2022 秋季本科时间序列

# 第 5 次作业答案

11 月 6 日

1. 解：

由 $(1-\phi\mathscr{L})X_t = Y_t$ 有 $X_t = \frac{Y_t}{1-\phi\mathscr{L}} = \sum_{i=0}^{\infty}\phi^i Y_{t-i}$

则有

$$\eta_k = \operatorname{cov}\left(\sum_{i=0}^{\infty}\phi^i Y_{t+k-i}, \sum_{i=0}^{\infty}\phi^i Y_{t-i}\right) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\phi^i\phi^j\gamma(k+j-i)$$

$\gamma_k$ 存在一致上界，即 $|\gamma_k| \le \gamma_0$，则有：

$$|\eta_k| \le \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\phi^i\phi^j\gamma_0 = \frac{\gamma_0}{(1-\phi)^2}$$

故 $|\eta_k| < \infty$，故 $X_t$ 为平稳序列。

2. 解：

$$\begin{aligned}
X_t &= \frac{1}{1-\frac{\mathscr{L}}{z_1}}\cdots\frac{1}{1-\frac{\mathscr{L}}{z_p}}\epsilon_t \\
&= \sum_{i=0}^{\infty}\left(\frac{\mathscr{L}}{z_i}\right)^i \frac{1}{1-\frac{\mathscr{L}}{z_0}}\cdots\frac{1}{1-\frac{\mathscr{L}}{z_p}}\epsilon_t \\
&= \sum_{i=0}^{\infty}\left(\frac{1}{z_1}\right)^i \frac{1}{1-\frac{\mathscr{L}}{z_2}}\cdots\frac{1}{1-\frac{\mathscr{L}}{z_p}}\epsilon_{t-i} \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\left(\frac{1}{z_1}\right)^i\left(\frac{1}{z_0}\right)^j \frac{1}{1-\frac{\mathscr{L}}{z_3}}\cdots\frac{1}{1-\frac{\mathscr{L}}{z_p}}\epsilon_{t-i-j} \\
&= \cdots \\
&= \sum_{i_1=0}^{\infty}\sum_{i_2=0}^{\infty}\cdots\sum_{i_p=0}^{\infty}\left(\frac{1}{z_1}\right)^{i_1}\cdots\left(\frac{1}{z_p}\right)^{i_p}\epsilon_{t-\left(i_1+\cdots+i_p\right)} \\
&= a_0\epsilon_t + a_1\epsilon_{t-1} + \cdots
\end{aligned}$$

则 $i_1 = \cdots = i_p = 0$ 时有 $a_0 = 1$,
$i_1 + \cdots + i_p = 1$ 时有 $a_1 = \sum_{k=1}^{p} \frac{1}{z_k}$,
$i_1 + \cdots + i_p = 1$ 时有 $a_2 = \sum_{i=1}^{p} \sum_{j=1}^{p} \frac{1}{z_i z_j}$,
由 $z_1, \cdots, z_p$ 为 $1 - \phi_1 z - \cdots - \phi_p z^p = 0$ 的根有 $\frac{1}{z_1} \cdots \frac{1}{z_p}$ 为 $z^p - \phi_1 z^{p-1} - \cdots - \phi_p = 0$ 的根, 则由推广的韦达定理不难发现

$$a_1 = \sum_{i=1}^{p} \frac{1}{z_i} = \phi_1$$

3. 解：

由 $X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t$ 有

$$\begin{bmatrix} \phi_1 & \phi_2 & 0 \\ \phi_2 & \phi_1 & -1 \\ 1 & -\phi_1 & -\phi_2 \end{bmatrix} \begin{bmatrix} \gamma(0) \\ \gamma(1) \\ \gamma(2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sigma_\epsilon^2 \end{bmatrix}$$

其中系数矩阵的行列式为

$$\begin{vmatrix} \phi_1 & \phi_2 & 0 \\ \phi_2 & \phi_1 & -1 \\ 1 & -\phi_1 & -\phi_2 \end{vmatrix} = (\phi_1 + \phi_2 - 1)(\phi_2 - \phi_1 - 1)(\phi_2 + 1)$$

$AR(2)$ 过程 $(1 - \phi_1 \mathscr{L} - \phi_2 \mathscr{L}^2) X_t = \epsilon_t$ 特征多项式 $1 - \phi_1 z - \phi_2^2 z = 0$
则有两特征根 $z_1, z_2$

$$z_1 z_2 = -\frac{1}{\phi_2}, z_1 + z_2 = -\frac{\phi_1}{\phi_2}$$

即可得 $\phi_1, \phi_2$ 关于 $z_1, z_2$ 表达式

$$\phi_1 = \frac{1}{z_1} + \frac{1}{z_2}, \phi_2 = -\frac{1}{z_1 z_2}$$

平稳则特征根 $|z_1| > 1, |z_2| > 1$,
则显然

$$(\phi_1 + \phi_2 - 1)(\phi_2 - \phi_1 - 1)(\phi_2 + 1) \neq 0$$

故 $\Phi$ 可逆，即 $AR(2)$ 过程平稳则系数矩阵 $\Phi$ 可逆,
若使得 $AR(2)$ 过程平稳则由 $|z_1| > 1, |z_2| > 1$ 有

$$\phi_1 = \frac{1}{z_1} + \frac{1}{z_2} \in (0, 2), \phi_2 = -\frac{1}{z_1 z_2} \in (-1, 1)$$

同时满足 $1 - \phi_1 - \phi_2 > 0$ 及 $1 + \phi_1 - \phi_2 > 0$

4. 解：

$$X_t = 0.9X_{t-1} - 0.2X_{t-2} + \epsilon_t$$

分别在等式两边同乘 $X_t, X_{t-1}, X_{t-2}$ 并取期望有

$$\begin{aligned}
\gamma(0) &= 0.9\gamma(1) - 0.2\gamma(2) + \sigma_\epsilon^2 \\
\gamma(1) &= 0.9\gamma(0) - 0.2\gamma(1) \\
\gamma(2) &= 0.9\gamma(1) - 0.2\gamma(0)
\end{aligned}$$

转换为矩阵形式有

$$\begin{bmatrix} 0 \\ 0 \\ \sigma_\epsilon^2 \end{bmatrix} = \begin{bmatrix} 0.9 & -1.2 & 0 \\ -0.2 & 0.9 & -1 \\ 1 & -0.9 & 0.2 \end{bmatrix} \begin{bmatrix} \gamma(0) \\ \gamma(1) \\ \gamma(2) \end{bmatrix}$$

解得

$$\gamma(0) = \frac{50}{21}\sigma_\epsilon^2, \gamma(1) = \frac{25}{14}\sigma_\epsilon^2, \gamma(2) = \frac{95}{84}\sigma_\epsilon^2$$

在 $X_{t+k} = \phi_1 X_{t+k-1} - \phi_2 X_{t+k-2} + \epsilon_{t+k}$ 两边同乘 $X_t$ 得

$$\gamma_X^2(k) = \phi_1\gamma_X^2(k-1) - \phi_2\gamma_X^2(k-2)$$

即有

$$\begin{bmatrix} \gamma(k) \\ \gamma(k-1) \end{bmatrix} = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \gamma(k-1) \\ \gamma(k-2) \end{bmatrix}$$

代入本题参数

$$\Lambda = \begin{bmatrix} \phi_1 & \phi_2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0.9 & -0.2 \\ 1 & 0 \end{bmatrix}$$

求特征值

$$\mathbb{P}(\lambda) = |\lambda I - \Lambda| = \begin{bmatrix} \lambda - 0.9 & 0.2 \\ -1 & \lambda \end{bmatrix} = \lambda^2 - 0.9\lambda + 0.2 = 0$$

解得

$$\lambda_1 = 0.4, \lambda_2 = 0.5$$

则有

$$\begin{bmatrix} \gamma(k) \\ \gamma(k-1) \end{bmatrix} = \begin{bmatrix} 0.4 & 0.5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0.4^{k-1} & \\ & 0.5^{k-1} \end{bmatrix} \begin{bmatrix} 0.4 & 0.5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \gamma(1) \\ \gamma(0) \end{bmatrix}$$

故有

$$\gamma(k) = \frac{25}{21}(7 \times 0.5^k - 5 \times 0.4^k)$$

5. 解：

由 $AR(2)$ 过程 $X_t = 2\phi X_{t-1} - \phi^2 X_{t-2} + \epsilon_t$ 有

$$\begin{bmatrix} 0 \\ 0 \\ \sigma_\epsilon^2 \end{bmatrix} = \begin{bmatrix} 2\phi & -\phi^2 - 1 & 0 \\ -\phi^2 & 2\phi & -1 \\ 1 & -2\phi & \phi^2 \end{bmatrix} \begin{bmatrix} \gamma(0) \\ \gamma(1) \\ \gamma(2) \end{bmatrix}$$

故有

$$\begin{bmatrix} \gamma(0) \\ \gamma(1) \\ \gamma(2) \end{bmatrix} = \begin{bmatrix} \frac{1}{1-3\phi^2+3\phi^4-\phi^6}\sigma_\epsilon^2 \\ \frac{2\phi}{1-3\phi^2+3\phi^4-\phi^6}\sigma_\epsilon^2 \\ \frac{\phi^2}{1-3\phi^2+3\phi^4-\phi^6}\sigma_\epsilon^2 \end{bmatrix}$$

同时由

$$\begin{bmatrix} \gamma(k) \\ \gamma(k-1) \end{bmatrix} = \begin{bmatrix} 2\phi & -\phi^2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \gamma(k-1) \\ \gamma(k-2) \end{bmatrix}$$

令

$$\Lambda = \begin{bmatrix} 2\phi & -\phi^2 \\ 1 & 0 \end{bmatrix}$$

$$\mathbb{P}(\lambda) = |\lambda I - \Lambda| = \lambda^2 - 2\phi\lambda + \phi^2$$

因此，

$$\lambda_1 = \lambda_2 = \phi$$

若尔当分解，

$$\Lambda = \begin{bmatrix} 2\phi & -\phi^2 \\ 1 & 0 \end{bmatrix} = T \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} T^{-1}$$

$$\Lambda T = \begin{bmatrix} \phi & 1 \\ 0 & \phi \end{bmatrix}$$

令 $T = [T_1, T_2]$，有

$$[\Lambda T_1, \Lambda T_2] = [T_1, T_2] \begin{bmatrix} \phi & 1 \\ 0 & \phi \end{bmatrix}$$

故可得 $T_1 = [\phi, 1]^T, T_2 = [1, 0]^T$，因此

$$\Lambda = \begin{bmatrix} \phi & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \phi & 1 \\ 0 & \phi \end{bmatrix} \begin{bmatrix} \phi & 1 \\ 1 & 0 \end{bmatrix}^{-1}$$

数学归纳法易推得

$$\begin{bmatrix} \phi & 1 \\ 0 & \phi \end{bmatrix}^k = \begin{bmatrix} \phi^k & k\phi^{k-1} \\ 0 & \phi^k \end{bmatrix}$$

故 $\Lambda^k$ 有

$$\Lambda^k = \begin{bmatrix} \phi & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \phi^k & k\phi^{k-1} \\ 0 & \phi^k \end{bmatrix} \begin{bmatrix} \phi & 1 \\ 1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} (k+1)\phi^k & -k\phi^{k+1} \\ k\phi^{k-1} & (1-k)\phi^k \end{bmatrix}$$

则

$$\begin{bmatrix} \gamma(k) \\ \gamma(k-1) \end{bmatrix} = \Lambda^k \begin{bmatrix} \gamma(1) \\ \gamma(0) \end{bmatrix} = \phi^{k-1} \begin{bmatrix} (k+1)\phi & -k\phi^2 \\ k & (1-k)\phi \end{bmatrix} \begin{bmatrix} \gamma(1) \\ \gamma(0) \end{bmatrix}$$

6. 解：

由 $X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t + \theta\epsilon_{t-1}$ 有：

$$\begin{cases} \gamma(0) = \phi_1\gamma(1) + \phi_2\gamma(2) + \sigma_\epsilon^2 \\ \gamma(1) = \phi_1\gamma(0) + \phi_2\gamma(1) + \theta\sigma_\epsilon^2 \\ \gamma(2) = \phi_1\gamma(1) + \phi_2\gamma(0) \end{cases}$$

转换为矩阵乘积形式为：

$$\begin{bmatrix} \sigma_\epsilon^2 \\ \theta\sigma_\epsilon^2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -\phi_1 & -\phi_2 \\ -\phi_1 & 1-\phi_2 & 0 \\ -\phi_2 & -\phi_1 & 1 \end{bmatrix} \begin{bmatrix} \gamma(0) \\ \gamma(1) \\ \gamma(2) \end{bmatrix}$$

若 $\phi_1 = 0.9, \phi_2 = -0.2, \theta = 0.5, \sigma_\epsilon^2 = 1$，则有：

$$\begin{bmatrix} \gamma(0) \\ \gamma(1) \\ \gamma(2) \end{bmatrix} = \begin{bmatrix} \frac{65}{21} \\ \frac{115}{42} \\ \frac{155}{84} \end{bmatrix}$$

则有

$$\begin{bmatrix} \gamma(k) \\ \gamma(k-1) \end{bmatrix} = \begin{bmatrix} 0.4 & 0.5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0.4^{k-1} & \\ & 0.5^{k-1} \end{bmatrix} \begin{bmatrix} 0.4 & 0.5 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} \gamma(1) \\ \gamma(0) \end{bmatrix}$$

最终可得

$$\gamma(k) = 15 \times 0.5^k - \frac{250}{21} \times 0.4^k$$

7. 解：

由

$$\sigma_X^2 = \mathbb{E}\left[(X_t - \mathbb{E}X_t)^2\right]$$

$$\hat{\sigma}_X^2 = \frac{1}{X}\sum_{t=1}^{T}(X_t - \mathbb{E}X_t)^2 = \frac{1}{T}\sum_{t=1}^{T} X_t^2$$

故有 $\mathbb{E}X_t = 0$，同时根据假设 $U$ 与 $\epsilon_t$ 独立有 $\mathbb{E}U + \mathbb{E}\epsilon_t = 0$，则有：

$$\begin{aligned}
\mathbb{E}\hat{\sigma}_X^2 &= \frac{1}{T}\mathbb{E}\left(\sum_{t=1}^{T} X_t^2\right) \\
&= \frac{1}{T}\mathbb{E}\left[(U_1+\epsilon_1)^2 + \cdots + (U_t+\epsilon_t)^2\right] \\
&= \frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T} U_t^2 + \sum_{t=1}^{T}\epsilon_t^2 + 2\sum_{t=1}^{T} U_t\epsilon_t\right] \\
&= \frac{1}{T}\left[T\mathbb{E}U^2 + T\mathbb{E}\epsilon^2 + 2T\mathbb{E}U\mathbb{E}\epsilon\right] \\
&= \sigma_U^2 + \sigma_\epsilon^2
\end{aligned}$$

$$\begin{aligned}
\hat{\sigma}_X^2 &= \frac{1}{T}\sum_{t=1}^{T} X_t^2 \\
&= \frac{1}{T}\sum_{t=1}^{T}(U+\epsilon_t)^2 \\
&= \frac{1}{T}\left(TU^2 + 2U\sum_{t=1}^{T}\epsilon_t + \sum_{t=1}^{T}\epsilon_t^2\right) \\
&= U^2 + \frac{2U}{T}\sum_{t=1}^{T}\epsilon_t + \frac{1}{T}\sum_{t=1}^{T}\epsilon_t^2
\end{aligned}$$

当 $t \to \infty$ 时，由大数定律有 $\frac{1}{T}\sum_{t=1}^{T}\epsilon_t \to \mathbb{E}(\epsilon_t) = 0$，$\frac{1}{T}\sum_{t=1}^{T}\epsilon_t^2 \to \sigma_\epsilon^2$，因此

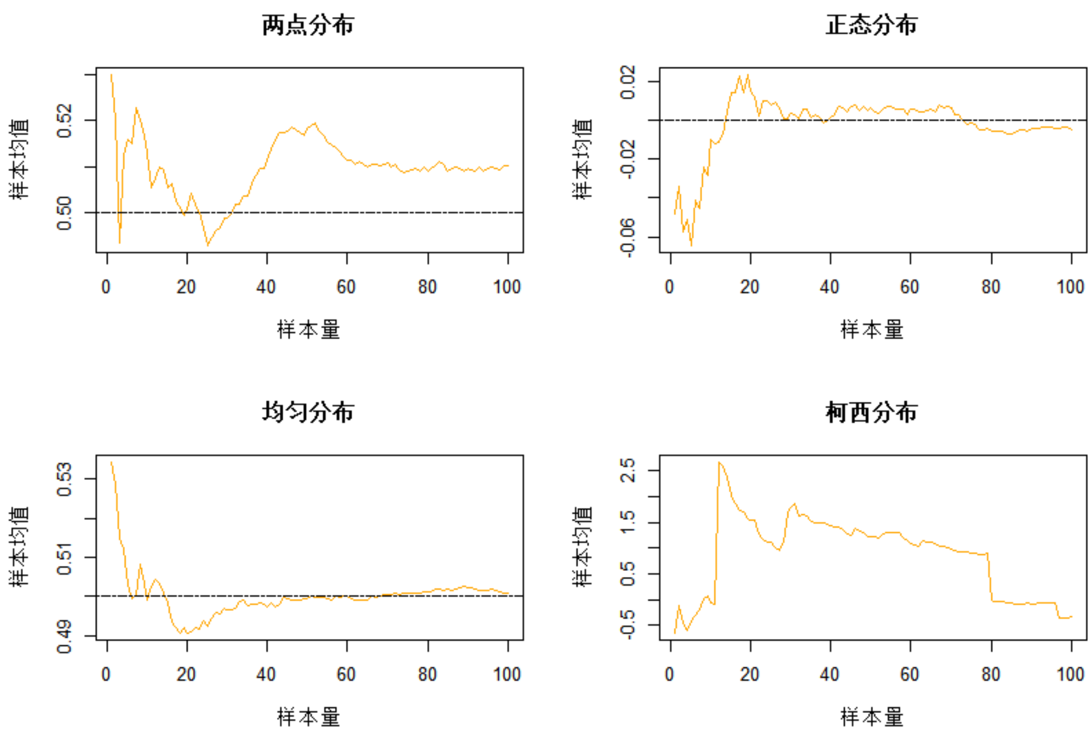
$$\lim_{t\to\infty}\hat{\sigma}_X^2 = U^2 + \sigma_\epsilon^2$$

8. (a) 代码如下：

```
library(tidyverse)
library(patchwork)
library(purrr)

```

```
N <- 100
K <- 100
set.seed(20221106)
bern_sq <- purrr::rbernoulli(N*K, p = 0.5) %>% as.numeric()
unif_sq <- runif(N*K, min = 0, max = 1)
norm_sq <- rnorm(N*K, sd = 1, mean = 0)
cauchy_sq <- rcauchy(N*K, location = 0, scale = 1)

sample_sq <- function(sq){
 mean <- vector(mode = "numeric", length = K)
 for (k in 1:K){
  sign_min <-  1
  sign_max <- k * 100
  sub_sample <- sq[sign_min:sign_max]
  mean[k] <- mean(sub_sample)
 }
 mean
}

opar <- par(no.readonly = T)
par(mfrow = c(2,2))
sq <- sample_sq(bern_sq)
plot(sq, type = "l", col = "orange",
xlab = "样本量", ylab = "样本均值",
main = "两点分布")
abline(h = 0.5, lty = 6)

sq <- sample_sq(norm_sq)
plot(sq, type = "l", col = "orange",
xlab = "样本量", ylab = "样本均值",
main = "正态分布")
abline(h = 0, lty = 6)

sq <- sample_sq(unif_sq)
plot(sq, type = "l", col = "orange",
```

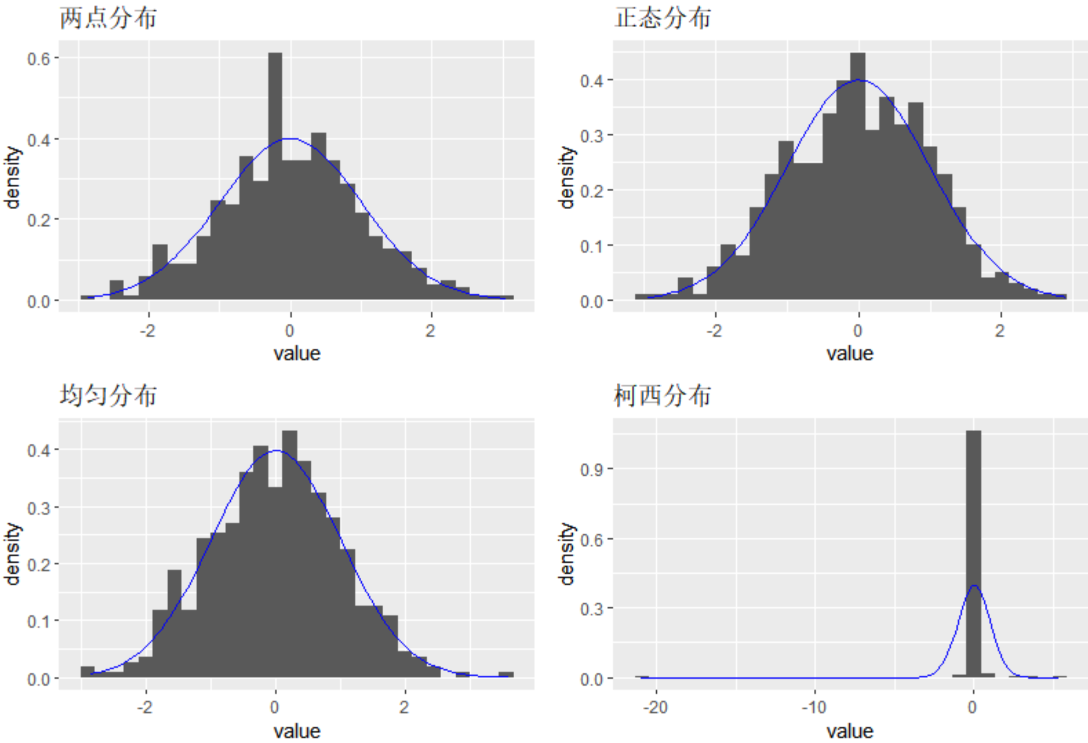
```r
xlab = "样本量", ylab = "样本均值",
main = "均匀分布")
abline(h = 0.5, lty = 6)

sq <- sample_sq(cauchy_sq)
plot(sq, type = "l", col = "orange",
xlab = "样本量", ylab = "样本均值",
main = "柯西分布")
```




(b) 代码如下：

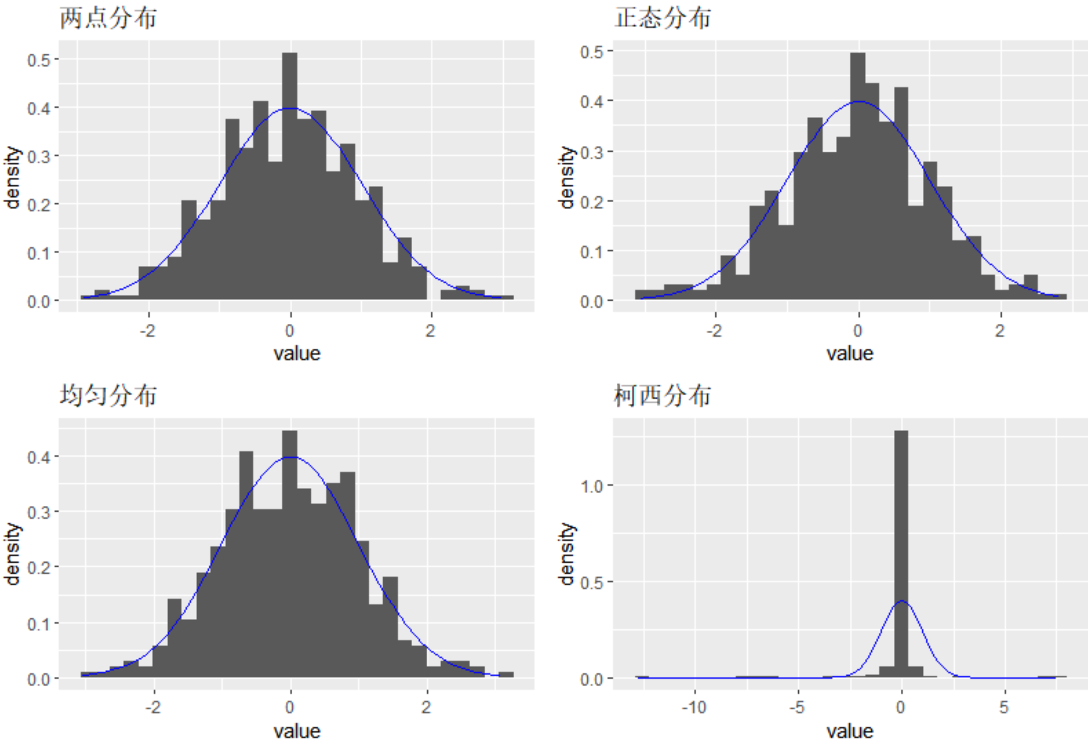
```r
N <- 100
K <- 500
bern_sq <- purrr::rbernoulli(N*K, p = 0.5) %>% as.numeric()
unif_sq <- runif(N*K, min = 0, max = 1)
norm_sq <- rnorm(N*K, sd = 1, mean = 0)
cauchy_sq <- rcauchy(N*K, location = 0, scale = 1)

sample_be_sq <- function(sq, K){
 mean <- vector(mode = "numeric",length = K)
```

```
 for (k in 1:K){
  sign_min <-  (k-1) *100 + 1
  sign_max <- k * 100
  sub_sample <- sq[sign_min:sign_max]
  mean[k] <- mean(sub_sample)
 }
 mean
}
sample_be_sq_norm <- function(sq) {
 k <- length(sq)
 mu_hat <- (1/k)*sum(sq, na.rm = T)
 sigma_hat <- (1/k)* sum((sq - mu_hat)^2)
 standard_sq <- (sq - mu_hat) * sqrt(1/ sigma_hat)
 return(standard_sq)
}
clt_plot <- function(sq,dis){
 tibble(sq = sq) %>%
 ggplot() +
 geom_histogram(aes(x=sq,y=..density..)) +
 stat_function(fun = dnorm, color = "blue") +
 labs(x = "value", y = "density")+
 ggtitle(dis)
}
sample_be_sq(bern_sq,500) %>% sample_be_sq_norm()
 %>% clt_plot(dis="两点分布")+
sample_be_sq(norm_sq,500) %>% sample_be_sq_norm()
 %>% clt_plot(dis="正态分布")+
sample_be_sq(unif_sq,500) %>% sample_be_sq_norm()
 %>% clt_plot(dis="均匀分布")+
sample_be_sq(cauchy_sq,500) %>% sample_be_sq_norm()
 %>% clt_plot(dis="柯西分布")
```




(c) 代码如下：

```
N <- 500
K <- 500
bern_sq <- purrr::rbernoulli(N*K, p = 0.5) %>% as.numeric()
unif_sq <- runif(N*K, min = 0, max = 1)
norm_sq <- rnorm(N*K, sd = 1, mean = 0)
cauchy_sq <- rcauchy(N*K, location = 0, scale = 1)

sample_be_sq(bern_sq,500) %>% sample_be_sq_norm()
 %>% clt_plot(dis="两点分布")+
sample_be_sq(norm_sq,500) %>% sample_be_sq_norm()
 %>% clt_plot(dis="正态分布")+
sample_be_sq(unif_sq,500) %>% sample_be_sq_norm()
 %>% clt_plot(dis="均匀分布")+
sample_be_sq(cauchy_sq,500) %>% sample_be_sq_norm()
 %>% clt_plot(dis="柯西分布")
```




(d) 对于柯西分布，中心极限定理并不适用，中心极限定理的含义是，如果样本量足够大，则变量均值的采样分布将近似于正态分布，而与该变量在总体中的分布无关。对于柯西分布，由于其期望并不存在，样本均值也不存在，故中心极限定理并不适用于柯西分布。