2022 秋季本科时间序列

第 7 次作业答案

12 月 1 日

1. (a) i.

$$\begin{aligned}\boldsymbol{P}_X\boldsymbol{\xi} &= \boldsymbol{X}(\boldsymbol{X}^\intercal\boldsymbol{X})^{-1}\boldsymbol{X}^\intercal\boldsymbol{X}\boldsymbol{a}\\ &= \boldsymbol{X}\boldsymbol{I}\boldsymbol{a}\\ &= \boldsymbol{X}\boldsymbol{a}\\ &= \boldsymbol{\xi}\end{aligned}$$

ii.

$$\begin{aligned}(\boldsymbol{X}\boldsymbol{a})^\intercal(\boldsymbol{I}-\boldsymbol{P}_X)\boldsymbol{\zeta} &= \boldsymbol{a}^\intercal\boldsymbol{X}^\intercal(\boldsymbol{\zeta}-\boldsymbol{X}(\boldsymbol{X}^\intercal\boldsymbol{X})^{-1}\boldsymbol{X}^\intercal\boldsymbol{\zeta})\\ &= \boldsymbol{a}^\intercal\boldsymbol{X}^\intercal\boldsymbol{\zeta}-\boldsymbol{a}^\intercal\boldsymbol{X}^\intercal\boldsymbol{X}(\boldsymbol{X}^\intercal\boldsymbol{X})^{-1}\boldsymbol{X}^\intercal\boldsymbol{\zeta}\\ &= \boldsymbol{a}^\intercal\boldsymbol{X}^\intercal\boldsymbol{\zeta}-\boldsymbol{a}^\intercal\boldsymbol{I}\boldsymbol{X}^\intercal\boldsymbol{\zeta}\\ &= 0\end{aligned}$$

(b) i. 由 OLS 估计系数表达式可知，$\boldsymbol{Y}$ 对 $\boldsymbol{W}$ 回归的系数为 $(\boldsymbol{W}^\intercal\boldsymbol{W})^{-1}\boldsymbol{W}^\intercal\boldsymbol{Y}$，而 $\boldsymbol{Z}$ 对 $\boldsymbol{W}$ 回归的系数为 $(\boldsymbol{W}^\intercal\boldsymbol{W})^{-1}\boldsymbol{W}^\intercal\boldsymbol{Z}$，故两组回归对应的残差向量分别为

$$\begin{aligned}(\boldsymbol{I}-\boldsymbol{W}(\boldsymbol{W}^\intercal\boldsymbol{W})^{-1}\boldsymbol{W}^\intercal)\boldsymbol{Y} = (\boldsymbol{I}-\boldsymbol{P}_W)\boldsymbol{Y} = \tilde{\boldsymbol{Y}}\\ (\boldsymbol{I}-\boldsymbol{W}(\boldsymbol{W}^\intercal\boldsymbol{W})^{-1}\boldsymbol{W}^\intercal)\boldsymbol{Z} = (\boldsymbol{I}-\boldsymbol{P}_W)\boldsymbol{Z} = \tilde{\boldsymbol{Z}}\end{aligned}$$

ii. 由 $\hat{\boldsymbol{e}} = \boldsymbol{Y}-\boldsymbol{X}\hat{\beta} = \boldsymbol{Y}-\boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y} = (\boldsymbol{I}-\boldsymbol{P}_X)\boldsymbol{Y}$，由 (a) 的结论可得 $\hat{\boldsymbol{e}}$ 与 $\boldsymbol{X}$ 的每个列向量相互垂直，故垂直于 $\boldsymbol{Z}$ 与 $\boldsymbol{W}(\boldsymbol{Z}^T\hat{e} = \boldsymbol{W}^T\hat{e} = 0)$。

iii. $\boldsymbol{P}_W\hat{\boldsymbol{e}} = \boldsymbol{W}(\boldsymbol{W}^\intercal\boldsymbol{X})^{-1}\boldsymbol{W}^\intercal\hat{\boldsymbol{e}}$，由 ii 可知 $\boldsymbol{W}^\intercal\boldsymbol{e} = 0$，故得所证。

iv. 只需要说明 $\hat{\boldsymbol{\delta}}$ 正好满足回归 $\tilde{\boldsymbol{Y}} = \tilde{\boldsymbol{Z}}\boldsymbol{\delta}+\boldsymbol{u}$ 的 OLS 系数估计所需满足的一阶条件 $\tilde{\boldsymbol{Z}}^\intercal\tilde{Y}-\tilde{\boldsymbol{Z}}^\intercal\tilde{\boldsymbol{Z}}\boldsymbol{\delta} = 0$。为此，首先注意到 $\hat{\boldsymbol{e}} = \boldsymbol{Y}-\boldsymbol{X}\hat{\boldsymbol{\beta}} = \boldsymbol{Y}-\boldsymbol{Z}\hat{\boldsymbol{\delta}}-\boldsymbol{W}\hat{\boldsymbol{\theta}}$，两端同时乘以 $\boldsymbol{I}-\boldsymbol{P}_W$ 可得

$$(\boldsymbol{I}-\boldsymbol{P}_W)\hat{\boldsymbol{e}} = (\boldsymbol{I}-\boldsymbol{P}_W)\boldsymbol{Y}-(\boldsymbol{I}-\boldsymbol{P}_W)\boldsymbol{Z}\hat{\boldsymbol{\delta}}-(\boldsymbol{I}-\boldsymbol{P}_W)\boldsymbol{W}\hat{\boldsymbol{\theta}}$$

由 i–iii 结论可知，上式可以进一步写为

$$\hat{\boldsymbol{e}} = \tilde{\boldsymbol{Y}}-\tilde{\boldsymbol{Z}}\hat{\boldsymbol{\delta}}$$

再在两端乘以 $\tilde{\boldsymbol{Z}}^\intercal$ 可得

$$0 = \tilde{\boldsymbol{Z}}\hat{\boldsymbol{e}} = \tilde{\boldsymbol{Z}}^\intercal\tilde{\boldsymbol{Y}}-\tilde{\boldsymbol{Z}}^\intercal\tilde{\boldsymbol{Z}}\hat{\boldsymbol{\delta}} = \tilde{\boldsymbol{Z}}^\intercal(\tilde{\boldsymbol{Y}}-\tilde{\boldsymbol{Z}}\hat{\boldsymbol{\delta}})$$

即得所证。

(c) 设 $\boldsymbol{Y}=\boldsymbol{I}\boldsymbol{\alpha}+\boldsymbol{X}\boldsymbol{\beta}+\boldsymbol{e}$，同理 FWL 定理定义回归残差 $\tilde{\boldsymbol{Y}}$ 和 $\tilde{\boldsymbol{X}}$

$$\tilde{\boldsymbol{Y}}=(\boldsymbol{I}-\boldsymbol{P}_I)\boldsymbol{Y}$$
$$\tilde{\boldsymbol{X}}=(\boldsymbol{I}-\boldsymbol{P}_I)\boldsymbol{X}$$

则可推出

$$\tilde{\boldsymbol{Y}}=(\boldsymbol{I}-\boldsymbol{P}_I)\boldsymbol{Y}=\boldsymbol{Y}-\overline{\boldsymbol{Y}}$$
$$\tilde{\boldsymbol{X}}=(\boldsymbol{I}-\boldsymbol{P}_I)\boldsymbol{X}=\boldsymbol{X}-\overline{\boldsymbol{X}}$$

于是可得 $\tilde{\boldsymbol{Y}}=\tilde{\boldsymbol{X}_t}^T\boldsymbol{\beta}+\boldsymbol{u}_t$ 的 OLS 估计所得 $\hat{\boldsymbol{\beta}}$ 等于 $\boldsymbol{Y}=\boldsymbol{\alpha}+\boldsymbol{X}_t^T\boldsymbol{\beta}+\boldsymbol{e}_t$ 中 $\hat{\boldsymbol{\beta}}$ 值

2. (a)

$$\hat{\boldsymbol{Y}}=\boldsymbol{Z}\hat{\boldsymbol{\delta}}+\boldsymbol{W}\hat{\boldsymbol{\theta}}=\boldsymbol{X}\hat{\boldsymbol{\beta}}=\boldsymbol{X}(\boldsymbol{X}^\intercal\boldsymbol{X})^{-1}\boldsymbol{X}^\intercal\boldsymbol{Y}=\boldsymbol{P_X}\boldsymbol{Y}$$
$$\bar{\boldsymbol{Y}}=\boldsymbol{Z}\hat{\boldsymbol{\delta}}=\boldsymbol{Z}(\boldsymbol{Z}^\intercal\boldsymbol{Z})^{-1}\boldsymbol{Z}^\intercal\boldsymbol{Y}=\boldsymbol{P_Z}\boldsymbol{Y}$$

(b)

$$\begin{aligned}(\hat{\boldsymbol{Y}}-\bar{\boldsymbol{Y}})^\intercal\boldsymbol{Z}&=[(\boldsymbol{P_X}-\boldsymbol{P_Z})\boldsymbol{Y}]^\intercal\boldsymbol{Z}\\&=\boldsymbol{Y}^\intercal(\boldsymbol{P_X}-\boldsymbol{P_Z})^\intercal\boldsymbol{Z}\\&=\boldsymbol{Y}^\intercal[(\boldsymbol{I}-\boldsymbol{P_Z})-(\boldsymbol{I}-\boldsymbol{P_X})]^\intercal\boldsymbol{Z}\\&=\boldsymbol{Y}^\intercal(\boldsymbol{I}-\boldsymbol{P_Z})^\intercal\boldsymbol{Z}-\boldsymbol{Y}^\intercal(\boldsymbol{I}-\boldsymbol{P_X})^\intercal\boldsymbol{Z}\end{aligned}$$

由第 1 题可知：$(\boldsymbol{I}-\boldsymbol{P_Z})^\intercal\boldsymbol{Z}=0,[(\boldsymbol{I}-\boldsymbol{P_X})\boldsymbol{Y}]^\intercal\boldsymbol{Z}=0$
因此 $(\hat{\boldsymbol{Y}}-\bar{\boldsymbol{Y}})^\intercal\boldsymbol{Z}=0$
故得证 $(\hat{\boldsymbol{Y}}-\bar{\boldsymbol{Y}})^\intercal\bar{\boldsymbol{Y}}=(\hat{\boldsymbol{Y}}-\bar{\boldsymbol{Y}})^\intercal\boldsymbol{P_Z}\boldsymbol{Y}=(\hat{\boldsymbol{Y}}-\bar{\boldsymbol{Y}})^\intercal\boldsymbol{Z}(\boldsymbol{Z}^\intercal\boldsymbol{Z})^{-1}\boldsymbol{Z}^\intercal\boldsymbol{Y}=0$

(c) 已知 $(\hat{\boldsymbol{Y}}-\bar{\boldsymbol{Y}})^\intercal\bar{\boldsymbol{Y}}=0$
要证 $\hat{\boldsymbol{Y}}^\intercal\hat{\boldsymbol{Y}}\geq\bar{\boldsymbol{Y}}^\intercal\bar{\boldsymbol{Y}}$ 即证 $\hat{\boldsymbol{Y}}^\intercal\hat{\boldsymbol{Y}}\geq\hat{\boldsymbol{Y}}^\intercal\bar{\boldsymbol{Y}}$

$$\begin{aligned}\hat{\boldsymbol{Y}}^\intercal(\hat{\boldsymbol{Y}}-\bar{\boldsymbol{Y}})&=(\hat{\boldsymbol{Y}}-\bar{\boldsymbol{Y}}+\bar{\boldsymbol{Y}})^\intercal(\hat{\boldsymbol{Y}}-\bar{\boldsymbol{Y}})\\&=(\hat{\boldsymbol{Y}}-\bar{\boldsymbol{Y}})^\intercal(\hat{\boldsymbol{Y}}-\bar{\boldsymbol{Y}})+\bar{\boldsymbol{Y}}^\intercal(\hat{\boldsymbol{Y}}-\bar{\boldsymbol{Y}})\\&=(\hat{\boldsymbol{Y}}-\bar{\boldsymbol{Y}})^\intercal(\hat{\boldsymbol{Y}}-\bar{\boldsymbol{Y}})\geq 0\end{aligned}$$

当且仅当 $\hat{\boldsymbol{Y}}-\bar{\boldsymbol{Y}}\neq 0$ 时，$\hat{\boldsymbol{Y}}^\intercal\hat{\boldsymbol{Y}}>\bar{\boldsymbol{Y}}^\intercal\bar{\boldsymbol{Y}}$ 严格成立

(d) 由 c 可知，当解释变量只有 $\boldsymbol{Z}$ 时，估计结果为 $\bar{\boldsymbol{Y}}$，当增加解释变量 $\boldsymbol{W}$ 时，估计结果为 $\hat{\boldsymbol{Y}}$

$$\hat{\boldsymbol{R}}^2-\bar{\boldsymbol{R}}^2=\frac{\hat{\boldsymbol{Y}}^\intercal\hat{\boldsymbol{Y}}}{\boldsymbol{Y}^\intercal\boldsymbol{Y}}-\frac{\bar{\boldsymbol{Y}}^\intercal\bar{\boldsymbol{Y}}}{\boldsymbol{Y}^\intercal\boldsymbol{Y}}=\frac{\hat{\boldsymbol{Y}}^\intercal\hat{\boldsymbol{Y}}-\bar{\boldsymbol{Y}}^\intercal\bar{\boldsymbol{Y}}}{\boldsymbol{Y}^\intercal\boldsymbol{Y}}\geq 0$$

因此 $\boldsymbol{Y}$ 对 $\boldsymbol{X}=[X_1,\ldots,X_n]$ 回归中增加解释变量 X_{n+1} 时，R^2 单调递增