2023 秋季本科时间序列

第 2 次作业答案

10 月 8 日

1. 由作业 1 已知，$P(X=i,Y=j)=\frac{i+j}{n^2(n+1)}$

   则 X 的边缘分布为 $P(X=i)=\sum_{j=1}^{n}P(X=i,Y=j)=\sum_{j=1}^{n}\frac{i+j}{n^2(n+1)}=\frac{ni+\sum_{j=1}^{n}j}{n^2(n+1)}=\frac{i}{n(n+1)}+\frac{1}{2n}$

   则 Y 的边缘分布为 $P(Y=j)=\sum_{i=1}^{n}P(X=i,Y=j)=\sum_{i=1}^{n}\frac{i+j}{n^2(n+1)}=\frac{nj+\sum_{i=1}^{n}i}{n^2(n+1)}=\frac{j}{n(n+1)}+\frac{1}{2n}$

   $P(X=i,Y=j)\neq P(X=i)\times P(Y=j)$，故 X 和 Y 不独立

2. 由题已知，$f_x(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}},f_y(y)=\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}$

   由 X 和 Y 相互独立

   故 $f(x,y)=f_x(x)\times f_y(y)=\frac{1}{2\pi}e^{-\frac{x^2}{2}-\frac{y^2}{2}}$

$$
\begin{aligned}
P(Z<t)&=P\left(\frac{X}{Y}<t\right)\\
&=P(X-tY<0,Y>0)+P(X-tY>0,Y<0)\\
&=2P(X-tY<0,Y>0)
\end{aligned}
$$

   其中 $P(X-tY<0,Y>0)=\int_0^\infty\int_{-\infty}^{ty}f(x,y)\mathrm{d}x\mathrm{d}y$

   令 $u=\frac{x}{y}$

$$
\begin{aligned}
P(X-tY<0,Y>0) &= \int_{-\infty}^{z}\int_{0}^{\infty} yf(x,y)\mathrm{d}y\mathrm{d}u \\
&= \frac{1}{2\pi}\int_{-\infty}^{z}\int_{0}^{\infty} ye^{-\frac{(u^2+1)y^2}{2}}\mathrm{d}y\mathrm{d}u \\
&= \frac{1}{2\pi}\int_{-\infty}^{z}\frac{1}{u^2+1}\left(-e^{-\frac{y^2}{2}}\Big|_{0}^{+\infty}\right)\mathrm{d}u \\
&= \frac{1}{2\pi}\int_{-\infty}^{z}\frac{1}{u^2+1}\mathrm{d}u \\
&= \frac{1}{2\pi}\left(\arctan z+\frac{\pi}{2}\right)
\end{aligned}
$$

故 $P(Z<t)=2P(X-tY<0,Y>0)=\frac{1}{\pi}\arctan z+\frac{1}{2}$，$f_z(z)=\frac{1}{\pi}\frac{1}{(1+z^2)}$

$\mathbb{E}(z)=\int_{-\infty}^{\infty} zf_z(z)\mathrm{d}z=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{z}{(1+z^2)}\mathrm{d}z=\frac{1}{2\pi}\ln(z^2+1)\Big|_{-\infty}^{+\infty}$，式子不收敛，故 Z 没有一阶矩

$\mathbb{E}(z^2)=\int_{-\infty}^{\infty} z^2f_z(z)\mathrm{d}z=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{z^2}{(1+z^2)}\mathrm{d}z=\frac{1}{\pi}\int_{-\infty}^{\infty}\left(1-\frac{1}{1+z^2}\right)\mathrm{d}z=\frac{1}{\pi}(z-\arctan z)\Big|_{-\infty}^{\infty}$

$\mathbb{E}(z^2)$ 不收敛，故 Z 没有二阶矩

故 Z 不存在一阶矩和二阶矩

3. $X\sim U\left[a,b\right],f_x(x)=\frac{1}{b-a}$

   故 $\mathbb{E}(X)=\frac{a+b}{2},\mathbb{E}(X^2)=\int_{-\infty}^{\infty}f(x)x^2\mathrm{d}x=\frac{a^2+b^2+ab}{3}$

   由

$$
\begin{cases}
\mathbb{E}(X)=\frac{a+b}{2}=\frac{1}{N}\sum_{i=1}^{N}x_i=\overline{x}\\
\mathbb{E}(X^2)=\frac{a^2+ab+b^2}{3}=\frac{1}{N}\sum_{i=1}^{N}x_i^2
\end{cases}
$$

   可得

$$
\begin{cases}
\hat{a}=\overline{x}-\sqrt{3}\times\sqrt{\frac{N-1}{N}}S\\
\hat{b}=\overline{x}+\sqrt{3}\times\sqrt{\frac{N-1}{N}}S
\end{cases}
$$

   由 $\{X_t\}$ 是独立同分布序列

   可知 $\bar{x}\xrightarrow{a.s.}\mathbb{E}(X)=\frac{a+b}{2},S\xrightarrow{a.s.}Var(X)=\frac{(a-b)^2}{12}$

   故 $\hat{a}\xrightarrow{a.s.}\mathbb{E}(X)-\sqrt{3}Var(X)=a,\hat{b}\xrightarrow{a.s.}\mathbb{E}(X)+\sqrt{3}Var(X)=b$

   故该矩估计具有一致性

4. 样本对应的似然函数为：

$$
L(\lambda|X)=\lambda^n e^{-\lambda\sum_{i=1}^{n}x_i}
$$

两端同时取自然对数，可得：

$$\ln L(\lambda) = n\ln(\lambda) - \lambda \sum_{i=1}^{n} x_i$$

等式两边取一阶导数，可得

$$\frac{\mathrm{d}\ln L(\lambda)}{\mathrm{d}\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i$$

等式两边取二阶导数，可得

$$\frac{\mathrm{d}^2 \ln L(\lambda)}{\mathrm{d}\lambda^2} = -\frac{n}{\lambda^2} < 0$$

则似然函数在一阶导数等于 0 时取最大值：

$$\frac{n}{\lambda} - \sum_{i=1}^{n} x_i = 0 \Longrightarrow \hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i}$$

X 为指数分布，则

$$\mathbb{E}(x) = \int_{-\infty}^{\infty} x f(x)\mathrm{d}x = \int_{-\infty}^{\infty} x\lambda e^{-\lambda x}\mathrm{d}x = \frac{1}{\lambda} \Longrightarrow \lambda = \frac{1}{\mathbb{E}(X)}$$

由 $\{X_t\}$ 是独立同分布序列

可知 $\bar{x} \xrightarrow{a.s.} \mathbb{E}(X) \Longrightarrow \frac{1}{\bar{x}} \xrightarrow{a.s.} \frac{1}{\mathbb{E}(X)}$

故 $\hat{\lambda} = \frac{n}{\sum_{i=1}^{n} x_i} \xrightarrow{a.s.} \frac{1}{\mathbb{E}(X)} = \lambda$

故该估计具有一致性

5. (a) $\mathbb{E}(\hat{S}_N^2) = \frac{1}{N}\mathbb{E}\left[\sum_{i=1}^{N}(X_i-\mu)^2)\right] = \frac{1}{N}\sum_{i=1}^{N}\mathbb{E}(X_i-\mu)^2 = \frac{1}{N}\times N\times\sigma^2 = \sigma^2, \therefore \mathbb{E}(\hat{S}_N^2) = \sigma^2$

$$\lim_{N\to\infty} \hat{S}_N^2 = \lim_{N\to\infty}\left[\frac{1}{N}\sum_{i=1}^{N}(X_i-\mu)^2)\right] = \lim_{N\to\infty}\left[\frac{\sum_{i=1}^{N} X_i^2}{N} - 2\mu\frac{\sum_{i=1}^{N} X_i}{N} + \mu^2\right]$$

由大数定律可以得出，$\frac{1}{N}\sum_{i=1}^{N} X_i \xrightarrow{a.s.} \mathbb{E}(X) = \mu, \frac{1}{N}\sum_{i=1}^{N} X_i^2 \xrightarrow{a.s.} \mathbb{E}(X^2)$

$\lim_{N\to\infty} \hat{S}_N^2 = \mathbb{E}(X^2) - \mu^2 = \sigma^2 \Longrightarrow \hat{S}_N^2 \xrightarrow{a.s.} \sigma^2$

(b)

$$\mathbb{E}(\hat{\sigma}_N^2) = \frac{1}{N-1}\mathbb{E}\left[\sum_{i=1}^{N}\left(X_i - \frac{1}{N}\sum_{i=1}^{N} X_i\right)^2\right] = \frac{1}{N-1}\mathbb{E}\left[\sum_{i=1}^{N} X_i^2 - 2\hat{\mu}_N\sum_{i=1}^{N} X_i + N\hat{\mu}_N^2\right]$$

$\mathbb{E}\left(\hat{\mu}_N\right) = \mathbb{E}\left(\frac{\sum X_i}{N}\right) = \frac{N\times\mu}{N} = \mu$

$\mathbb{E}(\hat{\mu}_N^2) = Var(\hat{\mu}_N) + [\mathbb{E}(\hat{\mu}_N)]^2 = \frac{1}{N^2}\sum_{i=1}^N Var(X_i) + \mu^2 = \frac{1}{N}\sigma^2 + \mu^2$

$$\begin{aligned}
\mathbb{E}(\hat{\sigma}_N^2) &= \frac{1}{N-1}\left[\mathbb{E}\left(\sum_{i=1}^N X_i^2\right) - 2N\mathbb{E}(\hat{\mu}_N^2) + N\mathbb{E}(\hat{\mu}_N^2)\right] \\
&= \frac{1}{N-1}\left[\sum_{i=1}^N \mathbb{E}(X_i^2) - N\mathbb{E}(\hat{\mu}_N^2)\right] \\
&= \frac{1}{N-1}\left[N(\sigma^2+\mu^2) - \sigma^2 - N\mu^2\right] \\
&= \frac{1}{N-1}\times(N-1)\times\sigma^2 \\
&= \sigma^2
\end{aligned}$$

$$\begin{aligned}
\lim_{N\to\infty}\hat{\sigma}_N^2 &= \lim_{N\to\infty}\frac{1}{N-1}\left(\sum_{i=1}^N X_i^2 - 2\hat{\mu}_N\sum_{i=1}^N X_i + N\hat{\mu}_N^2\right) \\
&= \lim_{N\to\infty}\frac{1}{N-1}\sum_{i=1}^N\left(X_i^2 - N\hat{\mu}_N^2\right) \\
&= \lim_{N\to\infty}\frac{N}{N-1}\frac{\sum_{i=1}^N X_i^2}{N} - \lim_{N\to\infty}\frac{N}{N-1}\hat{\mu}_N^2
\end{aligned}$$

由 $\frac{\sum_{i=1}^N X_i^2}{N} \xrightarrow{a.s.} \mathbb{E}(X^2), \hat{\mu}_N^2 \xrightarrow{a.s.} \mu$

可得 $\lim_{N\to\infty}\hat{\sigma}_N^2 = \mathbb{E}(X^2) - \mu^2 = \sigma^2$