2022 秋季本科时间序列

第 9 次作业答案

12 月 16 日

1. (a) 若 $\mathbf{\Phi}$ 的特征值模长小于 1，则 $\boldsymbol{X}_t$ 平稳。

$$|\mathbf{\Phi}-\lambda \boldsymbol{I}| = \begin{vmatrix} 0.3-\lambda & 0.5 \\ -0.2 & 1-\lambda \end{vmatrix} = \lambda^2 - 1.3\lambda + 0.4 = 0$$

解得 $\lambda_1 = 0.8, \lambda_2 = 0.5$, 故满足平稳性要求。$\mathbb{E}\boldsymbol{X}_t = \boldsymbol{c} + \mathbf{\Phi}\mathbb{E}\boldsymbol{X}_{t-1}$, 则有

$$\mathbb{E}\boldsymbol{X}_t = (\boldsymbol{I}-\mathbf{\Phi})^{-1}\boldsymbol{c} = \begin{bmatrix} 0 & 5 \\ -2 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

(b) 把 $\lambda_1 = 0.5$ 代入得，

$$(\mathbf{\Phi}-\lambda \boldsymbol{I})\boldsymbol{x}_1 = \begin{bmatrix} -0.5 & 0.5 \\ -0.2 & 0.2 \end{bmatrix}\boldsymbol{x}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \boldsymbol{x}_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

同理，当 $\lambda = 0.8$ 时，解得 $\boldsymbol{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$，故 $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix}, \boldsymbol{A}^{-1} = \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{5}{3} \end{bmatrix}$

$$\mathbf{\Phi}^i = \boldsymbol{A}\mathbf{\Lambda}^i\boldsymbol{A}^{-1} = \begin{bmatrix} 5 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0.5^i & 0 \\ 0 & 0.8^i \end{bmatrix}\begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ -\frac{2}{3} & \frac{5}{3} \end{bmatrix} = \begin{bmatrix} -\frac{2}{3}\times 0.8^i + \frac{5}{3}\times 0.5^i & \frac{5}{3}\times 0.8^i - \frac{5}{3}\times 0.5^i \\ -\frac{2}{3}\times 0.8^i + \frac{2}{3}\times 0.5^i & \frac{5}{3}\times 0.8^i - \frac{2}{3}\times 0.5^i \end{bmatrix}$$

$\boldsymbol{X}_t$ 的 $MA(\infty)$ 展开为 $\boldsymbol{X}_t = (\boldsymbol{I}+\mathbf{\Phi}+\cdots+\mathbf{\Phi}^n)\boldsymbol{c} + \sum_{i=0}^{n}\mathbf{\Phi}^i\epsilon_{t-i}, n\to\infty$，则

$$\begin{aligned}\mathrm{var}\boldsymbol{X}_t &= \sum_{i=0}^{\infty}\mathbf{\Phi}^i\mathbf{\Omega}\mathbf{\Phi}^{Ti} \\ &= \sum_{i=0}^{\infty}\boldsymbol{A}\mathbf{\Lambda}^i\boldsymbol{A}^{-1}\mathbf{\Omega}(\boldsymbol{A}^{-1})^T\mathbf{\Lambda}^{Ti}\boldsymbol{A}^T \\ &= \begin{bmatrix} \frac{250}{27} & \frac{295}{27} \\ \frac{295}{27} & \frac{448}{27} \end{bmatrix}\end{aligned}$$

(c) 设 $\boldsymbol{\Omega} = \boldsymbol{A}\boldsymbol{D}\boldsymbol{A}^T$，其中 $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}, \boldsymbol{D} = \begin{bmatrix} b & 0 \\ 0 & c \end{bmatrix}$，解得 $a = 1, b = 1, c = 3$，令 $\boldsymbol{P} = \boldsymbol{A}\boldsymbol{D}^{\frac{1}{2}}$，则 $\boldsymbol{P}^T = \boldsymbol{D}^{\frac{1}{2}}\boldsymbol{A}^T$，则 $\boldsymbol{\Omega} = \boldsymbol{P}\boldsymbol{P}^T$

$$\boldsymbol{P} = \boldsymbol{A}\boldsymbol{D}^{\frac{1}{2}} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & \sqrt{3} \end{bmatrix}, \boldsymbol{\Omega} = \begin{bmatrix} 1 & 0 \\ 1 & \sqrt{3} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & \sqrt{3} \end{bmatrix}$$

$$\boldsymbol{X}_{t+s} = (\boldsymbol{I} - \boldsymbol{\Phi})^{-1}\boldsymbol{c} + \sum_{i=0}^{\infty} \boldsymbol{\Phi}^i \boldsymbol{P}\boldsymbol{u}_{t+s-i}, \frac{\partial \boldsymbol{X}_{t+s}}{\partial \boldsymbol{u}_{kt}} = \boldsymbol{\Phi}^s \boldsymbol{p}_k$$

则

$$\frac{\partial \boldsymbol{X}_{1t+s}}{\partial \boldsymbol{u}_{1t}} = 0.8^s$$
$$\frac{\partial \boldsymbol{X}_{2t+s}}{\partial \boldsymbol{u}_{1t}} = 0.8^s$$
$$\frac{\partial \boldsymbol{X}_{1t+s}}{\partial \boldsymbol{u}_{2t}} = \sqrt{3}(\frac{5}{3} \times 0.8^s - \frac{5}{3} \times 0.5^s)$$
$$\frac{\partial \boldsymbol{X}_{2t+s}}{\partial \boldsymbol{u}_{2t}} = \sqrt{3}(\frac{5}{3} \times 0.8^s - \frac{2}{3} \times 0.5^s)$$

2. $X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \varepsilon_t$, 其中 $\phi_1 = 1.1, \phi_2 = -0.3$

$$\begin{bmatrix} X_{t+1} \\ X_t \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} X_t \\ X_{t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \end{bmatrix}$$

对任意的 $s \geq 2$

$$\begin{bmatrix} \hat{X}_{t+s|t} \\ \hat{X}_{t+s-1|t} \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \hat{X}_{t+s-1|t} \\ \hat{X}_{t+s-2|t} \end{bmatrix} = \boldsymbol{A}\begin{bmatrix} \hat{X}_{t+s-1|t} \\ \hat{X}_{t+s-2|t} \end{bmatrix} = \boldsymbol{A}^s\begin{bmatrix} \hat{X}_{t|t} \\ \hat{X}_{t-1|t} \end{bmatrix} = \boldsymbol{A}^s\begin{bmatrix} X_t \\ X_{t-1} \end{bmatrix}$$

则

$$det(\boldsymbol{A} - \lambda\boldsymbol{I}) = det\begin{bmatrix} \phi_1 - \lambda & \phi_2 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 - \phi_1\lambda - \phi_2 = \lambda^2 - 1.1\lambda + 0.3 = 0$$

解得 $\lambda_1 = 0.6, \lambda_2 = 0.5$

$$(\boldsymbol{A} - \lambda_1\boldsymbol{I})\boldsymbol{C}_1 = \begin{bmatrix} \phi_1 - \lambda_1 & \phi_2 \\ 1 & -\lambda_1 \end{bmatrix}\boldsymbol{C}_1 = 0 \quad \boldsymbol{C}_1 = \begin{bmatrix} \lambda_1 \\ 1 \end{bmatrix}$$

$$(\boldsymbol{A} - \lambda_2\boldsymbol{I})\boldsymbol{C}_2 = \begin{bmatrix} \phi_1 - \lambda_2 & \phi_2 \\ 1 & -\lambda_2 \end{bmatrix}\boldsymbol{C}_2 = 0 \quad \boldsymbol{C}_2 = \begin{bmatrix} \lambda_2 \\ 1 \end{bmatrix}$$

因此

$$\boldsymbol{C} = \begin{bmatrix} \boldsymbol{C}_1 & \boldsymbol{C}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}$$

$$\boldsymbol{C}^{-1}=\frac{\boldsymbol{C}^*}{det\boldsymbol{C}}=\frac{1}{\lambda_1-\lambda_2}\begin{bmatrix}1 & \lambda_2\\ -1 & \lambda_1\end{bmatrix}$$

$$\boldsymbol{A}^s=\boldsymbol{C}\begin{bmatrix}\lambda_1^s & 0\\ 0 & \lambda_2^s\end{bmatrix}\boldsymbol{C}^{-1}$$

$$\begin{bmatrix}\hat{X}_{t+s|t}\\ \hat{X}_{t+s-1|t}\end{bmatrix}=\boldsymbol{A}^s\begin{bmatrix}X_t\\ X_{t-1}\end{bmatrix}=\frac{1}{\lambda_1-\lambda_2}\begin{bmatrix}\lambda_1^{s+1}-\lambda_2^{s+1} & -\lambda_1^{s+1}\lambda_2+\lambda_1\lambda_2^{s+1}\\ \lambda_1^s-\lambda_2^s & -\lambda_1^s\lambda_2+\lambda_1\lambda_2^s\end{bmatrix}\begin{bmatrix}X_t\\ X_{t-1}\end{bmatrix}$$

因此 $\hat{X}_{t+s|t}=\frac{1}{\lambda_1-\lambda_2}[(\lambda_1^{s+1}-\lambda_2^{s+1})X_t+(-\lambda_1^s\lambda_2+\lambda_1\lambda_2^s)X_{t-1}]$，其中 $\lambda_1=0.6,\lambda_2=0.5$

又 $|\lambda_1|<1,|\lambda_2|<1$

所以 $\lim_{s\to\infty}\lambda_1^s=0,\lim_{s\to\infty}\lambda_2^s=0$

$$\lim_{s\to\infty}\boldsymbol{A}^s=\lim_{s\to\infty}\boldsymbol{C}\begin{bmatrix}\lambda_1^s & 0\\ 0 & \lambda_2^s\end{bmatrix}\boldsymbol{C}^{-1}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

$$\lim_{s\to\infty}\begin{bmatrix}\hat{X}_{t+s|t}\\ \hat{X}_{t+s-1|t}\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

即 $\lim_{s\to\infty}\hat{X}_{t+s|t}=0$

3.

$$X_t=\varepsilon_t+\theta\varepsilon_{t-1}+\psi\varepsilon_{t-2}$$

$$X_{t+s}=\varepsilon_{t+s}+\theta\varepsilon_{t+s-1}+\psi\varepsilon_{t+s-2}$$

当 $s=1$ 时，$X_{t+1}=\varepsilon_{t+1}+\theta\varepsilon_t+\psi\varepsilon_{t-1}\quad \hat{X}_{t+1|t}=\theta\varepsilon_t+\psi\varepsilon_{t-1}$

当 $s=2$ 时，$X_{t+2}=\varepsilon_{t+2}+\theta\varepsilon_{t+1}+\psi\varepsilon_t\quad \hat{X}_{t+2|t}=\psi\varepsilon_t$

当 $s=3$ 时，$X_{t+3}=\varepsilon_{t+3}+\theta\varepsilon_{t+2}+\psi\varepsilon_{t+1}\quad \hat{X}_{t+3|t}=0$

当 $s\geq 3$ 时，不包含 t 及之前信息，有 $\hat{X}_{t+s|t}=0$

综上可得

$$\hat{X}_{t+s|t}=\begin{cases}\theta\varepsilon_t+\psi\varepsilon_{t-1} & s=1\\ \psi\varepsilon_t & s=2\\ 0 & s\geq 3\end{cases}$$