2023秋季本科时间序列

第1次作业答案

9月22日

1. 根据分布函数定义，$F(x)$可写为

   $F(x)=P(X\le x)=\int_{-\infty}^{x}f(t)dt=\int_{-\infty}^{0}f(t)dt+\int_{0}^{x}f(t)dt$

   取$a,b\in[-\infty,+\infty]$, $\quad x+\triangle x\in[a,b]$

   令$g(x)=\int_{0}^{x}f(t)dt$, $\quad$则$g(x+\triangle x)-g(x)=\int_{x}^{x+\triangle x}f(t)dt$

   由$f(x)$在R上可积，得$f(x)$在R上有界，即$|f(x)|\le M$

   则$|g(x+\triangle x)-g(x)|\le\int_{x}^{x+\triangle x}|f(t)|\,dt\le M\triangle x$， $\therefore\lim_{\triangle x\to 0}(g(x+\triangle x)-g(x))=0$

   $\therefore\int_{0}^{x}f(t)dt$在$[a,b]$上连续， $\therefore F(x)=\int_{-\infty}^{x}f(t)dt=\int_{-\infty}^{0}f(t)dt+\int_{0}^{x}f(t)dt$连续

2. (a) $\because$ 协方差矩阵为半正定矩阵且为对称矩阵,方差为非负数

   $\therefore ad-bc\ge 0, b=c, a>0, b>0$

   (b) i. 由题已知，$Var(\mathrm{X})=E(X^2)-[E(\mathrm{X})]^2$

   $\because$ $[X,Y]^{\intercal}$ 服从2元联合正态分布

   $\therefore f(x,y)=\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2}-2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}+\frac{(y-\mu_y)^2}{\sigma_y^2}\right]}$

   则

$$
\begin{aligned}
E(x^2)&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x^2f(x,y)dxdy\\
&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{x^2}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2}-2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}+\frac{(y-\mu_y)^2}{\sigma_y^2}\right]}dxdy
\end{aligned}
$$

上式指数部分可改写为

$$-\frac{1}{2}(\rho\frac{x-\mu_x}{\sigma_x\sqrt{1-\rho^2}}-\frac{y-\mu_y}{\sigma_y\sqrt{1-\rho^2}})^2-\frac{(x-\mu_x)^2}{2\sigma_x^2}$$

则

$$\begin{aligned}E(X^2)&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{x^2}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}e^{\left[-\frac{1}{2}(\rho\frac{x-\mu_x}{\sigma_x\sqrt{1-\rho^2}}-\frac{y-\mu_y}{\sigma_y\sqrt{1-\rho^2}})^2-\frac{(x-\mu_x)^2}{2\sigma_x^2}\right]}dxdy\\&=\int_{-\infty}^{\infty}\frac{x^2}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}}dx\int_{-\infty}^{\infty}e^{\left[-\frac{1}{2}(\rho\frac{x-\mu_x}{\sigma_x\sqrt{1-\rho^2}}-\frac{y-\mu_y}{\sigma_y\sqrt{1-\rho^2}})^2\right]}dy\end{aligned}$$

对积分变量Y作如下替换：

$$t=(\rho\frac{x-\mu_x}{\sigma_x\sqrt{1-\rho^2}}-\frac{y-\mu_y}{\sigma_y\sqrt{1-\rho^2}})$$

则上式可化为：

$$\begin{aligned}E(x^2)&=\int_{-\infty}^{\infty}\frac{x^2}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}}dx\int_{-\infty}^{\infty}e^{-\frac{t^2}{2}}\sigma_y\sqrt{1-\rho^2}dt\\&=\int_{-\infty}^{\infty}\frac{x^2}{\sqrt{2\pi}\sigma_x}e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}}dx\\&=\frac{\sigma_x^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty}z^2e^{-\frac{z^2}{2}}dz\\&=\frac{\sigma_x^2}{\sqrt{2\pi}}\times\sqrt{2\pi}\\&=\sigma_x^2=a\end{aligned}$$

所以$Var(X)=E(x^2)-(E(X))^2=a$

ii. 由题已知，$Var(Y)=E(Y^2)-[E(Y)]^2$

$\because [X,Y]^\intercal$ 服从2元联合正态分布

$\therefore f(x,y)=\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2}-2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}+\frac{(y-\mu_y)^2}{\sigma_y^2}\right]}$

则

$$\begin{aligned}E(Y^2)&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}y^2f(x,y)dxdy\\&=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{y^2}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2}-2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}+\frac{(y-\mu_y)^2}{\sigma_y^2}\right]}dxdy\end{aligned}$$

上式指数部分可改写为

$$-\frac{1}{2}(\rho\frac{y-\mu_y}{\sigma_y\sqrt{1-\rho^2}}-\frac{x-\mu_x}{\sigma_x\sqrt{1-\rho^2}})^2-\frac{(y-\mu_y)^2}{2\sigma_y^2}$$

则

$$E(Y^2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{y^2}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}e^{\left[-\frac{1}{2}(\rho\frac{y-\mu_y}{\sigma_y\sqrt{1-\rho^2}}-\frac{x-\mu_x}{\sigma_x\sqrt{1-\rho^2}})^2-\frac{(y-\mu_y)^2}{2\sigma_y^2}\right]}dxdy$$

$$=\int_{-\infty}^{\infty}\frac{y^2}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}e^{-\frac{(y-\mu_y)^2}{2\sigma_y^2}}dy\int_{-\infty}^{\infty}e^{\left[-\frac{1}{2}(\rho\frac{y-\mu_y}{\sigma_y\sqrt{1-\rho^2}}-\frac{x-\mu_x}{\sigma_x\sqrt{1-\rho^2}})^2\right]}dx$$

对积分变量X作如下替换：

$$t=(\rho\frac{y-\mu_y}{\sigma_y\sqrt{1-\rho^2}}-\frac{x-\mu_x}{\sigma_x\sqrt{1-\rho^2}})$$

则上式可化为：

$$E(Y^2)=\int_{-\infty}^{\infty}\frac{y^2}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}e^{-\frac{(y-\mu_y)^2}{2\sigma_y^2}}dy\int_{-\infty}^{\infty}e^{-\frac{t^2}{2}}\sigma_x\sqrt{1-\rho^2}dt$$

$$=\int_{-\infty}^{\infty}\frac{y^2}{\sqrt{2\pi}\sigma_y}e^{-\frac{(y-\mu_y)^2}{2\sigma_y^2}}dy$$

$$=\frac{\sigma_y^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty}z^2e^{-\frac{z^2}{2}}dz$$

$$=\frac{\sigma_y^2}{\sqrt{2\pi}}\times\sqrt{2\pi}$$

$$=\sigma_y^2=d$$

所以$Var(Y)=E(Y^2)-(E(Y))^2=d$

iii. cov(x,y)=E(XY)-E(X)E(Y)=E(XY)

$$E(XY)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}xyf(x,y)dxdy$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{xy}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_x)^2}{\sigma_x^2}-2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}+\frac{(y-\mu_y)^2}{\sigma_y^2}\right]}dxdy$$

上式指数部分可改写为:

$$\frac{(x-\mu_x)^2}{\sigma_x^2}-2\rho\frac{(x-\mu_x)(y-\mu_y)}{\sigma_x\sigma_y}+\frac{(y-\mu_y)^2}{\sigma_y^2}=(\frac{x-\mu_x}{\sigma_x}-\rho\frac{(y-\mu_y)}{\sigma_y})^2+(\sqrt{1-\rho^2}\frac{y-\mu_y}{\sigma_y})^2$$

对积分变量X作如下替换:

$$u=\frac{1}{\sqrt{1-\rho^2}}(\frac{x-\mu_x}{\sigma_x}-\rho\frac{(y-\mu_y)}{\sigma_y})$$

$$v=\frac{y-\mu_y}{\sigma_y}$$

则

$$
\begin{aligned}
cov(x,y) &= \frac{\sigma_x\sigma_y}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(uv\sqrt{1-\rho^2}+\rho v^2)e^{-\frac{1}{2}(u^2+v^2)}dudv \\
&= \frac{\sigma_x\sigma_y}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(uv\sqrt{1-\rho^2}e^{-\frac{1}{2}(u^2+v^2)}+\rho v^2 e^{-\frac{1}{2}(u^2+v^2)})dudv \\
&= \frac{\sigma_x\sigma_y}{2\pi}\times 2\pi\times\rho = \rho\sigma_x\sigma_y \\
&= b = c
\end{aligned}
$$

(c)

$$
\begin{aligned}
f_x(x) &= \int_{-\infty}^{\infty}\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}e^{\left[-\frac{1}{2}(\rho\frac{x-\mu_x}{\sigma_x\sqrt{1-\rho^2}}-\frac{y-\mu_y}{\sigma_y\sqrt{1-\rho^2}})^2-\frac{(x-\mu_x)^2}{2\sigma_x^2}\right]}dy \\
&= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}}\int_{-\infty}^{\infty}e^{\left[-\frac{1}{2}(\rho\frac{x-\mu_x}{\sigma_x\sqrt{1-\rho^2}}-\frac{y-\mu_y}{\sigma_y\sqrt{1-\rho^2}})^2\right]}dy \\
&= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}}\int_{-\infty}^{\infty}e^{-\frac{t^2}{2}}\sigma_y\sqrt{1-\rho^2}dt \\
&= \frac{1}{\sqrt{2\pi}\sigma_x}e^{-\frac{(x-\mu_x)^2}{2\sigma_x^2}} \\
&= \frac{1}{\sqrt{2\pi a}}e^{-\frac{x^2}{2a}}
\end{aligned}
$$

$$
\begin{aligned}
f_y(y) &= \int_{-\infty}^{\infty}\frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}e^{\left[-\frac{1}{2}(\rho\frac{y-\mu_y}{\sigma_y\sqrt{1-\rho^2}}-\frac{x-\mu_x}{\sigma_x\sqrt{1-\rho^2}})^2-\frac{(y-\mu_y)^2}{2\sigma_y^2}\right]}dx \\
&= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}e^{-\frac{(y-\mu_y)^2}{2\sigma_y^2}}\int_{-\infty}^{\infty}e^{\left[-\frac{1}{2}(\rho\frac{y-\mu_y}{\sigma_y\sqrt{1-\rho^2}}-\frac{x-\mu_x}{\sigma_x\sqrt{1-\rho^2}})^2\right]}dx \\
&= \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}}e^{-\frac{(y-\mu_y)^2}{2\sigma_y^2}}\int_{-\infty}^{\infty}e^{-\frac{t^2}{2}}\sigma_x\sqrt{1-\rho^2}dt \\
&= \frac{1}{\sqrt{2\pi}\sigma_y}e^{-\frac{(y-\mu_y)^2}{2\sigma_y^2}} \\
&= \frac{1}{\sqrt{2\pi d}}e^{-\frac{y^2}{2d}}
\end{aligned}
$$

3. $(a)\because \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{i+j}{C}=1$

$\therefore C=\sum_{i=1}^{n}\sum_{j=1}^{n}(i+j)=\sum_{i=1}^{n}\sum_{j=1}^{n}i+\sum_{i=1}^{n}\sum_{j=1}^{n}j=\frac{n(n+1)}{2}\times n+\frac{n(n+1)}{2}\times n=n^2(n+1)=n^3+n^2$

(b) $\because X, Y$在题目中具有对称性

$$\therefore E(X) = E(Y) = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{i(i+j)}{n^2(n+1)} = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{j(i+j)}{n^2(n+1)}$$

$$= \frac{1}{n^2(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{n}(i^2+ij)$$

$$= \frac{1}{n^2(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{n}i^2 + \frac{1}{n^2(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{n}ij$$

$$= \frac{1}{n^2(n+1)} \times n \times \frac{n(n+1)(2n+1)}{6} + \frac{1}{n^2(n+1)} \times (\frac{n(n+1)}{2})^2$$

$$= \frac{(2n+1)}{6} + \frac{n+1}{4}$$

$$= \frac{7n+5}{12}$$

$$E(X^2) = E(Y^2) = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{i^2(i+j)}{n^2(n+1)}$$

$$= \frac{1}{n^2(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{n}(i^3+i^2j)$$

$$= \frac{1}{n^2(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{n}i^3 + \frac{1}{n^2(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{n}i^2j$$

$$= \frac{1}{n^2(n+1)} \times n \times (\frac{n(n+1)}{2})^2 + \frac{1}{n^2(n+1)} \times n \times \frac{n(n+1)}{2} \times \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(n+1)}{4} + \frac{(n+1)(2n+1)}{6} = \frac{(n+1)(5n+1)}{12}$$

$$Var(X) = Var(Y) = E(X^2) - (E(X))^2 = \frac{(n+1)(5n+1)}{12} - (\frac{7n+5}{12})^2 = \frac{(n-1)(11n+13)}{144}$$

$$E(XY) = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{ij(i+j)}{n^2(n+1)} = \frac{1}{n^2(n+1)}\sum_{i=1}^{n}\sum_{j=1}^{n}i^2j + ij^2 = \frac{(n+1)(2n+1)}{6}$$

$$Cov(X,Y) = E(XY) - E(X)E(Y) = \frac{(n+1)(2n+1)}{6} - (\frac{7n+5}{12})^2 = \frac{-(n-1)^2}{144}$$

4. (a)

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{i+j+k}{C} = 1$$

$$C = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}(i+j+k) = \frac{3n^3(n+1)}{2}$$

(b)

$$
\begin{aligned}
E(XYZ) &= \frac{1}{C}\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n}(i+j+k)ijk \\
&= \frac{1}{C}\times 3\times \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} i^2jk \\
&= \frac{(n+1)^2(2n+1)}{12}
\end{aligned}
$$