} q_f(\gamma', \omega'(\gamma')|\gamma)(\gamma \tilde{a}_l'(\gamma', \tilde{b}_l') - \delta \tilde{a}_l) \leq (1 - \delta + \delta\kappa)\tilde{a}_l,$$

$$\tilde{a}_l'(\gamma', \tilde{b}_l') + \tilde{b}_l' \geq \tilde{\mathcal{A}}_f(\gamma', \tilde{b}_l'). \tag{B.9}$$

$$\text{with } \tilde{b}_l' = \tilde{B}_l(\gamma, \tilde{a}_l, \tilde{b}_l) \text{ given}$$

C Further Theory Development

In this section we present other properties of the Fund contract. We start with the inverse Euler equation which is a key concept determining the dynamic of consumption in the contract.

Proposition C.1 (Inverse Euler Equation)**.** *In the Fund contract, the inverse Euler equation is given by*

$$\mathbb{E}\left[\frac{1}{u_c(c(\theta', x', b_l'))} \frac{1 + \nu_l(\theta', x', b_l')}{1 + \nu_b(\theta', x', b_l')} \middle| \theta\right] = \eta \frac{1}{u_c(c(\theta, x, b_l))},$$

and risk sharing is imperfect.

We obtain the inverse Euler equation by means of the first-order condition on consumption and the law of motion of the relative Pareto weight. This equation gives the intertemporal dynamic of consumption. If none of the constraints are ever binding (i.e. $\nu_b = \nu_l = 0$), it becomes

$$\mathbb{E}\left[\frac{1}{u_c(c(\theta', x', b'))} \middle| \theta\right] \leq \frac{1}{u_c(c(\theta, x, b))},$$

with strict inequality if $\eta < 1$, in our case. We therefore obtain a positive martingale, which by the supermartingale theorem, converges almost surely to 0. This is what the literature has called immiseration.

Thus, with $\eta < 1$, when none of the constraints are binding, consumption decreases. However, this reduction cannot go on indefinitely given the sovereign's participation constraint. This constraint puts a lower bound to the supermartingale and therefore acts as a stopper for immiseration. Conversely, the Fund's constraint puts an upper bound to the supermartingale which prevents consumption to increase indefinitely. As a result, in a contract with two-sided limited enforcement constraints and impatient borrower, risk sharing is only partial. The contract cannot converge to the first-best allocation characterised by constant consumption over time.

The long-run property of the Fund contract is related to the definition of an ergodic set of relative Pareto weights, x. The term ergodic refers to the fact that the relative Pareto weights in this set are aperiodic and recurrent with non-zero probability. In other words, the economy will move around the same set of relative Pareto weights over time and over histories. The following definition relies on the model in detrended form presented in Appendix B.

Definition C.1 (Steady State). *Given a Markov chain of γ with a unique ergodic set in Γ, a* Steady State Equilibrium *is defined by an ergodic set with a lower bound $\underline{x} = \min_{\gamma\in\Gamma}\{x : \widetilde{V}^b(\gamma, x, \tilde{b}_l) = \widetilde{V}^{af}(\gamma)\}$ and an upper bound $\overline{x} = \max_{\gamma\in\Gamma}\{x : \widetilde{V}^b(\gamma, x, \tilde{b}_l) = \widetilde{V}^{af}(\gamma)\}$, satisfying $\underline{x} < \overline{x}$, for the relative Pareto weights.*[44]

The lower bound of the ergodic set is determined by the lowest achievable relative Pareto weight in the contract. It represents the lowest value that the sovereign accepts in the contract, which keeps it away from immiseration. The upper bound represents the highest relative Pareto weight that makes the sovereign's constraint bind; therefore it is the highest weight that the lender may need to accept. We can further characterise the bounds of the ergodic set with the following lemma, validating their independence on b_l.[45]

Lemma C.1 (Bounds of the Ergodic set). *The bounds of the ergodic set solely depend on the current growth state, θ, thus for the detrended form, solely depend on γ.*

This lemma states that the bounds of the ergodic set are independent of b_l. In other words, the sovereign's participation constraint is solely determined by the realised growth state. From Definition C.1, the bounds of the ergodic set depend on the binding borrower's constraint. Thus, as the value of default is independent of b_l, so does the constraint.

We end this subsection with a result relating to the level of debt in the economy with the present value of the budget constraint. This leads to the following lemma.

[44] The value functions marked with $\widetilde{V}$ are the detrended value functions presented in Appendix B. In Section 6, Figure 3 shows (in gray) the ergodic set of our calibrated economy.

[45] It should be noted that if the sovereign and the Fund are equally patient (i.e. $\eta = 1$), then the upper bound would be determined by $\min_{\gamma\in\Gamma}\{x : \widetilde{V}^l(\gamma, x, \tilde{b}_l) = Z + \tilde{b}_l\}$, which depends on the endogenous b_l.

Lemma C.2 (Debt and Budget Constraint). *At any period t for $\bar{a}_{p,t} = 0$,*

$$a_t(\theta^t) + b_t = \mathbb{E}_t \sum_{j=0}^{\infty} Q_f(\theta^{t+j}, \omega(\theta^{t+j})|\theta^t) \times [c(\theta^{t+j}, a(\theta^{t+j}), b(\theta^{t+j})) - Y(\theta^{t+j}, a(\theta^{t+j}), b(\theta^{t+j}))],$$

$$a_{l,t}(\theta^t) + b_{l,t} = \mathbb{E}_t \sum_{j=0}^{\infty} Q_f(\theta^{n+j}, \omega(\theta^{n+j})|\theta^n) \times [c_f(\theta^{n+j}, a(\theta^{n+j}), b(\theta^{n+j})) + c_p(\theta^{n+j}, a(\theta^{n+j}), b(\theta^{n+j}))]),$$

with $Y(\theta^t, x(\theta^t), b(\theta^t)) \equiv \theta(\theta_t) f(n(\theta^t, x(\theta^t), b(\theta^t)))$ for all t and θ^t.

D Proofs

Proof of Proposition 1

We need to prove that (B.5) has a unique solution which in finite time reaches a 'golden path' with a stationary growth distribution. We show first that existence and uniqueness follows from Theorem 3 in Marcet and Marimon (2019) . They make the following assumptions: A1 a well defined Markov chain process for γ,[46] A2 continuity in $\{c, n\}$ and measurability in γ, A3 non-empty feasible sets, A4 uniform boundedness, A5 convex technologies, A6 concavity for the lender and strict concavity for the sovereign, and a *Strict Interiority Condition* **SIC**. Assumption A1, A2, A5 and A6 are trivially met in the economies described in Sections 2. Since feasible c and n are bounded, payoffs functions are bounded as well. This combined with the fact that the outside options are also bounded ensure that A4 is met. Whether a feasible contract exists (i.e. A3) as the statement of Proposition 1, it amounts to show that for every θ there is a $\underline{b}_l(\theta) > 0$ for which a feasible contract exists, then by monotonicity, it also exists if $b_{l,0}(\theta) \leq \underline{b}_l(\theta)$. However, given that the borrower is more impatient and risk averse than the Fund, and its participation constraint is the value of being in the IMD economy with $b(\theta^t) = 0$, while $Z \leq 0$, there is a $\underline{b}_l(\theta_0) = \tilde{\underline{b}}_l(\theta_0) > 0$ for which a feasible contract exists. Similarly, the same argument shows that assumption 1 ensures that the *Strict Interiority Condition* **SIC** is also met.

It should be noted that Theorem 3 in Marcet and Marimon (2019) is the recursive, saddle-point, representation corresponding, in our framework, to the original contract problem (6). While we have renormalized the co-state variables — to $(x, 1)$ — and detrended allocations, multipliers and states and co-states. Nevertheless, given that $\theta_t = \gamma_t \theta_{t-1}$, we normalize

[46] Being a discrete process A1b, stated in their theorem, is redundant.

$\gamma - 1 = 1$ and given that multipliers are uniformly bounded, the theorem also applies to our normalized and detrended version.

Regarding uniqueness, since the *contraction mapping theorem* also applies and there is a unique value function (B.7) and, furthermore, given the strict concavity assumptions of U and f, the allocation is unique.

Regarding the steady state, as defined in Definition C.1, the lower bound of the ergodic set is determined by the lowest achievable relative Pareto weight in the contract. It represents the lowest value that the sovereign accepts in the contract. The upper bound represents the highest relative Pareto weight that makes the sovereign's constraint bind; therefore it is the highest weight that the lender may need to accept. This means that every time the highest productivity shock hits (i.e. γ_{max}), the sovereign climbs to the top of the ergodic set. In opposition, for a sufficiently long string of lowest productivity shock (i.e. γ_{min}), the sovereign eventually hits the bottom of the set — owing to immiseration with $\eta < 1$. Hence, in the detrended version of the model, the lower bound is defined by $\underline{x} = \min_{\gamma \in \Gamma}\{x : \widetilde{V}^b(\gamma, x, \tilde{b}_l) = \widetilde{V}^{af}(\gamma)\}$, while the upper bound corresponds to $\overline{x} = \max_{\gamma \in \Gamma}\{x : \widetilde{V}^b(\gamma, x, \tilde{b}_l) = \widetilde{V}^{af}(\gamma)\}$.

To show the existence of a unique stationary equilibrium, one shows that the dynamic of the contract satisfies the conditions given by Stokey et al. (1989, Theorem 12.12). Set $\ddot{x}$ as the midpoint of $[\underline{x}, \overline{x}]$ and define the transition function $\mathcal{Q} : [\underline{x}, \overline{x}] \times \mathcal{X}([\underline{x}, \overline{x}]) \rightarrow \mathbb{R}$ as

$$\mathcal{Q}(x, G) = \sum_{\theta'|\theta} \pi(\theta'|\theta)\mathbb{I}\{x' \in G\}$$

We want to show is that $\ddot{x}$ is a mixing point such that for $N \geq 1$ and $\iota > 0$ one has that $\mathcal{Q}(\underline{x}, [x, \overline{x}])^N \geq \iota$ and $\mathcal{Q}(\overline{x}, [\underline{x}, x])^N \geq \iota$. Starting at $\overline{x}$, for a sufficiently long but finite series of γ_{min}, the relative Pareto weight transit to $\underline{x}$. Hence for some $N < \infty$, $\mathcal{Q}(\overline{x}, [\underline{x}, \ddot{x}])^N \geq \pi(\gamma_{min}|\gamma_{min})^N > 0$. Moreover, starting at $\underline{x}$, after drawing $N < \infty$ γ_{max}, the relative Pareto weight transit to $\overline{x}$ meaning that $\mathcal{Q}(\underline{x}, [\ddot{x}, \overline{x}])^N \geq \pi(\gamma_{max}|\gamma_{max})^N > 0$. Setting $\iota = \min\{\pi(\gamma_{min}|\gamma_{min})^N, \pi(\gamma_{max}|\gamma_{max})^N\}$ makes $\ddot{x}$ a mixing point and the above theorem applies. □

Proof of Corollary 1

The proof follows the argument of Thomas and Worrall (1994) and Zhang (1997). The participation constraint of the sovereign — i.e. (4) — ensures that the value of the sovereign in the contract is at most equal to its outside option. Hence, the sovereign is at most indifferent between defaulting or not and therefore never enters *full* default. □

Proof of Proposition 2

We conduct a proof by construction. The combination of the first-order conditions of (13) with respect to c and $a'(\theta', b')$ gives the sovereign's Euler equation for the Fund's securities

$$q_f(\theta', \omega'(\theta')|\theta)u_c(c) - \upsilon_b(\theta') = \beta\pi(\theta'|\theta)u_c(c')\left[(1-\delta+\delta\kappa) + \delta\sum_{\theta''|\theta'} q_f(\theta'', \omega''(\theta'')|\theta')\right], \quad \text{(D.1)}$$

where υ_b is the multiplier attached to the sovereign's endogenous borrowing limit in (15). Conversely, the first-order conditions with respect to c and b' gives the sovereign's Euler equation for the private bonds

$$q_p(\theta, \bar{\omega}')u_c(c) - \sum_{\theta'|\theta} \upsilon_b(\theta') = \beta\sum_{\theta'|\theta} \pi(\theta'|\theta)u_c(c')\big[(1-\delta+\delta\kappa) + \delta q_p(\theta', \bar{\omega}'')\big]. \quad \text{(D.2)}$$

Taking the first-order conditions of (17) with respect to b_l',

$$q_p(\theta, \bar{\omega}') = \frac{\sum_{\theta'|\theta} \pi(\theta'|\theta)(1-\delta+\delta\kappa+\delta q_p(\theta', \bar{\omega}''))}{1+r}, \quad \text{(D.3)}$$

which corresponds to the price without default and without binding constraint of the Fund. Taking the first-order conditions of (20) with respect to c and $a_l'(\theta', b_l')$ gives the Fund's Euler equation

$$q_f(\theta', \omega'(\theta')|\theta) - \varphi_f(\theta') = \tfrac{1}{1+r}\pi(\theta'|\theta)\big[(1-\delta+\delta\kappa) + \delta\textstyle\sum_{\theta''|\theta'} q_f(\theta'', \omega''(\theta'')|\theta')\big], \quad \text{(D.4)}$$

where φ_f is the multipliers attached to the Fund's endogenous limit. Given this, we now have to distinguish three cases:[47]

1. The sovereign's and the Fund's participation constraints are not binding. The lenders' Euler equations read respectively

$$q_f(\theta', \omega'(\theta')|\theta) = \frac{\pi(\theta'|\theta)}{1+r}\left[(1-\delta+\delta\kappa) + \delta\sum_{\theta''|\theta'} q_f(\theta'', \omega''(\theta'')|\theta')\right],$$

$$q_p(\theta, \bar{\omega}') = \sum_{\theta'|\theta} \frac{\pi(\theta'|\theta)}{1+r}[(1-\delta+\delta\kappa) + \delta q_p(\theta', \bar{\omega}'')] = \frac{1-\delta+\delta\kappa}{1+r-\delta},$$

   and the sovereign's Euler equations are

$$q_f(\theta', \omega'(\theta')|\theta) = \beta\pi(\theta'|\theta)\frac{u_c(c')}{u_c(c)}\left[(1-\delta+\delta\kappa) + \delta\sum_{\theta''|\theta'} q_f(\theta'', \omega''(\theta'')|\theta')\right],$$

[47] Recall that, under Assumption 1, it is not possible that the two participation constraints bind at the same time.

$$q_p(\theta, \bar{\omega}') = \beta \sum_{\theta'|\theta} \pi(\theta'|\theta) \frac{u_c(c')}{u_c(c)} [(1 - \delta + \delta\kappa) + \delta q_p(\theta', \bar{\omega}'')].$$

If none of the two constraints is ever binding,

$$\begin{aligned}
\sum_{\theta'|\theta} q_f(\theta', \omega'(\theta')|\theta) &= \beta \sum_{\theta'|\theta} \pi(\theta'|\theta) \frac{u_c(c')}{u_c(c)} \left[(1 - \delta + \delta\kappa) + \delta \sum_{\theta''|\theta'} q_f(\theta'', \omega''(\theta'')|\theta') \right] \\
&= \sum_{\theta'|\theta} \pi(\theta'|\theta) \frac{1}{1+r} \left[(1 - \delta + \delta\kappa) + \delta \sum_{\theta''|\theta'} q_f(\theta'', \omega''(\theta'')|\theta') \right], \\
q_p(\theta, \bar{\omega}') &= \beta \sum_{\theta'|\theta} \pi(\theta'|\theta) \frac{u_c(c')}{u_c(c)} [(1 - \delta + \delta\kappa) + \delta q_p(\theta', \bar{\omega}'')] \\
&= \sum_{\theta'|\theta} \pi(\theta'|\theta) \frac{1}{1+r} [(1 - \delta + \delta\kappa) + \delta q_p(\theta', \bar{\omega}'')],
\end{aligned}$$

It then follows that $Q_p(\theta, \bar{\omega}') = \sum_{\theta'|\theta} Q_f(\theta', \omega'(\theta')|\theta) = \frac{1-\delta+\delta\kappa}{1+r-\delta}$.

2. The sovereign's participation constraint binds and the Fund's participation constraint is not binding.

   The lenders' Euler equations are respectively

$$\begin{aligned}
q_f(\theta', \omega'(\theta')|\theta) &= \frac{\pi(\theta'|\theta)}{1+r} \left[(1 - \delta + \delta\kappa) + \delta \sum_{\theta''|\theta'} q_f(\theta'', \omega''(\theta'')|\theta') \right], \\
q_p(\theta, \bar{\omega}') &= \sum_{\theta'|\theta} \frac{\pi(\theta'|\theta)}{1+r} [(1 - \delta + \delta\kappa) + \delta q_p(\theta', \bar{\omega}'')],
\end{aligned}$$

   and the sovereign's Euler equations are

$$\begin{aligned}
q_f(\theta', \omega'(\theta')|\theta) u_c(c) - \varphi_b(\theta') &= \beta \pi(\theta'|\theta) \frac{u_c(c')}{u_c(c)} \left[(1 - \delta + \delta\kappa) + \delta \sum_{\theta''|\theta'} q_f(\theta'', \omega''(\theta'')|\theta') \right], \\
q_p(\theta, \bar{\omega}') u_c(c) - \sum_{\theta'|\theta} \varphi_b(\theta') &= \beta \sum_{\theta'|\theta} \pi(\theta'|\theta) \frac{u_c(c')}{u_c(c)} [(1 - \delta + \delta\kappa) + \delta q_p(\theta', \bar{\omega}'')].
\end{aligned}$$

   If the Fund's participation constraint never binds,

$$\sum_{\theta'|\theta} q_f(\theta', \omega'(\theta')|\theta) > \beta \sum_{\theta'|\theta} \pi(\theta'|\theta) \frac{u_c(c')}{u_c(c)} \left[(1 - \delta + \delta\kappa) + \delta \sum_{\theta''|\theta'} q_f(\theta'', \omega''(\theta'')|\theta') \right] \quad \text{and}$$

$$\sum_{\theta'|\theta} q_f(\theta', \omega'(\theta')|\theta) = \sum_{\theta''|\theta'} \frac{\pi(\theta'|\theta)}{1+r} \left[(1 - \delta + \delta\kappa) + \delta \sum_{\theta''|\theta'} q_f(\theta'', \omega''(\theta'')|\theta') \right],$$

   Moreover, $Q_p(\theta, \bar{\omega}') = \sum_{\theta'|\theta} Q_f(\theta', \omega'(\theta')|\theta) = \frac{1-\delta+\delta\kappa}{1+r-\delta}$.

3. The sovereign's participation constraint is not binding and the Fund's participation constraint binds.

   The lenders' Euler equations read respectively

$$q_f(\theta', \omega'(\theta')|\theta) - \varphi_f(\theta') = \frac{\pi(\theta'|\theta)}{1+r}\left[(1-\delta+\delta\kappa) + \delta \sum_{\theta''|\theta'} q_f(\theta'', \omega''(\theta'')|\theta')\right],$$

$$q_p(\theta, \bar{\omega}') = \sum_{\theta'|\theta} \frac{\pi(\theta'|\theta)}{1+r}[(1-\delta+\delta\kappa) + \delta q_p(\theta', \bar{\omega}'')],$$

   The sovereign's Euler equations are

$$q_f(\theta', \omega'(\theta')|\theta) = \beta\pi(\theta'|\theta)\frac{u_c(c')}{u_c(c)}\left[(1-\delta+\delta\kappa) + \delta \sum_{\theta''|\theta'} q_f(\theta'', \omega''(\theta'')|\theta')\right],$$

$$q_p(\theta, \bar{\omega}') = \beta \sum_{\theta'|\theta} \pi(\theta'|\theta)\frac{u_c(c(\theta', \omega'))}{u_c(c)}[(1-\delta+\delta\kappa) + \delta q_p(\theta', \bar{\omega}'')].$$

   If the sovereign's participation constraint never binds,

$$\sum_{\theta'|\theta} q_f(\theta', \omega'(\theta')|\theta) = \beta \sum_{\theta'|\theta} \pi(\theta'|\theta)\frac{u_c(c')}{u_c(c)}\left[(1-\delta+\delta\kappa) + \delta \sum_{\theta''|\theta'} q_f(\theta'', \omega''(\theta'')|\theta')\right] \quad \text{and}$$

$$\sum_{\theta'|\theta} q_f(\theta', \omega'(\theta')|\theta) > \sum_{\theta''|\theta'} \frac{\pi(\theta'|\theta)}{1+r}[(1-\delta+\delta\kappa) + \delta \sum_{\theta''|\theta'} q_f(\theta'', \omega''(\theta'')|\theta')].$$

   However, $Q_p(\theta, \bar{\omega}') < \sum_{\theta'|\theta} Q_f(\theta', \omega'(\theta')|\theta)$ cannot be an equilibrium. At this price the private lender is willing to hold an infinite amount of debt in the Fund and provide an infinite amount of assets to the sovereign. To avoid this arbitrage, it must be that $Q_p(\theta, \bar{\omega}') = \sum_{\theta'|\theta} Q_f(\theta', \omega'(\theta')|\theta) > \frac{1-\delta+\delta\kappa}{1+r-\delta}$.

Hence, in all possible states, $Q_p(\theta, \bar{\omega}') = \sum_{\theta'|\theta} Q_f(\theta', \omega'(\theta')|\theta) \geq \frac{1-\delta+\delta\kappa}{1+r-\delta}$. □

Proof of Proposition 3

As shown in Proposition 2, when (24) binds, $Q_p(\theta, \bar{\omega}') = Q_f(\theta, \bar{\omega}') > \frac{1}{1+r}$. At this price, private lenders do not want to lend to the sovereign as the present net discounted return of one unit of debt is negative. In other words, as the private lenders borrow on the international bond market at r, they are unwilling to save at $r_p(\theta, \bar{\omega}') < r$ because they cannot break even at such rates. There is therefore no trade in the private bond market meaning that $b' \geq \delta b$.

For the second part, we conduct a proof by construction. When (24) does not bind, the budget constraint reads

$$c + q_p(\theta, \bar{\omega}')(b' - \delta b) + \sum_{\theta'|\theta} q_f(\theta', \omega'(\theta')|\theta)(a'(\theta') - \delta a) = \theta f(n) + (1-\delta+\delta\kappa)(b+a).$$

Given that $\sum_{\theta'|\theta} q_f(\theta', \omega'(\theta')|\theta)\hat{a}(\theta') = 0$ and Proposition 2, it can be rewritten as

$$c + q_p(\theta, \bar{\omega}')(b' - \delta b) + q_f(\theta, \bar{\omega}')(\bar{a}' - \delta \bar{a}) = \theta f(n) + (1 - \delta + \delta\kappa)(b + a),$$
$$c + q(\theta, \bar{\omega}')(\bar{\omega}' - \delta(b + a)) = \theta f(n) + (1 - \delta + \delta\kappa)(b + a),$$

where $q(\theta, \bar{\omega}') \equiv q_p(\theta, \bar{\omega}') = q_f(\theta, \bar{\omega}')$ by Proposition 2. Having the same price and being equally accessible, private and Fund-provided bonds are prefect substitute, so that the decomposition of $\bar{\omega}'$ between b' and $\bar{a}'$ is indeterminate. □

Proof of Corollary 2

Observe that if $\delta = 0$, the entire part of b matures today. Hence, following Proposition 3, if there is a sudden stop of funding from private lenders, $b' \geq 0$ meaning that the Fund's participation constraint becomes independent of the value of the debt held in the private bond market. We are therefore back to the standard case of Ábrahám et al. (2021).

In opposition, when $\delta > 0$, only a faction $1 - \delta$ of b matures today. Hence, if there is a sudden stop of funding from private lenders, $b' \geq \delta b$ for $b < 0$ following Proposition 3. As a result, the Fund's participation constraint depends on the value of the debt held in the private bond market. Moreover, the larger is $-b$, the tighter is the Fund's constraint. In other words, the more debt is held in the private bond market, the lower is the risk sharing provided by the Fund contract. Hence, $\delta = 0$, is the average maturity that maximizes the risk-sharing in the Fund contract.

Focusing on the steady state, Lemma C.1 states that the bounds of the ergodic set are independent of b. Hence, if (24) does not bind in steady state for $\delta > 0$, δ is irrelevant. □

Proof of Proposition 4

The proof has the following steps: first, following Alvarez and Jermann (2000), we complete the characterization of the Fund contract asset structure; second we map the state in the Fund problem with the state in that the decentralized economy; third we map the initial conditions and participation constraints, and fourth, we complete the mapping between policies and value functions.

First, Fund's assets have prices given br (25); i.e.

$$q_f(\theta', x', b'|\theta) = \frac{\pi(\theta'|\theta)}{1+r}\left[(1 - \delta + \delta\kappa) + \delta \sum_{\theta''|\theta'} q_f(\theta'', x'', b''|\theta')\right] \max\left\{\frac{u_c(c(\theta', x', b'))}{u_c(c(\theta, x, b))}\eta, 1\right\}.$$

As shown in Lemma C.2, iterating over the budget constraint of the sovereign gives

$$a(\theta^t) + b(\theta^t) =$$

$$\mathbb{E}_t \sum_{j=0}^{\infty} Q(\theta^{t+j}, x(\theta^{t+j}), b(\theta^{t+j})|\theta^t)[c(\theta^{t+j}, x(\theta^{t+j}), b(\theta^{t+j})) - Y(\theta^{t+j}, x(\theta^{t+j}), b(\theta^{t+j}))], \quad \text{(D.5)}$$

where, $Y(\theta^t, x(\theta^t), b(\theta^t)) = \theta(\theta_t) f(n(\theta^t, x(\theta^t), b(\theta^t)))$ for all t and θ^t. Similarly, for $\bar{a}_p(\theta^t) = 0$, iterating over the consolidated budget constraint of the two lenders and denoting $c_l \equiv c_f + c_p$ leads to

$$\begin{aligned} a_l(\theta^t) + b_l(\theta^t) &= \mathbb{E}_t \sum_{j=0}^{\infty} Q(\theta^{t+j}, x(\theta^{t+j}), b(\theta^{t+j})|\theta^t) c_l(\theta^{t+j}, x(\theta^{t+j}), b(\theta^{t+j})) \quad \text{(D.6)} \\ &= \mathbb{E}_t \sum_{j=0}^{\infty} Q(\theta^{t+j}, x(\theta^{t+j}), b(\theta^{t+j})|\theta^t)[Y(\theta^{t+j}, x(\theta^{t+j}), b(\theta^{t+j})) \\ &\quad - c(\theta^{t+j}, x(\theta^{t+j}), b(\theta^{t+j}))] \\ &= -a(\theta^t) - b(\theta^t). \end{aligned}$$

The market clearing conditions in the Fund and the private bond market implies that $a_l(\theta^t) + a(\theta^t) = 0$ and $b(\theta^t) + b_l(\theta^t) = 0$, respectively, for all t and θ^t.

Second, up to this point, θ is a short-hand for (θ, x, b) and we need to map (s, x) into (θ, ω, b) or, equivalently, into (θ, a, b) or (θ, a_l, b_l), depending on whether we are referring to the sovereign or the Fund perspective. This map is given by the identification of the consumption policies and the Fund's consumption first-order condition:

$$u_c(c(\theta, a, b)) = u_c(c(\theta, x, b_l)) = \frac{1 + \nu_l(\theta, x, b_l)}{1 + \nu_b(\theta, x, b_l)} \frac{1}{x}.$$

This equality has three implications: *i*) since $u_c(c(\theta, a, b)) = \varkappa(\theta, a, b)$ the above equality also defines the Lagrange multiplier of the sovereign's budget constraint (14); *ii*) defines $c(\theta, a, b) = c(\theta, x, b_l)$, *i*), and *iii*) since the right hand side of the above equality is, by (8), equal to η/x' the law of motion of the co-state variable x maps into the borrower's Euler equation.

Third, we establish the correspondence between the initial conditions and participation constraints, between the Fund problem and the RCE. For the former, given (D.5) and (D.6) evaluated at $t = 0$, one can determine $\bar{a}'$ and b' using Definition 2, the budget constraint

$$c(\theta_0, a_0, b_0) + q_f(\theta_0, \omega_1)(\bar{a}' - \delta a_0) + \sum_{\theta_1|\theta_0} q_f(\theta_1, \omega_1(\theta_1)|\theta_0)\hat{a}'(\theta_1) + q_p(\theta_0, \bar{\omega}_1)(b' - \delta b_0)$$
$$\leq \theta_0 f(n) + (1 - \delta + \delta\kappa)(a_0 + b_0).$$

and the fact that $\sum_{\theta_1|\theta_0} q_f(\theta_1, \omega_1(\theta_1)|\theta_0)\hat{a}'(\theta_1) = 0$. Once, $\bar{a}'$ and b' are determined, one can find the holdings of Arrow securities $\hat{a}'(\theta', \theta_0, a_0, b_0)$ for all $\theta' \in \Theta$. We can then retrieve the entire portfolio recursively for $t > 0$.

Now, given Lemma C.1, we can define the relative Pareto weight for which the sovereign's and the Fund's participation constraints bind in (θ, b) as $\underline{\underline{x}}(\theta)$ and $\overline{\overline{x}}(\theta, b)$, respectively. Then set the endogenous borrowing limits such that

$$\mathcal{A}_b(\theta) = a(\theta, \underline{\underline{x}}(\theta), b) + b(\theta, \underline{\underline{x}}(\theta), b),$$
$$\mathcal{A}_f(\theta, b) = a_l(\theta, \overline{\overline{x}}(\theta, b), b) + b_l(\theta, \overline{\overline{x}}(\theta, b), b).$$

This definition implies that $a'(\theta', \theta, a, b) + b' \geq \mathcal{A}_b(\theta')$ and $a_l'(\theta', \theta, a, b) + b_l' \geq \mathcal{A}_f(\theta', b')$. Hence, the constructed asset holdings satisfy the competitive equilibrium constraints for both the lenders and the sovereign.

Therefore, if we identify $W^b(\theta, a, b) = V^b(\theta, s, b_l)$ we have shown that the Fund allocation of consumption, leisure and asset holdings is a solution to the sovereign's problem (13).

Fourth, we complete the mapping of policies and value functions. For the lenders, consumption is optimal if the asset portfolio is optimally determined. For this observe that

$$\begin{aligned}
q_f(\theta', \omega'(\theta')|\theta) &= \frac{1}{1+r}\pi(\theta'|\theta)\frac{u_c(c')}{u_c(c)}\eta\left[(1-\delta+\delta\kappa) + \delta\sum_{\theta''|\theta'} q_f(\theta'', \omega''(\theta'')|\theta')\right] \\
&\geq \frac{1}{1+r}\pi(\theta'|\theta)\left[(1-\delta+\delta\kappa) + \delta\sum_{\theta''|\theta'} q_f(\theta'', \omega''(\theta'')|\theta')\right] \\
&\quad \text{if } a'(\theta', \theta, a, b) + b' > \mathcal{A}_b(\theta'), \\
q_f(\theta', \omega'(\theta')|\theta) &= \frac{1}{1+r}\pi(\theta'|\theta)\left[(1-\delta+\delta\kappa) + \delta\sum_{\theta''|\theta'} q_f(\theta'', \omega''(\theta'')|\theta')\right] \\
&\geq \frac{1}{1+r}\pi(\theta'|\theta)\frac{u_c(c')}{u_c(c)}\eta\left[(1-\delta+\delta\kappa) + \delta\sum_{\theta''|\theta'} q_f(\theta'', \omega''(\theta'')|\theta')\right] \\
&\quad \text{if } a_l'(\theta', \theta, a, b) + b_l' > \mathcal{A}_f(\theta', b')
\end{aligned}$$

It we then identify $W^b(\theta, a, b) = V^b(\theta, x, b)$ and $W^p(\theta, a, b) + W^f(s, a, b) = V^l(s, x, b)$, the corresponding portfolios solve the private lenders' problem (17) and the decentralized Fund problem (20).

In sum, we obtain a map between (x, b) and $\omega = a + b$ for a given θ. More precisely, $B(\theta, x, b) = B(\theta, a, b)$, $c(\theta, a, b) = c(\theta, x, b)$, $c_p(\theta, a, b) = \tau_p(\theta, x, b)$, $c_f(\theta, a, b) = \tau_f(\theta, x, b)$, $c_p(\theta, a, b) + c_f(\theta, a, b) = \tau(\theta, x, b)$ and $n(\theta, a, b) = n(\theta, x, b)$. Moreover the endogenous limits of the sovereign and the lenders bind uniquely and exclusively when the participation constraints of the sovereign and the lenders bind, respectively. □

Proof of Proposition 5

As we said, the starting point is the first-order condition of the sovereign's problem (13): $u_c(c(\theta,a,b)) = \varkappa(\theta,a,b)$, where $\varkappa(\theta,a,b)$ is the Lagrange multiplier of the budget constraint (14). From the sovereign's and Fund's Euler equations we obtain the following intertemporal relation between these multipliers:

$$\varkappa(\theta,a,b) = \eta\frac{1+\mathring{\nu}_b(\theta',a',b')}{1+\mathring{\nu}_l(\theta',a',b')}\varkappa'(\theta',a',b'), \tag{D.7}$$

where $\mathring{\nu}_b(\theta',a',b')$ and $\mathring{\nu}_l(\theta',a',b')$ are normalized Lagrange multipliers of the endogenous limit constraints (15) and (22).

The proof has two steps: first, we derive (D.7) and map it into the co-state x and, second, we map policies and value functions from RCE to the Fund's problem (A.1).

First, note that (D.1) and (D.4) can be read as the Euler equations of (13) and (20), where $\upsilon_b(\theta',a',b')$ and $\varphi_f(\theta',a_l',b_l')$ are the Lagrange multipliers of the endogenous constraints (15) and (22). Let $A(\delta) \equiv \left[(1-\delta+\delta\kappa)+\delta\sum_{\theta''|\theta'} q_f(\theta'',\omega''(\theta'')|\theta')\right]$, and define $\mathring{\nu}_b(\theta',a',b')$ and $\mathring{\nu}_f(\theta',a',b')$ by

$$\upsilon_b(\theta',a_l',b_l') = \frac{\pi(\theta'|\theta)\beta A(\delta)}{\varkappa'(\theta',a',b')}\mathring{\nu}_b(\theta',a',b') \text{ and } \varphi_f(\theta',a_l',b_l') = \frac{\pi(\theta'|\theta)A(\delta)}{(1+r)\varkappa'(\theta',a',b')}\mathring{\nu}_f(\theta',a',b').$$

Then, substituting the first-order condition $\varkappa(\theta,a,b) = u_c(c(\theta,a,b))$ into (D.1) and (D.4), these Euler equations read

$$q_f(\theta',\omega'(\theta')|\theta)\varkappa(\theta,a,b) = \pi(\theta'|\theta)\beta A(\delta)(1+\mathring{\nu}_b(\theta',a',b'))\varkappa'(\theta',a',b') \tag{D.8}$$

$$q_f(\theta',\omega'(\theta')|\theta) = \pi(\theta'|\theta)\frac{A(\delta)}{1+r}(1+\mathring{\nu}_l(\theta',a',b')), \tag{D.9}$$

where we have used the fact that by Proposition 2 $\varphi_l(\theta',a_l',b_l') = \varphi_f(\theta',a_l',b_l')$. Dividing (D.8) by (D.9) we obtain (D.7). Therefore, we can define

$$x = \frac{1+\mathring{\nu}_l(\theta,a,b))}{1+\mathring{\nu}_b(\theta,a,b)}\frac{1}{\varkappa(\theta,a,b)},$$

which also defines the map from state (θ,a,b) to state (s,x,b), as well as the policy identities: $c(s,x,b) = c(\theta,a,b)$, $n(s,x,b) = (\theta,a,b)$ and transfers by $\tau = \theta f(n) - c$.

Second, the split of transfers $\tau_p(s,x,b)$ and $\tau_f(s,x,b)$ is given by (12) and (11) and since the limited enforcement values, $V^{af}(\theta)$ and $\theta^- Z + b_l$ are already specified in the decentralized economy, it only remains to identify the value functions: $V^b(\theta,x,b_l) = W^b(\theta,a,b)$,

$V^l(\theta, x, b_l) = W^p(\theta, a_l, b_l) + W^f(\theta, a_l, b_l)$, to finally define $FV(s, x, b_l) = xV^b(\theta, x, b_l) + V^l(\theta, x, b_l)$. It follows the allocation and assets of the RCE uniquely maps into the solution of the Fund problem (A.1). □

Proof of Corollary 3

By definition of $\mathcal{A}_b(\theta')$ in (16), the Fund implements the tight quantity restriction defined in Aguiar and Amador (2020, Proposition 9). Hence, the Fund rules out the borrowing equilibrium and implements the saving equilibrium as defined by the two authors.

The fact that the equilibrium displays no excess lending is a direct consequence of Proposition 3 and the occurrence of the negative spread coupled with the MIP of the Fund defined in Definition 2.

Form the perspective of the centralized Fund contract, existence and uniqueness is guaranteed by Assumption 1 — among other requirements. As shown in the proof of Proposition 1, this assumption ensures the existence of the saddle point function equation. Marcet and Marimon (2019, Theorem 3) additionally show that such functional equation is a contraction mapping which ensures existence and uniqueness of the Fund contracting outcome.

Form the perspective of the decentralized Fund contract, since, by Proposition 1, the solution is unique which RCE allocation (when Assumption 2 is satisfied) must also be unique. Note that in our environment, the Fund is a big player meaning that it is not a price taker unlike the continuum of private lenders. In other words, the Fund, as capacity announcer, can play the role of a coordination device for private lenders.

Finally, the Fund's MIP ensures that there is a unique split of $\bar{\omega}$ between the Fund and the private lenders. □

Proof of Proposition 6

Assume that there exists an $-\underline{\underline{a}}(\theta) = \ddot{\bar{a}}' < 0$ where, according to Definition 3, in a given state, say $\ddot{\theta}'$, for a total capacity announcement $\bar{\omega}' + \hat{a}'(\theta', d_p')$,

$$V^{ap}(\ddot{\theta}, \ddot{\bar{a}}' + \hat{a}'(\ddot{\theta}, 1)) = W^b(\ddot{\theta}, \ddot{\bar{a}}' + \hat{a}'(\ddot{\theta}, 0), \bar{\omega}' - \ddot{\bar{a}}', 0),$$

while for the remaining $\theta' \in \Theta \setminus \ddot{\theta}'$ for which $\pi(\theta'|\theta) > 0$,

$$V^{ap}(\theta', \ddot{\bar{a}}' + \hat{a}'(\theta', 1)) \leq W^b(\theta', \ddot{\bar{a}}' + \hat{a}'(\theta', 0), \bar{\omega}' - \ddot{\bar{a}}', 0).$$

As a result, if the Fund's MIP satisfies Definition 3, there is no *partial* default. Also, by monotonicity if $\bar{a}' > \ddot{\bar{a}}'$, then it is optimal for the sovereign to enter *partial* default in at least $\ddot{\theta}'$.

For the second part of the proposition, we want to additionally show that, for all θ' for which $\pi(\theta'|\theta) > 0$ and for which the Fund's participation constraint does not bind,

$$V^{ap}(\theta', \ddot{a}' + \hat{a}'(\theta', 1)) = W^b(\theta', \ddot{a}' + \hat{a}'(\theta', 0), \bar{\omega}' - \ddot{a}', 0). \tag{D.10}$$

If this is true, then by a simple monotonicity argument, when $\bar{a}' > \ddot{a}'$,

$$V^{ap}(\theta', \bar{a}' + \hat{a}'(\theta', 1)) > W^b(\theta', \bar{a}' + \hat{a}'(\theta', 0), \bar{\omega}' - \bar{a}', 0).$$

Conversely, when $\bar{a}' \leq \ddot{a}'$,

$$V^{ap}(\theta', \bar{a}' + \hat{a}'(\theta', 1)) < W^b(\theta', \bar{a}' + \hat{a}'(\theta', 0), \bar{\omega}' - \bar{a}', 0).$$

Thus, the proof of the second part of the proposition relies on whether (D.10) holds with equality for all θ' for which $\pi(\theta'|\theta) > 0$ and for which the Fund's participation constraint does not bind.

In appendix A, we show that the optimal consumption and leisure policies satisfy the first-order conditions of problem (A.1), $u_c(c) = \frac{1+\nu_l}{1+\nu_b}\frac{1}{x} = \frac{\eta}{x'}$, and $\theta f_n(n) = \frac{h_n(1-n)}{u_c(c)}$. As one can see, optimal consumption solely depends on x', while optimal labor depends on both x' and θ — and is therefore subject to change in *partial* default owing to the output penalty $\theta^d \leq \theta$ for the same x'. Given this, we consider two cases: when labor is exogenous (i.e. $U(c, n) = U(c)$) and when it is endogenous.

We start with the case of exogenous labor. One way to make the borrower indifferent between repayment and *partial* default is to construct a path of consumption, say $\{c_t(\theta^t)\}_{t=0}^{\infty}$, which remains unchanged irrespective of the repayment decision in any state θ. Given the aforementioned first-order condition and the fact that labor is exogenous, the sequence $\{c_t(\theta^t)\}_{t=0}^{\infty}$ directly relates to a unique sequence of relative Pareto weight, say $\{x_t(\theta^{t-1})\}_{t=0}^{\infty}$. From Proposition 4, we know that there is a direct correspondence between $x_{t+1}(\theta^t)$ and $\omega_t(\theta^t)$ for all t and θ^t. Moreover, as long as the the Fund's participation constraint does not bind, $x_{t+1}(\theta^t)$ for any t and θ^t is independent of the level of private debt b_t as shown in Lemma C.1.

Assume that the Fund's participation constraint never binds. From $\{x_t(\theta^{t-1})\}_{t=0}^{\infty}$, we can find the underlying assets under *partial* default, say $\{a_t^{ap}(\theta^t)\}_{t=0}^{\infty}$. That is, for a specific $x_{t+1}(\theta^t)$, we have an underlying $a_t^{ap}(\theta^t) = \bar{a}_t^{ap} + \hat{a}(\theta^t, 1)$. Similarly, we can find the underlying assets under repayment, say $\{\omega_t(\theta^t)\}_{t=0}^{\infty}$, where for a specific $x_{t+1}(\theta^t)$ we have a corresponding $\omega_t(\theta^t) = \bar{a}_t + b_t + \hat{a}(\theta^t, 0)$. As the Fund's constraint does not bind, $x_{t+1}(\theta^t)$ is independent on the split of $\bar{\omega}_t$ between b_t and $\bar{a}_t$. Now observe that in a *partial* default at t, the sovereign reneges b_t and repays $\bar{a}_t$. That is, by entering *partial* default, for $b_t < 0$, the sovereign would end up with an indebtedness of $\bar{a}_t$ instead of $\bar{a}_t + b_t$ and an insurance of $\hat{a}(\theta^t, 1)$ instead of $\hat{a}(\theta^t, 0)$. Thus, if one sets $\bar{a}_t > \bar{a}_t^{ap}$ for all t, then entering *partial* default, the sovereign has a lower liability towards the Fund than $-\bar{a}_t^{ap}$ which corresponds to a larger relative Pareto weight than $x_{t+1}(\theta^t)$ and therefore a larger consumption than under repayment. In opposition, if one sets $\bar{a}_t = \bar{a}_t^{ap}$ and $b_t = \ddot{b}_t = \bar{\omega}_t - \bar{a}_t^{ap}$ for all t, $x_{t+1}(\theta^t)$ — and therefore $c_t(\theta^t)$ — remains the same irrespective of the repayment decision.[48] Most importantly, this holds true for any state θ as $\bar{a}_t$ and b_t are not state contingent.[49]

There is therefore a perfect indifference in entering into *partial* default for any state, as repayment and *partial* default are related to the same sequence of relative Pareto weights for any t and θ^t. Note that if the Fund's participation constraint was binding, the relative Pareto weight $x_{t+1}(\theta^t)$ would depend on the level of private debt b_t under repayment, while under *partial* default it does not. Hence depending on the exact values of θ^d and λ, to obtain the same sequence of relative Pareto weight under *partial* default and repayment, $-\bar{a}_t$ might need to be larger, equal or lower than $-\bar{a}_t^{ap}$. This could potentially break the indifference we have just shown. That is why we only focus on the cases in which the Fund's participation constraint does not bind.

We turn to the case of endogenous labor analyzed in the main text. Given the default penalty upon *partial* default, the same sequence of relative Pareto weight, $\{x_t(\theta^{t-1})\}_{t=0}^{\infty}$, would lead to the same consumption sequence but a different labor sequence between repayment and *partial* default. In other words, the sovereign would not anymore be indifferent between repayment and *partial* default. To correct this, we have to generate two sequences of relative Pareto weight — one for repayment, say $\{x_t^r(\theta^{t-1})\}_{t=0}^{\infty}$, and one for *partial* default, say $\{x_t^d(\theta^{t-1})\}_{t=0}^{\infty}$ — such that the sequence of instantaneous utility in repayment, say $\{U(c_t^r(\theta^t), n_t^r(\theta^t))\}_{t=0}^{\infty}$, exactly equates the sequence of instantaneous utility in *partial* default, say $\{U(c_t^d(\theta^t), n_t^d(\theta^t))\}_{t=0}^{\infty}$. As in the case of exogenous labor, we then use the correspondence between $x_{t+1}(\theta^t)$ and $\omega_t(\theta^t)$ given in Proposition 4 and apply the same reasoning as before with the only difference that we need to consider the two sequences of relative Pareto weight instead of one.

As a result, (D.10) holds with equality for all θ' for which $\pi(\theta'|\theta) > 0$ and for which

[48] Note that $\bar{a}_t^{ap} > \bar{\omega}_t$ as we equalize the consumption in repayment and *partial* default and $b_t < 0$.

[49] Particularly, $\bar{\omega}' = \frac{\sum_{\theta'|\theta} q_f(\theta', a'(\theta', b', 0), b'|\theta)(a'(\theta', b', 0) + b')}{q_f(\theta, \bar{a}', b')}$ and $\bar{a}^{ap\prime} = \frac{\sum_{\theta'|\theta} q_f(\theta', a'(\theta', b', 1), 0|\theta) a'(\theta', 0, 1)}{q_f(\theta, \bar{a}', 0)}$.

the Fund's participation constraint does not bind meaning that the *partial* default decision — being optimal whenever $0 > \bar{a}' > \ddot{\bar{a}}'$ — is not state contingent when the DSA does not bind. $\square$

Proof of Corollary 4

Given Proposition 6, it holds that for all θ', and $\bar{a}'$ and b' such that $\bar{a}' \leq -\underline{\underline{a}}(\theta)$, $D_p(\theta', a', b') = 0$, and under Corollary 1, $D_f(\theta', a', b') = 0$. Moreover, when the Fund's participation constraint does not bind, for all θ' and for all $\bar{a}'$ and $b' < 0$ such that $\bar{a}' > -\underline{\underline{a}}(\theta)$, $D_p(\theta', a', b') = 1$ and $D_f(\theta', a', b') = 0$, which implies from (28) that for all θ and for all $\tilde{b}' < \bar{\omega}' + \underline{\underline{a}}(\theta)$, $q_p(\theta, \bar{a}', \tilde{b}') = 0$. $\square$

Proof of Proposition C.1

The first order condition on consumption reads $u_c(c) = \frac{1+\nu_l}{1+\nu_b}\frac{1}{x}$. The law of motion of the relative Pareto weight is given by $x' = \frac{1+\nu_b}{1+\nu_l}\eta x$. Combining those two equations one obtains

$$x' = \frac{1+\nu_b(\theta, x, b)}{1+\nu_l(\theta, x, b)}\eta x = \frac{1}{u_c(c(\theta', x', b'))}\frac{1+\nu_l(\theta', x', b')}{1+\nu_b(\theta', x', b')}. \tag{D.11}$$

Moreover, observe that using the above first-order condition

$$\frac{1+\nu_b(\theta, x, b)}{1+\nu_l(\theta, x, b)}\eta x = \eta\left[\frac{1}{u_c(c(\theta, x, b))}\frac{1+\nu_b(\theta, x, b)}{1+\nu_l(\theta, x, b)}\frac{1+\nu_l(\theta, x, b)}{1+\nu_b(\theta, x, b)}\right] = \eta\frac{1}{u_c(c(\theta, x, b))}.$$

Hence, one can rewrite (D.11) as

$$\eta\frac{1}{u_c(c(\theta, x, b))} = \frac{1}{u_c(c(\theta', x', b'))}\frac{1+\nu_l(\theta', x', b')}{1+\nu_b(\theta', x', b')}.$$

Taking expectations on both sides with respect to θ' leads to

$$\eta\frac{1}{u_c(c(\theta, x, b))} = \mathbb{E}\left[\frac{1}{u_c(c(\theta', x', b'))}\frac{1+\nu_l(\theta', x', b')}{1+\nu_b(\theta', x', b')}\middle|\theta\right].$$

This equation is the inverse Euler equation. It gives the dynamic of consumption over time and therefore the extent of insurance. If none of the constraint ever binds and $\eta = 1$, then the contract achieves full insurance. However, whenever one of those two point is no true, consumption is not constant across states. Insurance is thus only partial in our environment. $\square$

Proof of Lemma C.1

Recall that, in the detrended version of the model, the lower bound is defined by $\underline{x} = \min_{\gamma\in\Gamma}\{x : \widetilde{V}^b(\gamma, x, \tilde{b}) = \widetilde{V}^{af}(\gamma)\}$, while the upper bound corresponds to $\overline{x} = \max_{\gamma\in\Gamma}\{x : \widetilde{V}^b(\gamma, x, \tilde{b}) = \widetilde{V}^{af}(\gamma)\}$.

The key insight is to see that the sovereign's outside option is independent of the level of indebtedness, while the sovereign's value increases with the relative Pareto weight by definition. Assume now by contradiction that the lower bound $\underline{x}(\gamma, \tilde{b})$ is a function of γ and the level of debt $\tilde{b}$. That is for some $\ddot{b} \neq \tilde{b}$, $\underline{x}(\gamma, \tilde{b}) \neq \underline{x}(\gamma, \ddot{b})$. This implies that either $\widetilde{V}^b(\gamma, \underline{x}(\gamma, \tilde{b}), \tilde{b}) > \widetilde{V}^b(\gamma, \underline{x}(\gamma, \ddot{b}), \ddot{b})$ or $\widetilde{V}^b(\gamma, \underline{x}(\gamma, \tilde{b}), \tilde{b}) < \widetilde{V}^b(\gamma, \underline{x}(\gamma, \ddot{b}), \ddot{b})$ depending on which of the two relative Pareto weight is the largest. The former case leads to $\widetilde{V}^b(\gamma, \underline{x}(\gamma, \tilde{b}), \tilde{b}) > \widetilde{V}^{af}(\gamma)$, while the latter case leads to $\widetilde{V}^b(\gamma, \underline{x}(\gamma, \tilde{b}), \tilde{b}) < \widetilde{V}^{af}(\gamma)$. Both cases contradict the fact that $\underline{x}(\gamma, \tilde{b})$ is the relative Pareto weight for which the sovereign's constraint binds. It must therefore be that for all $\ddot{b} \neq \tilde{b}$, $\underline{x}(\gamma) = \underline{x}(\gamma, \tilde{b}) = \underline{x}(\gamma, \ddot{b})$. The same reasoning applies to the upper bound. □

Proof of Lemma C.2

Under Proposition 2, define

$$q(\theta^t, \bar{\omega}(\theta^t)) \equiv \sum_{\theta^{t+1}|\theta^t} q_f(\theta^{t+1}, \omega(\theta^{t+1})|\theta^t) = q_p(\theta^t, \bar{\omega}(\theta^t)),$$
$$Q(\theta^t, \bar{\omega}(\theta^t)) \equiv \sum_{\theta^{t+1}|\theta t} Q_f(\theta^{t+1}, \omega(\theta^{t+1})|\theta^t) = Q_p(\theta^t, \bar{\omega}(\theta^t)),$$

for all t and θ^t. Furthermore, the transversality condition of the borrower is:[50]

$$\lim_{j \to \infty} \mathbb{E}_t Q(\theta^{t+j}, \omega(\theta^{t+j})|\theta^t)[a(\theta^{t+j}) + b(\theta^{t+j})] = 0,$$

where

$$Q(\theta^{t+j}, \omega(\theta^{t+j})|\theta^t) = Q(\theta^{t+j}, \omega(\theta^{t+j})|\theta^{t+j-1}) \cdots Q(\theta^{t+1}, \omega(\theta^{t+1})|\theta^t).$$

Using the borrower's budget constraint and the price relationship, one gets

$$\begin{aligned}
&(a(\theta^t) + b(\theta^t))(1 - \delta + \delta\kappa + \delta q(\theta^t, \bar{\omega}(\theta^{t+1}))) = \\
&\qquad c(\theta^t, a(\theta^t), b(\theta^t)) + q(\theta^t, \bar{\omega}(\theta^{t+1}))a(\theta^{t+1}) + q(\theta^t, \bar{\omega}(\theta^{t+1}))b(\theta^{t+1}) - Y(\theta^t, a(\theta^t), b(\theta^t)),
\end{aligned}$$

where, $Y(\theta^t, a(\theta^t), b(\theta^t)) = \theta(\theta_t) f(n(\theta^t, a(\theta^t), b(\theta^t)))$ for all t and θ^t. Iterating forward the budget constraint and using the transversality condition as well as the equilibrium price relationship, one obtains

[50]The differentiability and strict concavity and convexity assumptions of the functional forms guarantee the local uniqueness of the policy and value functions. This in turn implies that the transversality conditions are satisfied.

$$a(\theta^t) + b(\theta^t) = \mathbb{E}_t \sum_{j=0}^{\infty} Q(\theta^{t+j}, \omega(\theta^{t+j})|\theta^t)[c(\theta^{t+j}, a(\theta^{t+j}), b(\theta^{t+j})) - Y(\theta^{t+j}, a(\theta^{t+j}), b(\theta^{t+j}))].$$

Similarly, the transversality condition of the lenders is:

$$\lim_{t\to\infty} \mathbb{E}_t Q(\theta^{t+1}, \omega(\theta^{t+1})|\theta^t)[a_l(\theta^{t+1}) + b_l(\theta^{t+1})] = 0.$$

Using the consolidated budget constraint of both lenders, one gets

$$(a_l(\theta^t) + b_l(\theta^t))(1 - \delta + \delta\kappa + \delta q(\theta^t, \bar{\omega}(\theta^{t+1}))) = c_f(\theta^t, a(\theta^t), b(\theta^t)) + c_p(\theta^t, a(\theta^t), b(\theta^t)) + q(\theta^t, \bar{\omega}(\theta^{t+1}))a_l(\theta^{t+1}) + q(\theta^t, \bar{\omega}(\theta^{t+1}))b_l(\theta^{t+1}).$$

Note that we only consider the case in which $\bar{a}_p(\theta^t) = 0$. Iterating forward the budget constraint and using the transversality condition as well as the equilibrium price relationship, one obtains

$$\begin{aligned} a_l(\theta^t) + b_l(\theta^t) &= \mathbb{E}_t \sum_{j=0}^{\infty} Q(\theta^{t+j}, \omega(\theta^{t+j})|\theta^t) c_l(\theta^{t+j}, a(\theta^{t+j}), b(\theta^{t+j})) \\ &= \mathbb{E}_t \sum_{j=0}^{\infty} Q(\theta^{t+j}, \omega(\theta^{t+j})|\theta^t)[Y(\theta^{t+j}, a(\theta^{t+j}), b(\theta^{t+j})) - c(\theta^{t+j}, a(\theta^{t+j}), b(\theta^{t+j}))] \\ &= a(\theta^t) + b(\theta^t). \end{aligned}$$

The market clearing conditions in the Fund and the private bond market implies that $a_l(\theta^t) + a(\theta^t) = 0$ and $b(\theta^t) + b_l(\theta^t) = 0$, respectively, for all t and θ^t. □

E Additional Details of the Calibration

E.1 Data Sources and Measurement

We calibrate the model for Italy. The main data sources and definitions of data variables are listed in Table E.1. The data frequency is quarterly, and the time periods are from 1992Q1 to 2019Q4, avoiding the interruption caused by COVID-19. Whenever the data souces contain the seasonally adjusted series for the relevant data variables, we use the them directly; otherwise, we seasonally adjust the data series using X11 algorithm with R package `seasonal`. For debt service and average maturity, we use annual series since quarterly ones are unavailable meanwhile we only need the sample avearge for our calibration.

To map the data to the model, we construct model consistent data measures as below.

Table E.1: Data Sources and Definitions

Series	Sources	Unit
Output	ECB[a]	1 million 2010 constant Euro
Total working hours	ECB[b]	1 thousand hours
Employment	Eurostat[c]	1000 persons
Government debt	Eurostat[d]	end-of-quarter percentage
Debt service	AMECO[e]	end-of-year percentage of GDP, annual
Fiscal surplus	Eurostat, Bank of Italy[f]	million Euro
Long-term bond yields	Eurostat[g]	percentage, nominal
Debt maturity	OECD, EuroStat, ESM[h]	years, annual
Labor share	AMECO[i]	percentage, annual

[a] Real GDP, chain linked volume; data in 1991Q1–2014Q2 under ESA95, and data in 2014Q3–2019Q4 under ESA10, with the latter series adjusted to match the former in the overlapping periods 1995Q1–2014Q2.

[b] Hours for total employment; same adjustment to data under ESA95 and ESA10 as for output.

[c] Total employment (Eurostat label `lfsi_emp_q_h`).

[d] General government consolidated gross debt (Eurostat label `gov_10q_ggdebt`); quarterly series available for 2000Q1 onwards, and for 1992Q1-1999Q4, interpolate annual series instead; measured as end-of-quarter debt stock to total GDP of previous 4 quarters.

[e] AMECO (label `UYIGE`) for 1995–2015; European Commission *General Government Data* (GDD 2002) for 1992–1995.

[f] Eurostat (net lending, label `gov_10q_ggnfa`) 1999Q1–2019Q4; Bank of Italy (financing of the gross borrowing requirement, including privatization receipts) 1992Q1–1998Q4.

[g] EMU convergence criterion bond yields (label `irt_lt_mcby_q`).

[h] See text below; ESM data are obtained from private correspondance.

[i] Compensation of employees (UWCD) plus gross operating surplus (UOGD) minus gross operating surplus adjusted for imputed compensation of self-employed (UQGD), then divided by nominal GDP (UVGD).

Labor input For the aggregate labor input n_t, we use two series, the aggregate working hours H_t and the total employment E_t. We calculate the normalized labor input as $n_t = H_t/(E_t \times 5200)$, assuming 100 hours of allocatable time per worker per week. However, for second order data moment computations, we use H_t directly, since the per worker annual working hours do not show a significant cyclical pattern and both the level and the trend do not affect the computation of the moments.

Fiscal position and private consumption We hold the premise of fitting the *observed* fiscal behavior of Italy, so that we use directly the *data measures* of primary surplus to calibrate the model, and correspondingly, define the model consistent measure of consumption as the difference between output and primary surplus, since in the model, primary surplus ps is equal to output y minus consumption c. We have raw data on quarterly fiscal *surplus* instead of *primary surplus*. To arrive the latter from the former, we add back interest payment of the government to fiscal surplus. To be more precise, we first calculate fiscal suplus to GDP ratio (nominal quarterly GDP obtained from CEIC for Italy). Second, we obtain quarterly interest payment to GDP ratio from Eurostat (label `gov_10q_ggnfa`) for 1999Q1 onwards, and use the end-of-year annual value (obtained from AMECO and European Commission *General Government Data*) for each quarter in the year as a proxy for 1992Q1–1998Q4. Third, we add fiscal surplus to GDP and interest payment to GDP to arrive at primary surplus to GDP, and conduct seasonal adjustment to the series. And finally, we obtain the level of quarterly (*real*) primary surplus by multiplying the seasonally adjusted primary surplus to GDP ratio to (*real*) output in the same quarter.

Government debt, spread, and maturity Following Bocola et al. (2019) and Ábrahám et al. (2021), we calibrate the model to match the total public debt of Italy.

For the nominal risk free rate, we use the annualized short-term (3M) interest rates in the Euro money market (obtaied from EuroStat with label `irt_st_q`) for 1999Q1–2019Q4, and the annulized short-term (3M) bond return of Germany (obtained from EuroStat with label `irt_h_mr3_q`) for 1992Q1–1998Q4, before the start of Euro. To convert the nominal risk-free rate into real rate, we subtract GDP deflator of Germany from the former series. To arrive at a meaningful measure of the *real* spread, i.e., a spread unaffected by expected inflation hence rightly reflecting credit risk, we split the sample into to two parts. After the introduction of Euro, we can directly use the spread between the long-term nominal bond yields and the nominal risk-free rate, since all rates are denominated in Euro and thus subject to the same inflation expectation. For the period before Euro, we follow Ábrahám et al. (2021) and use spot and forward exchange rates (retrieved from Datastream) to convert the German nominal risk free rate into Italy's local currency, hence deriving a synthetic local currency risk free rate, and finally take the difference between the local nominal long-term bond yield with the synthetic risk free rate.

The information on the maturity structure of the government debt for Italy is not comprehensive. We manage to obtain government debt maturity data over 1990–2015 for Italy from all sources listed in Table E.1.

E.2 Estimation Results

Panel (a) of Figure E.1 plots the sample productivity series for Italy used for our calibration of the productivity shock process. It is clear that the during the 2008 Global Financial Crisis, there was prominent negative growth in productivities. This distinctive feature in the productivity dyanmics is also the main motivation for the use of Markov regime switching model (29) to calibrate the productivity shock. Correspondingly, Panel (b) shows that a 2-regime specification capture the crisis dynamics very well, with the smoothed regime probabilities reach almost 1 during the sudden drop periods observed in Panel (a).

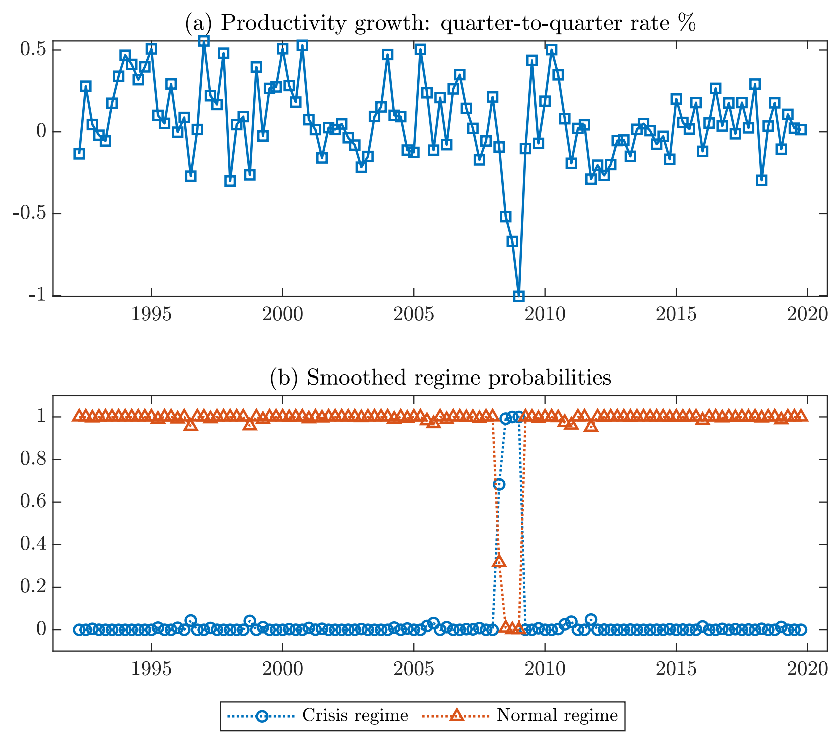

Figure E.1: Data sample and the estimated smoothed regime probabilities

The final estimation results are summarized in Table E.2. Note that we identify regime 1 as the crisis regime, and regime 2 as the normal regime. To overcome the local maximum problem of the highly nonlinear likelihood function, we randomize initializations of the EM algorithm of 1,000 times.

Table E.2: Parameters of the regime switching productivity process

	$\mu(\varsigma)$	$\rho(\varsigma)$	$\sigma(\varsigma)$	P	$\varsigma' = 1$	$\varsigma' = 2$	invariant dist.
$\varsigma = 1$	-0.0336	0.9018	0.0009	$\varsigma = 1$	0.6633	0.3367	0.0372
$\varsigma = 2$	0.0009	0.2167	0.0020	$\varsigma = 2$	0.0130	0.9870	0.9628

Notes: ς denotes the current regime of productivity shock, and ς' denotes that of the next period.

F Welfare Calculations

This section describes how the welfare gains depicted in Table 3 are computed. Similar to Ábrahám et al. (2021), define value of the sovereign for a sequence $\{c(\theta^t), n(\theta^t)\}$ starting from an initial state at $t = 0$ as

$$V^b(\{c(\theta^t), n(\theta^t)\}) = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c(\theta^t), n(\theta^t)) = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\log(c(\theta^t)) + \gamma \frac{(1 - n(\theta^t))^{\sigma_n} - 1}{1 - \sigma_n} \right],$$

where the last equality is obtained from the functional form considered in Section 6. We denote the sovereign's allocations with the Fund by $\{c^f(\theta^t), n^f(\theta^t)\}$ and the allocations without the Fund by $\{c^i(\theta^t), n^i(\theta^t)\}$. The value for the borrower with and without the Fund is given by $W^{bf}(\theta, \omega) = W^{bf}(\{c^f(\theta^t), n^f(\theta^t)\})$ and $V^{bi}(\theta, b) = V^{bi}(\{c(^i\theta^t), n^i(\theta^t)\})$, respectively.[51] To properly compare the two economies, we consider the point where $\omega = b \equiv o$. Thus (θ, o) represents the initial state for both economies. Now define $V^{bi}(\theta, o; \chi) \equiv V^{bi}(\{(1+\chi)c^i(\theta^t), n^i(\theta^t)\})$, where $\chi(\theta, o)$ represents the consumption-equivalent welfare gain of the Fund's intervention. It then directly follows that the welfare gain is computed in the following way $V^{bi}(\theta, o; \chi) = W^{bf}(\theta, o)$. Given the above functional form, we have that $\frac{\log(1+\chi)}{1-\beta} + V^{bi}(\theta, o) = V^{bf}(\theta, o)$. The welfare gain therefore boils down to $\chi(\theta, o) = \exp\left[(V^{bf}(\theta, o) - V^{bi}(\theta, o))(1 - \beta)\right] - 1$. We concentrate our analysis to the case in which $o = 0$.

Welfare decomposition

Following Ábrahám et al. (2021), we can decompose the welfare gains into four main components. As the Fund avoids default, it avoids the output penalty and the market exclusions. Those are the first two sources of welfare gains. In addition, as one can see from the two last columns of Table 3, the Fund enlarges the debt capacity of the sovereign. Finally, the

[51] Note that in equilibrium $W^{bf}(\theta, a, b) \equiv W^{bf}(\theta, \omega)$.

Fund provides state-contingent transfer, whereas the economy without the Fund only has access to non-contingent bonds. Table F.3, presents the decomposition of the welfare gains for each of the depicted growth states and zero initial debt. As one can see, the main source of welfare gains is the larger debt capacity followed by the state contingency and the circumvention of output penalty. Note that debt capacity and state contingency are closely linked one another. Without state-contingent transfers, the sovereign could not sustain a larger indebtedness.

Table F.3: Welfare Decomposition at Zero Initial Debt

State	No penalty (%)	Immediate return to market (%)	Greater debt capacity (%)	State-contingent insurance (%)
$\gamma = \gamma_{min}$	4.57	2.00	85.43	8.00
$\gamma = \gamma_{med}$	4.02	1.51	88.18	6.28
$\gamma = \gamma_{max}$	3.91	1.26	86.82	8.01

G Additional Tables and Figures

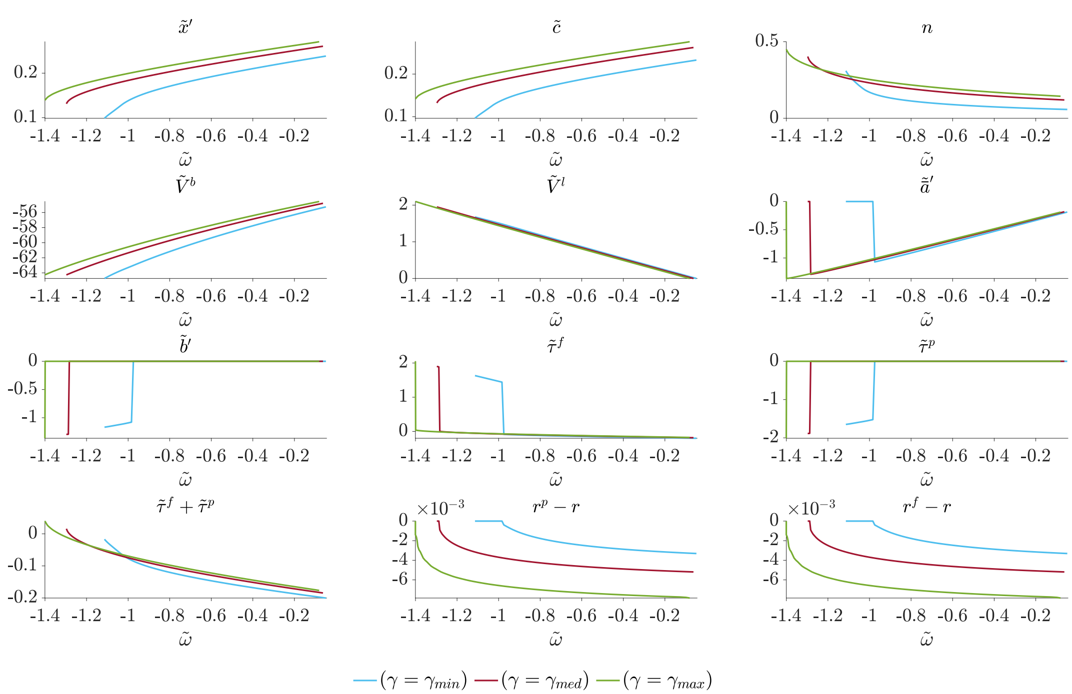

Figure G.2: Optimal Policies with Zero Private Debt as Function of $(\gamma, \tilde{\omega})$

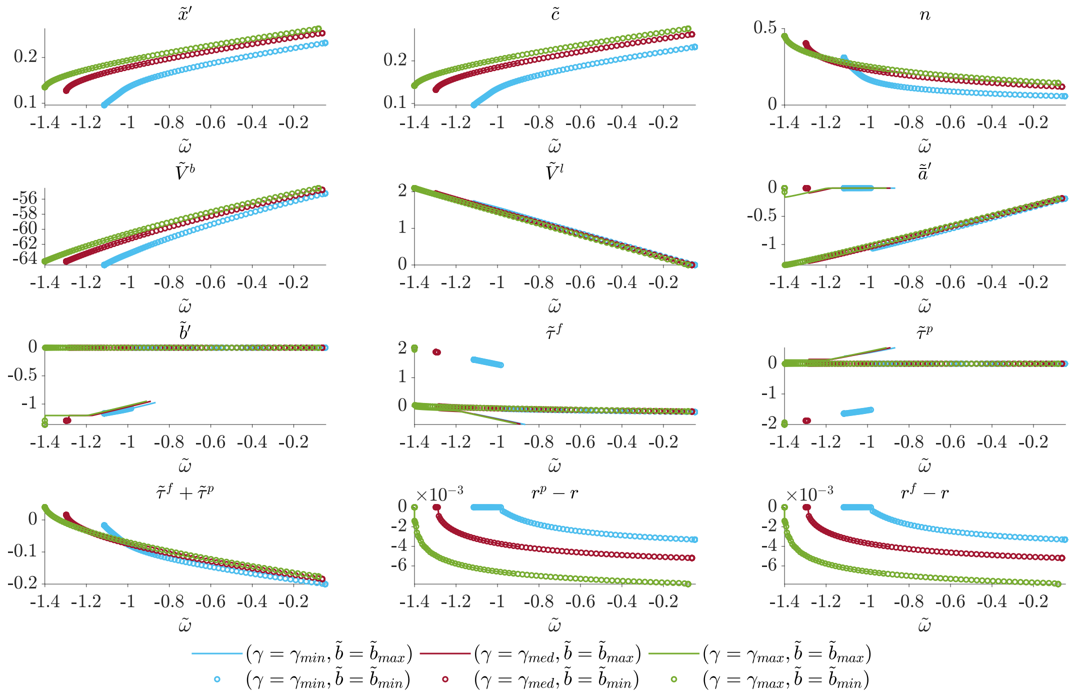

Figure G.3: Optimal Policies for Different Levels of Private Debt as Function of $(\gamma, \tilde{\omega})$

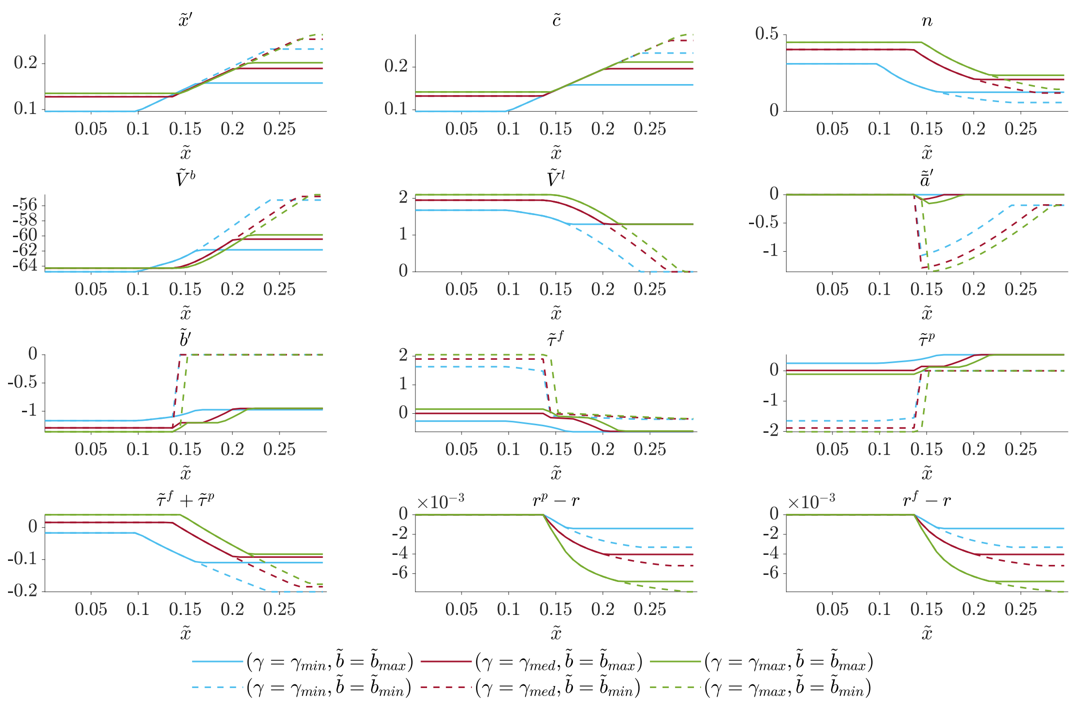

Figure G.4: Optimal Policies for Different Levels of Private Debt as Function of $(\gamma, \tilde{x})$

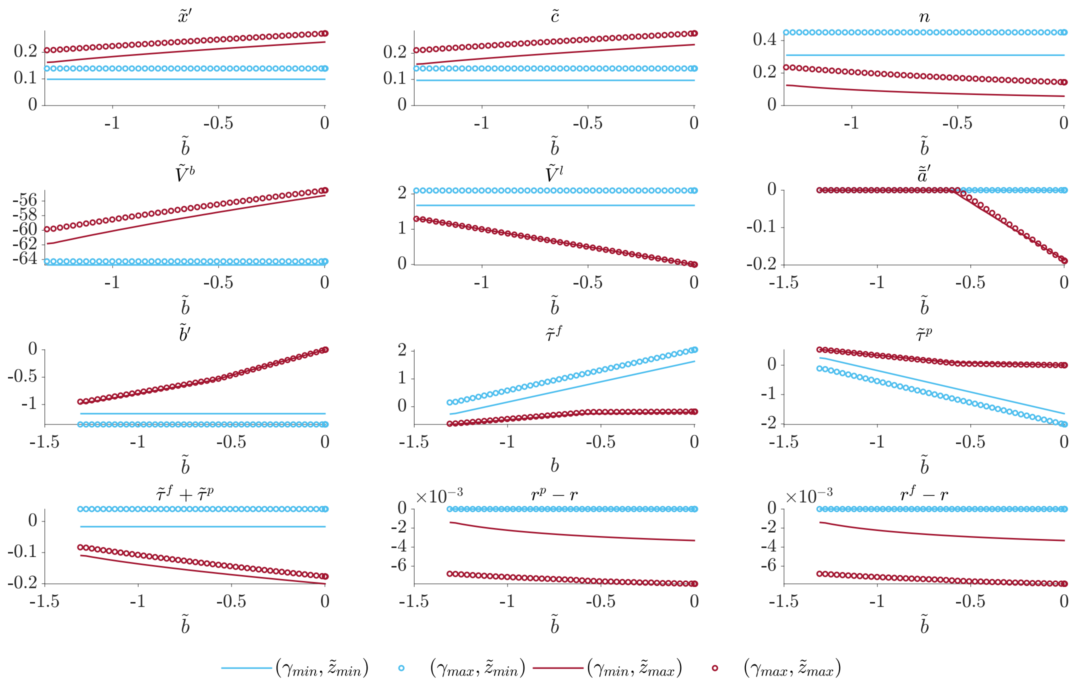

Figure G.5: Optimal Policies as Function of $(\gamma, \tilde{b})$

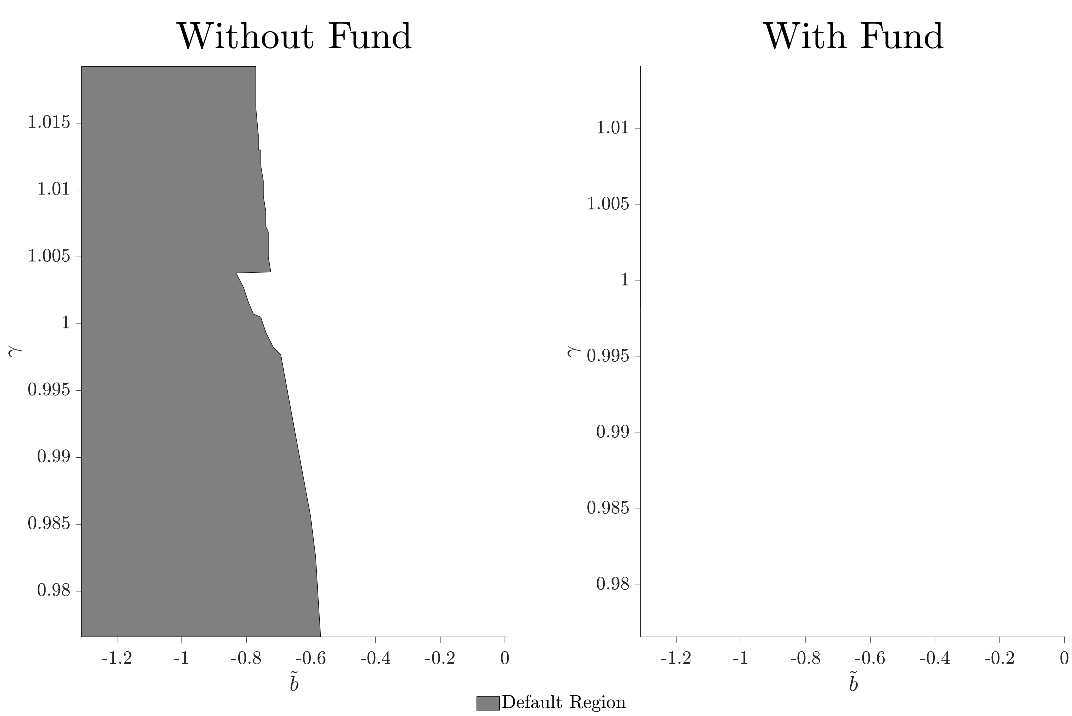

Figure G.6: Default Set as a Function of $(\gamma, \tilde{b})$

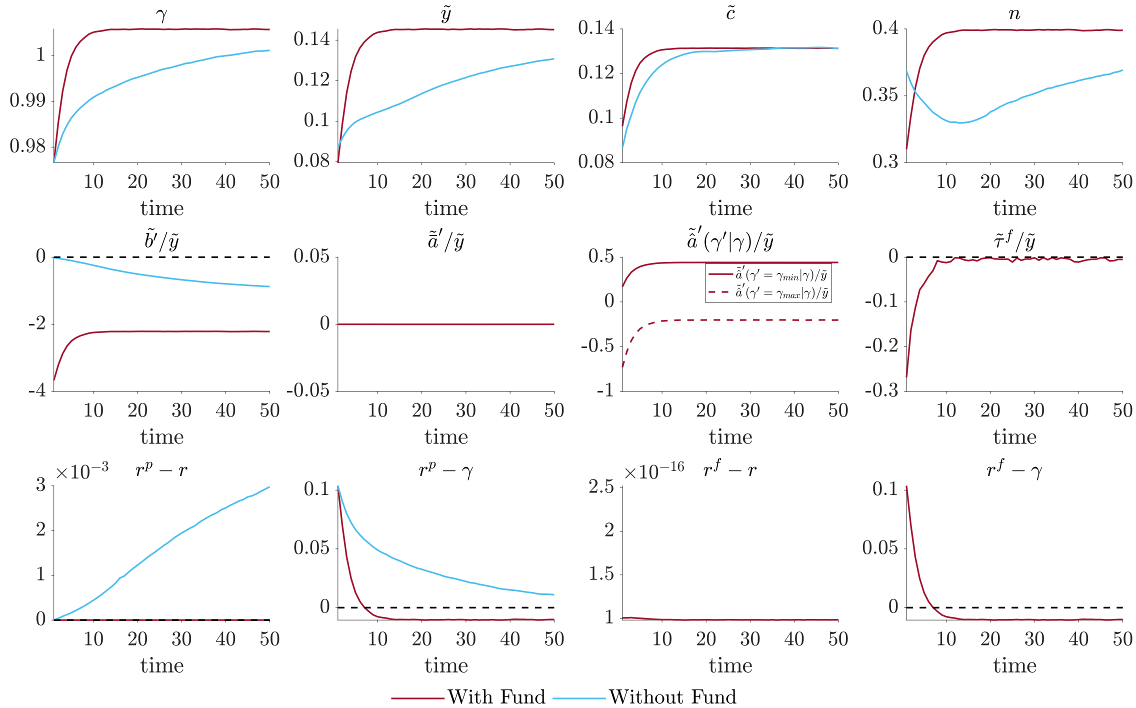

Figure G.7: Impulse Response Functions — Negative γ Shock

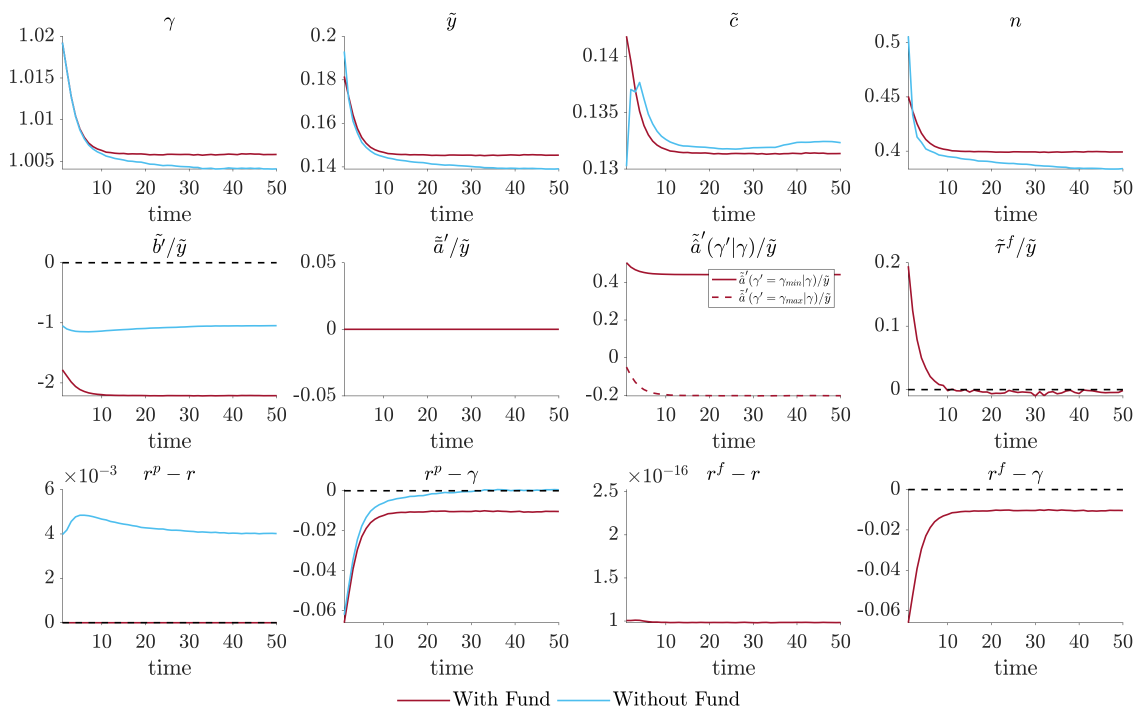

Figure G.8: Impulse Response Functions — Positive γ Shock

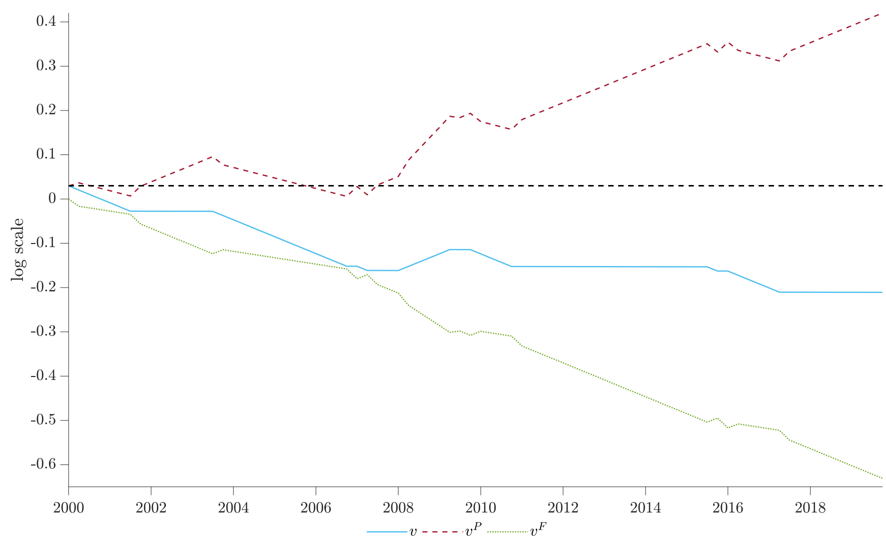

Figure G.9: Fund vs. Private Debt