2023 秋季本科时间序列

第 7 次作业答案

11 月 25 日

1. (a) 由于该过程满足平稳条件，因此对等式两端取期望：

$$\mathbb{E}X_t = \mu + \phi_1 \mathbb{E}X_{t-1} + \phi_2 \mathbb{E}X_{t-2}$$

化简可得：

$$\mathbb{E}X_t = \frac{\mu}{1-\phi_1-\phi_2} = 2$$

(b)

$$\begin{aligned}
\tilde{X}_t =& \phi_1 \tilde{X}_{t-1} + \phi_2 \tilde{X}_{t-2} + \varepsilon_t + \theta \varepsilon_{t-1} \\
=& \phi_1 \left(\phi_1 \tilde{X}_{t-2} + \phi_2 \tilde{X}_{t-3} + \varepsilon_{t-1} + \theta \varepsilon_{t-2} \right) + \phi_2 \tilde{X}_{t-2} + \varepsilon_t + \theta \varepsilon_{t-1} \\
=& \left({\phi_1}^2 + \phi_2 \right) \tilde{X}_{t-2} + \phi_1 \phi_2 \tilde{X}_{t-3} + \varepsilon_t + \left(\theta + \phi_1 \right) \varepsilon_{t-1} + \theta \phi_1 \varepsilon_{t-2} \\
=& \left({\phi_1}^3 + \phi_2 \phi_1 \right) \tilde{X}_{t-3} + \left({\phi_1}^2 \phi_2 + {\phi_2}^3 \right) \tilde{X}_{t-4} + \varepsilon_t + \left(\theta + \phi_1 \right) \varepsilon_{t-1} + \left(\phi_1 \theta + {\phi_1}^2 + \phi_2 \right) \varepsilon_{t-2}
\end{aligned}$$

因此：

$$\psi_0 = 1$$

$$\psi_1 = \theta + \phi_1$$

$$\psi_2 = \phi_1 \theta + {\phi_1}^2 + \phi_2$$

(c)

$$\mathbb{E}X_t \varepsilon_t = \psi_0 \sigma_\epsilon^2 = \sigma_\epsilon^2$$

$$\mathbb{E}X_t \varepsilon_{t-1} = \psi_1 \sigma_\epsilon^2 = \left(\theta + \phi_1 \right) \sigma_\epsilon^2$$

$$\mathbb{E}X_t \varepsilon_{t-2} = \psi_2 \sigma_\epsilon^2 = \left(\phi_1 \theta + {\phi_1}^2 + \phi_2 \right) \sigma_\epsilon^2$$

(d) 在等式两端依次乘以 X_t, X_{t-1}, X_{t-2} 并取期望，可得：

$$\begin{cases} \sigma_X^2(1) = \phi_1\sigma_X^2(0) + \phi_2\sigma_X^2(1) + \theta\sigma_\epsilon^2 \\ \sigma_X^2(2) = \phi_1\sigma_X^2(1) + \phi_2\sigma_X^2(0) \\ \sigma_X^2(0) = \phi_1\sigma_X^2(1) + \phi_2\sigma_X^2(2) + \left(\theta^2 + \phi_1\theta + 1\right)\sigma_\epsilon^2 \end{cases}$$

即：

$$\begin{bmatrix} \phi_1 & \phi_2 - 1 & 0 \\ \phi_2 & \phi_1 & -1 \\ 1 & -\phi_1 & -\phi_2 \end{bmatrix} \begin{bmatrix} \sigma_X^2(0) \\ \sigma_X^2(1) \\ \sigma_X^2(2) \end{bmatrix} = \begin{bmatrix} -\theta\sigma_\epsilon^2 \\ 0 \\ \left(\theta^2 + \phi_1\theta + 1\right)\sigma_\epsilon^2 \end{bmatrix}$$

设系数矩阵为 Φ, 则有：

$$\begin{bmatrix} \sigma_X^2(0) \\ \sigma_X^2(1) \\ \sigma_X^2(2) \end{bmatrix} = \frac{1}{\det(\Phi)} \begin{bmatrix} -\phi_1\phi_2 - \phi_1 & \phi_2^2 - \phi_2 & 1 - \phi_2 \\ \phi_2^2 - 1 & -\phi_1\phi_2 & \phi_1 \\ -\phi_1\phi_2 - \phi_1 & \phi_1^2 + \phi_2 - 1 & \phi_1^2 - \phi_2^2 + \phi_2 \end{bmatrix} \begin{bmatrix} -\theta\sigma_\epsilon^2 \\ 0 \\ \left(\theta^2 + \phi_1\theta + 1\right)\sigma_\epsilon^2 \end{bmatrix}$$

解得：

$$\begin{cases} \sigma_X^2(0) = \dfrac{1}{\det(\Phi)}\left(2\phi_1\theta - \phi_2\theta^2 - \phi_2 + \theta^2 + 1\right)\sigma_\epsilon^2 \\ \sigma_X^2(1) = \dfrac{1}{\det(\Phi)}\left(\phi_1^2\theta + \phi_1\theta^2 + \phi_1 + \theta\phi_2^2 + \theta\right)\sigma_\epsilon^2 \\ \sigma_X^2(2) = \dfrac{1}{\det(\Phi)}\left(\phi_1^2\theta^2 + \phi_1^3\theta - \phi_2^2\theta^2 + \phi_2\theta^2 - \phi_1\phi_2^2\theta + 2\phi_1\phi_2\theta + \phi_1\theta + \phi_1^2 - \phi_2^2 + \phi_2\right)\sigma_\epsilon^2 \end{cases}$$

其中，$\det(\Phi) = (\phi_1 + \phi_2 - 1)(\phi_2 - \phi_1 - 1)(\phi_2 + 1)$

(e)

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t + \theta\varepsilon_{t-1}$$

$$(1 - \phi_1\mathscr{L} - \phi_2\mathscr{L}^2)X_t = (1 + \phi_1\mathscr{L})\varepsilon_t$$

$$A(\mathscr{L})X_t = B(\mathscr{L})\varepsilon_t$$

$$X_t = A(\mathscr{L})^{-1}B(\mathscr{L})\varepsilon_t$$

令 $C(z) = A(z)^{-1}B(z) = \frac{1+\theta z}{1-\phi_1 z-\phi_2 z^2}$

$$f_x(\omega) = G(x) \times 1 = \sqrt{C(e^{i\omega})C(e^{-i\omega})} = \sqrt{\frac{B(e^{i\omega})B(e^{-i\omega})}{A(e^{i\omega})A(e^{-i\omega})}}$$

$$
\begin{aligned}
S_x(\omega) &= f_x^2(\omega)S_\varepsilon(\omega) \\
&= \frac{1}{2\pi}f_x^2(\omega) \\
&= \frac{1}{2\pi}\frac{B(e^{i\omega})B(e^{-i\omega})}{A(e^{i\omega})A(e^{-i\omega})} \\
&= \frac{1}{2\pi}\frac{(1+\theta e^{i\omega})(1+\theta e^{-i\omega})}{(1-\phi_1 e^{i\omega}-\phi_2 e^{i\omega})(1-\phi_1 e^{-i\omega}-\phi_2 e^{-i\omega})} \\
&= \frac{1}{2\pi}\frac{(1+\theta)^2+2\theta\cos(\omega)}{1+\phi_1^2+\phi_2^2-2\phi_1(1-\phi_2)\cos(\omega)-2\phi_2\cos(2\omega)}
\end{aligned}
$$

根据逆变换公式，$\sigma_x^2(k) = \int_{-\frac{1}{2}}^{\frac{1}{2}} S_x(\omega)e^{i2\pi\omega k}\mathrm{d}\omega$

$$
\begin{aligned}
\sigma_x^2(k) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} S_x(\omega)e^{i2\pi\omega k}\mathrm{d}\omega \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2\pi}\frac{(1+\theta)^2+2\theta\cos(\omega)}{1+\phi_1^2+\phi_2^2-2\phi_1(1-\phi_2)\cos(\omega)-2\phi_2\cos(2\omega)}e^{i2\pi\omega k}\mathrm{d}\omega \\
\sigma_x^2(0) &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{2\pi}\frac{(1+\theta)^2+2\theta\cos(\omega)}{1+\phi_1^2+\phi_2^2-2\phi_1(1-\phi_2)\cos(\omega)-2\phi_2\cos(2\omega)}e^{i2\pi\omega 0}\mathrm{d}\omega \\
&= \frac{\left(2\phi_1\theta-\phi_2\theta^2-\phi_2+\theta^2+1\right)}{(\phi_1+\phi_2-1)(\phi_2-\phi_1-1)(\phi_2+1)}\sigma_\epsilon^2
\end{aligned}
$$

(f)

$$
Y_t = X_t - \mu - \phi_1 X_{t-1} - \phi_2 X_{t-2}
$$

由 $\{\varepsilon_t\} \overset{\text{iid}}{\sim} N(0,\sigma_\varepsilon^2)$ 可得,

$$
\mathrm{var}(Y_t) = \mathrm{var}(\varepsilon_t + \theta\varepsilon_{t-1}) = (1+\theta^2)\sigma_\varepsilon^2
$$

$$
\mathrm{cov}(Y_t, Y_{t-1}) = \mathrm{cov}(\varepsilon_t + \theta\varepsilon_{t-1}, \varepsilon_{t-1} + \theta\varepsilon_{t-2}) = \theta\sigma_\varepsilon^2
$$

$$
\mathrm{cov}(Y_t, Y_{t-k}) = 0 (k \geq 2)
$$

故

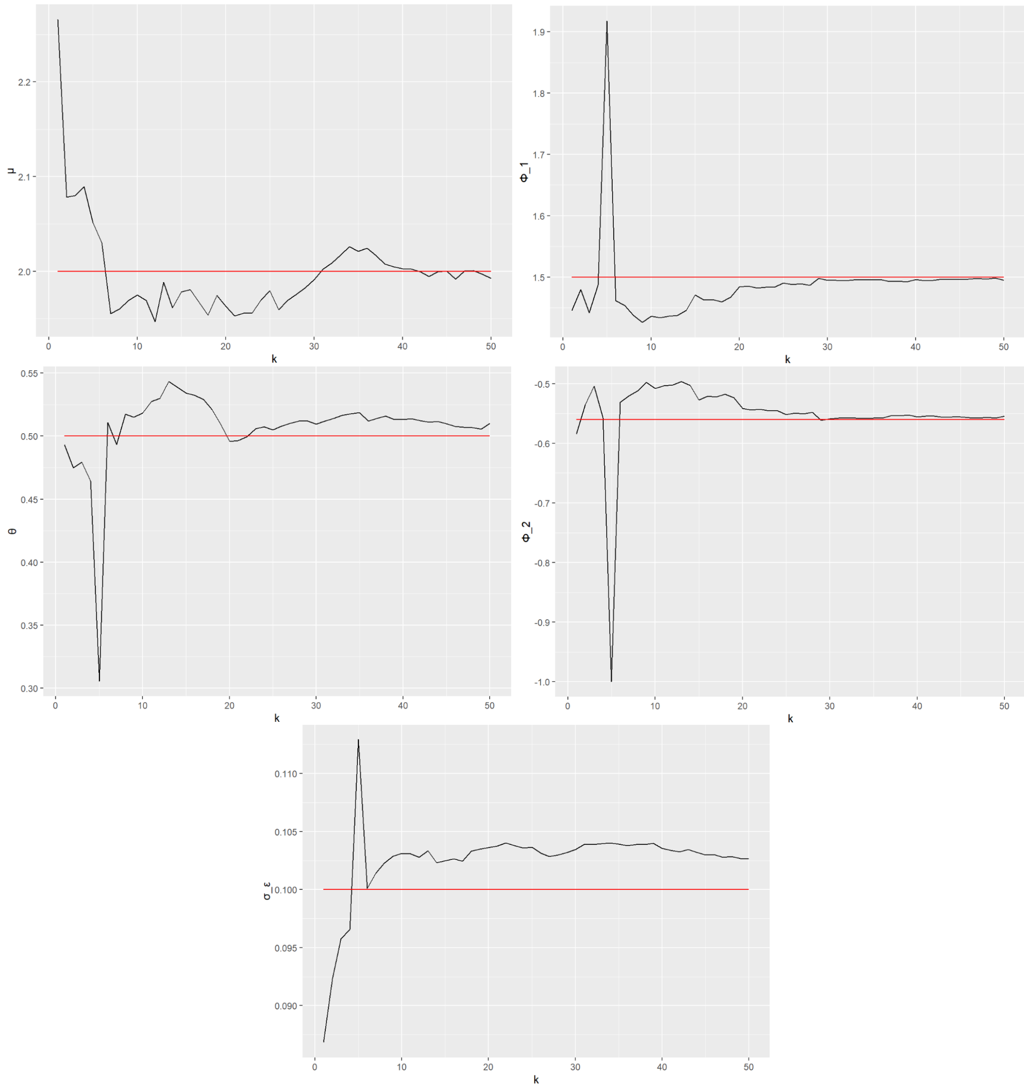
$$
\boldsymbol{\Sigma} = \begin{bmatrix}
1+\theta^2 & \theta & 0 & 0 & \cdots & 0 \\
\theta & 1+\theta^2 & \theta & 0 & \cdots & 0 \\
0 & \theta & 1+\theta^2 & \theta & \cdots & 0 \\
0 & 0 & \theta & 1+\theta^2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1+\theta^2
\end{bmatrix}
$$

(g) 代码如下：

```
mu <- 0.12
phi1 <- 1.5
phi2 <- -0.56
theta <- 0.5
sigma <- 0.1

set.seed(1)
epsilon <- rnorm(5000, mean = 0, sd = sigma)

X <- numeric(5000)
X[1] <- mu + phi1*2 + phi2*2 +epsilon[1]
X[2] <- mu + phi1*X[1] + phi2*2 + epsilon[2]
        + theta* epsilon[1]
for (t in 3:5000) {
   X[t] <- mu + phi1 * X[t-1] + phi2 * X[t-2] + epsilon[t]
        + theta * epsilon[t-1]
        }

estimates <- matrix(NA, nrow = 50, ncol = 5)

for (k in 1:50) {
    X_k <- X[1:(100*k)]
    fit <- arima(X_k, order = c(2, 0, 1), method = "ML")
 estimates[k,] <- c(fit$coef["intercept"], fit$coef["ar1"],
 fit$coef["ar2"], fit$coef["ma1"], sqrt(fit$sigma2))
     }
```

```
df_estimates <- tibble(
k = 1:50,
mu = estimates[,1],
Φ_1 = estimates[,2],
Φ_2 = estimates[,3],
theta = estimates[,4],
sigma = estimates[,5])

ggplot(df_estimates)
+ geom_line(aes(k, mu))
+ geom_line(aes(k, 2),color = 'red')
+ labs(x='k',y='mu')

ggplot(df_estimates)
+ geom_line(aes(k, Φ_1))
+ geom_line(aes(k, phi1),color = 'red')
+ labs(x = 'k',y = 'Φ_1')

ggplot(df_estimates)
+ geom_line(aes(k, Φ_2))
+ geom_line(aes(k, phi2),color = 'red')
+ labs(x = 'k',y = 'Φ_2')

ggplot(df_estimates)
+ geom_line(aes(k, theta))
+ geom_line(aes(k, theta),color = 'red')
```

```
+ labs(x = 'k',y = 'theta')

ggplot(df_estimates)
+ geom_line(aes(k, sigma))
+ geom_line(aes(k, sigma),color = 'red')
+ labs(x = 'k',y = 'sigma')
```

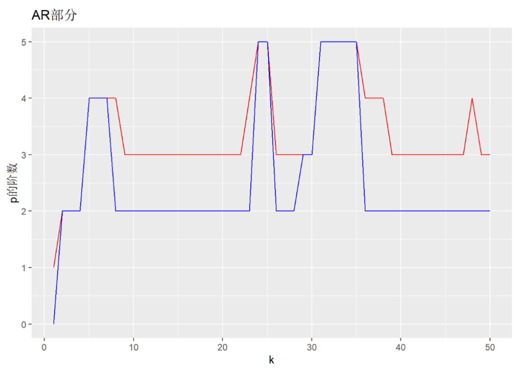


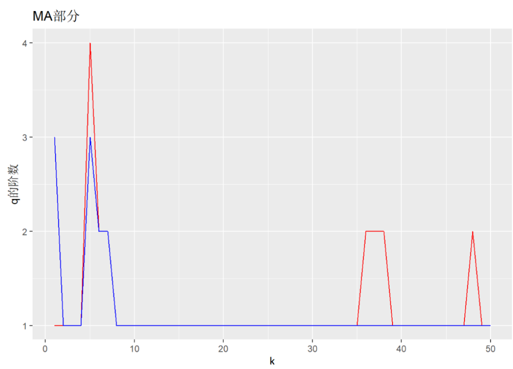
(h) 代码如下：

```
set.seed(1234)

```

```
mu <- 0.12
sigma <- 0.1
phi1 <- 1.5
phi2 <- -0.56
theta <- 0.5

epsilon <- rnorm(5000, mean = 0, sd = sigma)
X <- numeric(5000)
X[1] <- mu
X[2] <- mu
for (t in 3:5000) {
X[t]<-mu+phi1*X[t-1]+phi2*X[t-2]+epsilon[t]+theta*epsilon[t-1]
}

order_aic <- matrix(NA, nrow = 50, ncol = 2)
order_bic <- matrix(NA, nrow = 50, ncol = 2)

for (k in 1:50) {
 X_k <- X[1:(100*k)]

 fit_aic <- auto.arima(X_k, ic = "aic")
 fit_bic <- auto.arima(X_k, ic = "bic")

 order_aic[k,] <- c(fit_aic$arma[1], fit_aic$arma[2])
 order_bic[k,] <- c(fit_bic$arma[1], fit_bic$arma[2])
}
```

```
df_aic<-tibble(k=1:50, p=order_aic[,1],q = order_aic[,2])
df_bic<-tibble(k=1:50, p=order_bic[,1],q = order_bic[,2])

df_p <- tibble(k = 1:50, aic = df_aic$p, bic = df_bic$p)
df_q <- tibble(k = 1:50, aic = df_aic$q, bic = df_bic$q)

ggplot(df_p)+
geom_line(aes(k, aic), color = 'red')+
geom_line(aes(k, bic), color = 'blue')+
labs(title = 'AR部分',
x = 'k',
y = 'p的阶数')

ggplot(df_q)+
geom_line(aes(k, aic), color = 'red')+
geom_line(aes(k, bic), color = 'blue')+
labs(title = 'MA部分',
x = 'k',
y = 'q的阶数')
```





2.

$$\mathbb{E}(X_{t+1}, X_t) = \theta_1(1+\theta_2)$$

$$E(X_{t+2}, X_t) = \theta_2$$

故 X_{t+1} 与 X_t、X_{t-1} 相关，X_{t+2} 与 X_t 相关

当 $s \geq 3$ 时，$X_{t+3}X_t$、X_{t-1} 均不相关

(a)

$$\hat{X}_{t+1|t} = \mathscr{L}(X_{t+1}|X_t, X_{t-1}) = \hat{b}X$$

其中，$X = \begin{bmatrix} X_t \\ X_{t-1} \end{bmatrix}$

$$\begin{aligned}
\hat{b} &= \underset{a}{argmin}\mathbb{E}\left[Y - b^T X\right]^2 \\
&= (\mathbb{E}\left[XX^T\right])^{-1}\mathbb{E}\left[XX_{t+1}\right] \\
&= \begin{bmatrix} 1+\theta_1^2+\theta_2^2 & \theta_1(1+\theta_2) \\ \theta_1(1+\theta_2) & 1+\theta_1^2+\theta_2^2 \end{bmatrix} \begin{bmatrix} \theta_1(1+\theta_2) \\ \theta_2 \end{bmatrix} \\
&= \frac{1}{\det\Phi} \begin{bmatrix} 1+\theta_1^2+\theta_2^2 & -\theta_1(1+\theta_2) \\ -\theta_1(1+\theta_2) & 1+\theta_1^2+\theta_2^2 \end{bmatrix} \begin{bmatrix} \theta_1(1+\theta_2) \\ \theta_2 \end{bmatrix} \\
&= \frac{1}{\det\Phi} \begin{bmatrix} \theta_1(1+\theta_2)(1+\theta_1^2+\theta_2^2-\theta_2) \\ \theta_2(1+\theta_1^2+\theta_2)^2 - \theta_1^2(1+\theta_2)^2 \end{bmatrix} \\
&= \frac{1}{(1+\theta_1^2+\theta_2^2)^2 - \theta_1^2(1+\theta_2)^2} \begin{bmatrix} \theta_1(1+\theta_2)(1+\theta_1^2+\theta_2^2-\theta_2) \\ \theta_2(1+\theta_1^2+\theta_2)^2 - \theta_1^2(1+\theta_2)^2 \end{bmatrix}
\end{aligned}$$

(b)

$$\hat{X}_{t+2|t} = \hat{b}X_t$$

$$\begin{aligned}
\hat{b} &= \mathbb{E}(X_t^2)^{-1}\mathbb{E}(X_t X_{t-1}) \\
&= \frac{\theta_1(1+\theta_2)}{1+\theta_1^2+\theta_2^2}
\end{aligned}$$