2023 秋季本科时间序列

第 5 次作业答案

11 月 9 日

1. 解：

   $AR(2)$ 过程 $(1-\phi_1\mathscr{L}-\phi_2\mathscr{L}^2)X_t=\epsilon_t$ 特征多项式 $1-\phi_1 x-\phi_2 x^2=0$，则有两特征根 x_1, x_2:

   $$x_{1,2}=\frac{\phi_1 \pm \sqrt{\phi_1^2+4\phi_2}}{-2\phi_2}$$

   若使得 $AR(2)$ 过程平稳则 $|x_1|>1, |x_2|>1$, 记特征根的倒数为 z_1, z_2:

   $$z_1=\frac{1}{2}(\phi_1-\sqrt{\phi_1^2+4\phi_2}), z_2=\frac{1}{2}(\phi_1+\sqrt{\phi_1^2+4\phi_2})$$

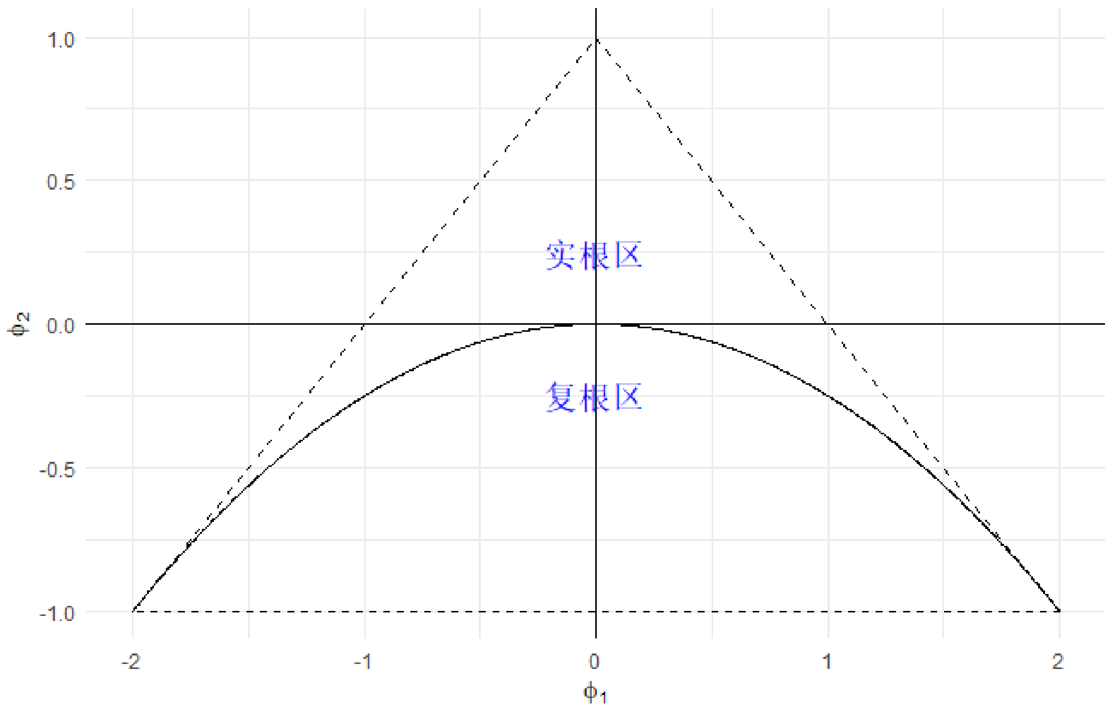
   (a) 当 $\phi_1^2+4\phi_2 \geq 0$ 时为实根，由于 $|z_1|<1, |z_2|<1$，则有 $-1<z_1<z_2<1$，即：

   $$\phi_1+\phi_2<1, \phi_2-\phi_1<1$$

   (b) 当根为复数根时，$|z_1|=|z_2|<1$，即：

   $$|z_1|^2=\frac{\phi_1^2+(-\phi_1^2-4\phi_2)}{4}=-\phi_2>1 \Rightarrow \phi_2>-1, \phi_1^2+4\phi_2<0$$

   因此，使得 X_t 为平稳过程的自回归系数取值范围 $(\phi_1, \phi_2) \in \mathbb{R}^2$ 的图示如下：



2. 解：

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t$$

$$X_{t-1} = \phi_1 X_{t-2} + \phi_2 X_{t-3} + \epsilon_{t-1}$$

$$X_{t-2} = \phi_1 X_{t-3} + \phi_2 X_{t-4} + \epsilon_{t-2}$$

$$\cdots\cdots$$

因此：

$$
\begin{aligned}
X_t &= \phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t \\
&= \phi_1(\phi_1 X_{t-2} + \phi_2 X_{t-3} + \epsilon_{t-1}) + \phi_2 X_{t-2} + \epsilon_t \\
&= (\phi_1^2 + \phi_2) X_{t-2} + \phi_1\phi_2 X_{t-3} + \phi_1 \epsilon_{t-1} + \epsilon_t \\
&= (\phi_1^2 + \phi_2)(\phi_1 X_{t-3} + \phi_2 X_{t-4} + \epsilon_{t-2}) + \phi_1\phi_2 X_{t-3} + \phi_1 \epsilon_{t-1} + \epsilon_t \\
&= (\phi_1^3 + 2\phi_1\phi_2) X_{t-3} + \phi_2(\phi_1^2 + \phi_2) X_{t-4} + (\phi_1^2 + \phi_2)\epsilon_{t-2} + \phi_1 \epsilon_{t-1} + \epsilon_t \\
&= (\phi_1^3 + 2\phi_1\phi_2)(\phi_1 X_{t-4} + \phi_2 X_{t-5} + \epsilon_{t-3}) + \phi_2(\phi_1^2 + \phi_2) X_{t-4} + (\phi_1^2 + \phi_2)\epsilon_{t-2} + \phi_1 \epsilon_{t-1} + \epsilon_t \\
&= (\phi_1^4 + 3\phi_1^2\phi_2 + \phi_2^2) X_{t-4} + \phi_2(\phi_1^3 + 2\phi_1\phi_2) X_{t-5} + (\phi_1^3 + 2\phi_1\phi_2)\epsilon_{t-3} + (\phi_1^2 + \phi_2)\epsilon_{t-2} + \phi_1 \epsilon_{t-1} + \epsilon_t
\end{aligned}
$$

3. 解：

系数矩阵的行列式为

$$\begin{vmatrix} \phi_1 & \phi_2 - 1 & 0 \\ \phi_2 & \phi_1 & -1 \\ 1 & -\phi_1 & -\phi_2 \end{vmatrix} = (\phi_1 + \phi_2 - 1)(\phi_2 - \phi_1 - 1)(\phi_2 + 1)$$

要使行列式非零，则有 $(\phi_1 + \phi_2 - 1)(\phi_2 - \phi_1 - 1)(\phi_2 + 1) \neq 0$，即行列式非零的条件为

$$\begin{cases} \phi_1 + \phi_2 - 1 \neq 0 \\ \phi_2 - \phi_1 - 1 \neq 0 \\ \phi_2 + 1 \neq 0 \end{cases}$$

由 1. 可知 AR(2) 过程平稳性条件，因此可得：AR(2) 过程平稳性条件是行列式非零的条件的充分不必要条件

用 Cramer 法则计算矩阵的逆：

$$\Phi^{-1} = \frac{1}{\det \Phi} \Phi^*$$

由于

$$\Phi = \begin{bmatrix} \phi_1 & \phi_2 - 1 & 0 \\ \phi_2 & \phi_1 & -1 \\ 1 & -\phi_1 & -\phi_2 \end{bmatrix}$$

则有：

$$\Phi_{11} = \begin{vmatrix} \phi_1 & -1 \\ -\phi_1 & -\phi_2 \end{vmatrix} = -\phi_1\phi_2 - \phi_1, \Phi_{12} = -\begin{vmatrix} \phi_2 & -1 \\ 1 & -\phi_2 \end{vmatrix} = \phi_2^2 - 1, \Phi_{13} = \begin{vmatrix} \phi_2 & \phi_1 \\ 1 & -\phi_1 \end{vmatrix} = -\phi_1\phi_2 - \phi_1$$

$$\Phi_{21} = -\begin{vmatrix} \phi_2 - 1 & 0 \\ -\phi_1 & -\phi_2 \end{vmatrix} = \phi_2^2 - \phi_2, \Phi_{22} = \begin{vmatrix} \phi_1 & 0 \\ 1 & -\phi_2 \end{vmatrix} = -\phi_1\phi_2, \Phi_{23} = -\begin{vmatrix} \phi_1 & \phi_2 - 1 \\ 1 & -\phi_1 \end{vmatrix} = \phi_1^2 + \phi_2 - 1$$

$$\Phi_{31} = \begin{vmatrix} \phi_2 - 1 & 0 \\ \phi_1 & -1 \end{vmatrix} = -\phi_2 + 1, \Phi_{32} = -\begin{vmatrix} \phi_1 & 0 \\ \phi_2 & -1 \end{vmatrix} = \phi_1, \Phi_{33} = \begin{vmatrix} \phi_1 & \phi_2 - 1 \\ \phi_2 & \phi_1 \end{vmatrix} = \phi_1^2 - \phi_2^2 + \phi_2$$

故矩阵的逆为：

$$\Phi^{-1}=\frac{1}{|\Phi|}\Phi^{*}=\frac{1}{(\phi_1+\phi_2-1)(\phi_2-\phi_1-1)(\phi_2+1)}\begin{bmatrix}-\phi_1\phi_2-\phi_1 & \phi_2^2-\phi_2 & -\phi_2+1\\ \phi_2^2-1 & -\phi_1\phi_2 & \phi_1\\ -\phi_1\phi_2-\phi_1 & \phi_1^2+\phi_2-1 & \phi_1^2-\phi_2^2+\phi_2\end{bmatrix}$$

4. 解：

$$X_t=0.9X_{t-1}-0.2X_{t-2}+\epsilon_t$$

分别在等式两边同乘 X_t, X_{t-1}, X_{t-2} 并取期望有

$$\sigma_X^2(0)=0.9\sigma_X^2(1)-0.2\sigma_X^2(2)+\sigma_\epsilon^2$$

$$\sigma_X^2(1)=0.9\sigma_X^2(0)-0.2\sigma_X^2(1)$$

$$\sigma_X^2(2)=0.9\sigma_X^2(1)-0.2\sigma_X^2(0)$$

转换为矩阵形式有

$$\begin{bmatrix}0\\0\\\sigma_\epsilon^2\end{bmatrix}=\begin{bmatrix}0.9 & -1.2 & 0\\ -0.2 & 0.9 & -1\\ 1 & -0.9 & 0.2\end{bmatrix}\begin{bmatrix}\sigma_X^2(0)\\\sigma_X^2(1)\\\sigma_X^2(2)\end{bmatrix}$$

解得

$$\sigma_X^2(0)=\frac{50}{21}\sigma_\epsilon^2, \sigma_X^2(1)=\frac{25}{14}\sigma_\epsilon^2, \sigma_X^2(2)=\frac{95}{84}\sigma_\epsilon^2$$

在 $X_{t+k}=\phi_1X_{t+k-1}+\phi_2X_{t+k-2}+\epsilon_{t+k}$ 两边同乘 X_t 得

$$\sigma_X^2(k)=\phi_1\sigma_X^2(k-1)+\phi_2\sigma_X^2(k-2)$$

即有

$$\begin{bmatrix}\sigma_X^2(k)\\\sigma_X^2(k-1)\end{bmatrix}=\begin{bmatrix}\phi_1 & \phi_2\\ 1 & 0\end{bmatrix}\begin{bmatrix}\sigma_X^2(k-1)\\\sigma_X^2(k-2)\end{bmatrix}$$

代入本题参数

$$\Lambda = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0.9 & -0.2 \\ 1 & 0 \end{bmatrix}$$

求特征值

$$\mathbb{P}(\lambda) = |\lambda I - \Lambda| = \begin{bmatrix} \lambda - 0.9 & 0.2 \\ -1 & \lambda \end{bmatrix} = \lambda^2 - 0.9\lambda + 0.2 = 0$$

解得

$$\lambda_1 = 0.4, \lambda_2 = 0.5$$

则有

$$\begin{bmatrix} \sigma_X^2(k) \\ \sigma_X^2(k-1) \end{bmatrix} = \begin{bmatrix} 0.4 & 0.5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0.4^{k-1} & \\ & 0.5^{k-1} \end{bmatrix} \begin{bmatrix} 0.4 & 0.5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \sigma_X^2(1) \\ \sigma_X^2(0) \end{bmatrix}$$

故有

$$\sigma_X^2(k) = \frac{25}{21}(7 \times 0.5^k - 5 \times 0.4^k)\sigma_\epsilon^2$$

5. 解：

由 AR(2) 过程 $X_t = X_{t-1} - 0.25X_{t-2} + \epsilon_t$:

$$\begin{bmatrix} 0 \\ 0 \\ \sigma_\epsilon^2 \end{bmatrix} = \begin{bmatrix} 1 & -1.25 & 0 \\ -0.25 & 1 & -1 \\ 1 & -1 & 0.25 \end{bmatrix} \begin{bmatrix} \sigma_X^2(0) \\ \sigma_X^2(1) \\ \sigma_X^2(2) \end{bmatrix}$$

解得

$$\sigma_X^2(0) = \frac{80}{27}\sigma_\epsilon^2, \sigma_X^2(1) = \frac{64}{27}\sigma_\epsilon^2, \sigma_X^2(2) = \frac{64}{27}\sigma_\epsilon^2$$

由

$$\begin{bmatrix} \sigma_X^2(k) \\ \sigma_X^2(k-1) \end{bmatrix} = \begin{bmatrix} 1 & -0.25 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sigma_X^2(k-1) \\ \sigma_X^2(k-2) \end{bmatrix}$$

令

$$\Lambda = \begin{bmatrix} 1 & -0.25 \\ 1 & 0 \end{bmatrix}$$

$$\mathbb{P}(\lambda) = |\lambda I - \Lambda| = \lambda^2 - \lambda + 0.25$$

解得

$$\lambda_1 = \lambda_2 = 0.5$$

故无法使用特征值分解

$$\Lambda = \begin{bmatrix} 1 & -0.25 \\ 1 & 0 \end{bmatrix} = T \begin{bmatrix} 0.5 & 1 \\ 0 & 0.5 \end{bmatrix} T^{-1}$$

$$\Lambda T = \begin{bmatrix} 0.5 & 1 \\ 0 & 0.5 \end{bmatrix}$$

令 $T = [T_1, T_2]$，有

$$[\Lambda T_1, \Lambda T_2] = [T_1, T_2] \begin{bmatrix} 0.5 & 1 \\ 0 & 0.5 \end{bmatrix}$$

故可得 $T_1 = [0.5, 1]^T, T_2 = [1, 0]^T$，即 $T = [T_1, T_2] = \begin{bmatrix} 0.5 & 1 \\ 1 & 0 \end{bmatrix}$

因此

$$\Lambda = \begin{bmatrix} 0.5 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0.5 & 1 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0.5 & 1 \\ 1 & 0 \end{bmatrix}^{-1}$$

由于

$$\begin{bmatrix} 0.5 & 1 \\ 0 & 0.5 \end{bmatrix}^k = \begin{bmatrix} 0.5^k & 0.5^{k-1}k \\ 0 & 0.5^k \end{bmatrix}$$

故 Λ^k 有

$$\Lambda^k = \begin{bmatrix} 0.5 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0.5^k & 0.5^{k-1}k \\ 0 & 0.5^k \end{bmatrix} \begin{bmatrix} \phi & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} (k+1)0.5^k & -0.5^{k+1}k \\ 0.5^{k-1}k & (1-k)0.5^k \end{bmatrix}$$

则

$$\begin{bmatrix} \sigma_X^2(k) \\ \sigma_X^2(k-1) \end{bmatrix} = \Lambda^k \begin{bmatrix} \sigma_X^2(1) \\ \sigma_X^2(0) \end{bmatrix} = 0.5^{k-1} \begin{bmatrix} 0.5(k+1) & -0.5^2 k \\ k & 0.5(1-k) \end{bmatrix} \begin{bmatrix} \sigma_X^2(1) \\ \sigma_X^2(0) \end{bmatrix}$$

即：

$$\sigma_X^2(k) = 0.5^{k-1}[0.5(k-1)\frac{64}{27}\sigma_\epsilon^2 - 0.25k\frac{80}{27}\sigma_\epsilon^2] = \frac{0.5^{k-1}(12k-32)\sigma_\epsilon^2}{27}$$

6. 解：

$$X_t = \mu + \phi_1 X_{t-1} + \cdots + \phi_p X_{t-p} + \epsilon_t$$

则有

$$EX_t = \frac{\mu}{A(1)}, A(z) = 1 - \sum_{i=1}^{p} \phi_i z^i$$

令

$$Y_t = X_t - \frac{\mu}{A(1)}$$

则

$$\sigma_Y^2(k) = \sigma_X^2(k)$$

$$\begin{aligned} X_t - \frac{\mu}{A(1)} &= \mu - \frac{\mu}{A(1)} + \sum_{i=1}^{p} \phi_i X_{t-i} + \epsilon_t \\ &= \frac{-\sum_{i=1}^{p} \phi_i \mu}{A(1)} + \sum_{i=1}^{p} \phi_i X_{t-i} + \epsilon_t \\ &= \sum_{i=1}^{p} \phi_i (X_{t-i} - \frac{\mu}{A(1)}) + \epsilon_t \end{aligned}$$

即

$$Y_t = \sum_{i=1}^{p} \phi_i Y_{t-i} + \epsilon_t$$

Yule-Walker 方程

$$\begin{bmatrix} \sigma_X^2(1) \\ \vdots \\ \sigma_X^2(p) \end{bmatrix} = \begin{bmatrix} \sigma_X^2(0) & \cdots & \sigma_X^2(p-1) \\ \vdots & \ddots & \vdots \\ \sigma_X^2(p-1) & \cdots & \sigma_X^2(0) \end{bmatrix} \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_p \end{bmatrix}$$

$$\sigma_X^2(0) = \sum_{i=1}^{p} \phi_i \sigma_X^2(i) + \sigma_\epsilon^2$$

下面将 Yule-Walker 方程重新排列为可由过程参数直接求解的形式

当 $p = 2$ 时

$$\begin{bmatrix} -1 & \phi_1 & \phi_2 \\ \phi_1 & \phi_2 - 1 & 0 \\ \phi_2 & \phi_1 & -1 \end{bmatrix} \begin{bmatrix} \sigma_X^2(0) \\ \sigma_X^2(1) \\ \sigma_X^2(2) \end{bmatrix} = \begin{bmatrix} -\sigma_\epsilon^2 \\ 0 \\ 0 \end{bmatrix}$$

当 $p = 3$ 时

$$\begin{bmatrix} -1 & \phi_1 & \phi_2 & \phi_3 \\ \phi_1 & \phi_2 - 1 & \phi_3 & 0 \\ \phi_2 & \phi_1 + \phi_3 & -1 & 0 \\ \phi_3 & \phi_2 & \phi_1 & -1 \end{bmatrix} \begin{bmatrix} \sigma_X^2(0) \\ \sigma_X^2(1) \\ \sigma_X^2(2) \\ \sigma_X^2(3) \end{bmatrix} = \begin{bmatrix} -\sigma_\epsilon^2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

当 $p = 4$ 时

$$\begin{bmatrix} -1 & \phi_1 & \phi_2 & \phi_3 & \phi_4 \\ \phi_1 & \phi_2 - 1 & \phi_3 & \phi_4 & 0 \\ \phi_2 & \phi_1 + \phi_3 & \phi_4 - 1 & 0 & 0 \\ \phi_3 & \phi_2 + \phi_4 & \phi_1 & -1 & 0 \\ \phi_4 & \phi_3 & \phi_2 & \phi 1 & -1 \end{bmatrix} \begin{bmatrix} \sigma_X^2(0) \\ \sigma_X^2(1) \\ \sigma_X^2(2) \\ \sigma_X^2(3) \\ \sigma_X^2(4) \end{bmatrix} = \begin{bmatrix} -\sigma_\epsilon^2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

对任意 p，其方程组的系数矩阵满足类似的形式：

$$\begin{bmatrix} -1 & \phi_1 & \cdots & \phi_{p-1} & \phi_p \\ \phi_1 & \phi_2 - 1 & \cdots & \phi_p & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_{p-1} & \phi_{p-2} + \phi_p & \cdots & -1 & 0 \\ \phi_p & \phi_{p-1} & \cdots & \phi_1 & -1 \end{bmatrix} \begin{bmatrix} \sigma_X^2(0) \\ \sigma_X^2(1) \\ \vdots \\ \sigma_X^2(p-1) \\ \sigma_X^2(p) \end{bmatrix} = \begin{bmatrix} -\sigma_\epsilon^2 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$