2022 秋季本科时间序列

第 3 次作业答案

10 月 4 日

1. (a) 证：

X_t 期望与方差分别为 μ, σ^2, 故样本均值 $\hat{\mu}$ 的期望为 $\mathbb{E}\left[\frac{1}{T}\sum_{t=1}^{T} X_t\right] = \frac{1}{T}\cdot T\mu = \mu$, 由独立同分布则 $\mathrm{cov}(X_s, X_t) = 0$, 则样本均值 $\hat{\mu}$ 的方差为

$$
\begin{aligned}
\mathrm{var}(\hat{\mu}) = \mathbb{E}\left(\hat{\mu}-\mu\right)^2 = \mathbb{E}\left(\frac{1}{T}\sum_{t=1}^{T} X_t - \mu\right)^2 \\
&= \frac{1}{T^2}\mathbb{E}\left(\sum_{t=1}^{T} X_t - T\mu\right)^2 \\
&= \frac{1}{T^2}\mathbb{E}\left[(X_1-\mu) + \ldots + (X_T-\mu)\right]^2 \\
&= \frac{1}{T^2}\mathbb{E}\left[(X_1-\mu)^2 + \ldots + (X_T-\mu)^2 + \sum_{t=1}^{T}\sum_{s=1,s\neq t}^{T}(X_s-\mu)(X_t-\mu)\right] \\
&= \frac{1}{T^2}\left(T\sigma^2 + 0\right) = \frac{\sigma^2}{T}
\end{aligned}
$$

则有

$$
\begin{aligned}
\mathbb{E}\hat{\sigma}^2 &= \frac{1}{T-1}\mathbb{E}\left[X_t^2 + \hat{\mu}^2 - 2X_t\hat{\mu}\right] \\
&= \frac{1}{T-1}\sum_{t=1}^{T}\left[\mathrm{var}(X) + (\mathbb{E}X)^2 + \mathrm{var}(\hat{\mu}) + (\mathbb{E}\hat{\mu})^2 - 2\mathbb{E}\left(\frac{X_t^2 + X_t\sum_{i=1,i\neq t}^{T} X_i}{T}\right)\right] \\
&= \frac{1}{T-1}\sum_{t=1}^{T}\left[\sigma^2 + \mu^2 + \frac{\sigma^2}{T} + \mu^2 - 2\frac{\sigma^2+\mu^2+(T-1)\mu^2}{T}\right] \\
&= \frac{1}{T-1}\sum_{t=1}^{T}\left(\frac{T-1}{T}\sigma^2\right) \\
&= \sigma^2
\end{aligned}
$$

无偏性得证.

(b) 证：

$$\hat{\sigma}^2 = \frac{1}{T-1}\sum_{t=1}^{T}\left(X_t - \hat{\mu}\right)^2 = \frac{1}{T-1}\sum_{t=1}^{T}\left(X_t^2 - 2\hat{\mu}X_t + \hat{\mu}^2\right) = \frac{1}{T-1}\sum_{t=1}^{T}\left(X_t^2 - \hat{\mu}^2\right)$$

由强大数定律 $\frac{1}{T}\sum_{t=1}^{T} X_t^2 \xrightarrow{a.s.} \mathbb{E}X_t^2 = \sigma^2 + \mu^2$

同时 $\hat{\mu}^2 \xrightarrow{a.s.} \mu^2, \frac{T}{T-1} \longrightarrow 1$

故 $\hat{\sigma}^2 = \frac{1}{T-1}\sum_{t=1}^{T}\left(X_t^2 - \hat{\mu}^2\right) \xrightarrow{a.s.} \sigma^2$

一致性得证.

2. 证：

$$\hat{\rho}_{XY} = \frac{\frac{1}{T-1}\sum_{t=1}^{T}(X_t - \bar{X})(Y_t - \bar{Y})}{\sqrt{\frac{1}{T-1}\sum_{t=1}^{T}(X_t - \bar{X})^2}\sqrt{\frac{1}{T-1}\sum_{t=1}^{T}(Y_t - \bar{Y})^2}}$$

其中 $\bar{X} = \frac{1}{T}\sum_{t=1}^{T} X_t, \bar{Y} = \frac{1}{T}\sum_{t=1}^{T} Y_t$,

设 $\mathbb{X} = \left(X_1 - \bar{X}, ..., X_T - \bar{X}\right)^T, \mathbb{Y} = \left(Y_1 - \bar{Y}, ..., Y_T - \bar{Y}\right)^T$,

由 Cauchy-Schwarz 不等式 $\left|x \cdot y\right| \le \|x\| \times \|y\|$,

$$\left|\hat{\rho}_{XY}\right| = \frac{|\mathbb{X}\mathbb{Y}|}{\|\mathbb{X}\| \cdot \|\mathbb{Y}\|} \le 1$$

得证.

3. 证：首先有 $\frac{1}{3}\sum_{t=1}^{3} X_t = 0$, 则有

$$\begin{aligned}\hat{\rho}(1) &= \frac{\hat{\sigma}_X^2(1)}{\hat{\sigma}_X^2(0)} \\ &= \frac{\frac{1}{T-1}\sum_{t=1}^{T-1}(X_t - \hat{\mu})(X_{t+1} - \hat{\mu})}{\frac{1}{T-1}(X_1^2 + X_2^2 + X_3^2)} \\ &= \frac{X_1X_2 + X_2X_3}{X_1^2 + X_2^2 + X_3^2} \\ &= \frac{-(X_1 + X_3)^2}{(X_1 + X_3)^2 + X_1^2 + X_3^2} \in (-1, 0)\end{aligned}$$

即 $\hat{\rho}(1)$ 取值不会超过 $[-1, 1]$, 设 $X_1 = 1, X_2 = -2, X_3 = 1$, 此时 $\hat{\rho}(1) = -\frac{2}{3}$.

4. (a) 解：递推 X_t 表达式,

$$X_1 = \mu + \rho X_0 + \epsilon_1 \sim N(\mu, \sigma_\epsilon^2)$$

$$X_2 = \mu + \rho(\mu + \rho X_0 + \epsilon_1) + \epsilon_2 = (1 + \rho)\mu + \rho^2 X_1 + \rho\epsilon_1 + \epsilon_2$$

...

$$X_t = (1 + \rho + ... + \rho^{t-1})\mu + \rho^t X_0 + (\epsilon_t + \rho\epsilon_{t-1} + ... + \rho^{t-1}\epsilon_1) = \sum_{k=0}^{t-1}\mu\rho^k + \rho^t X_0 + \sum_{k=1}^{t}\rho^{t-k}\epsilon_k$$

故

$$X_t \sim N\left(\sum_{k=0}^{t-1} \mu\rho^k, \sum_{k=0}^{t-1} \sigma_\epsilon^2 \rho^{2k}\right)$$

$$F_t = \mathbf{\Phi}\left(\frac{X - \sum_{k=0}^{t-1} \mu\rho^k}{\sqrt{\sum_{k=0}^{t-1} \sigma_\epsilon^2 \rho^{2k}}}\right)$$

(b) 解：

若 $\left|\rho\right| < 1, 1 + ... + \rho^{t-1} = \frac{\rho^{t-1}-1}{\rho-1}, \lim\limits_{t\to+\infty} \frac{\rho^{t-1}-1}{\rho-1} \to \frac{1}{1-\rho}$,

同时 $1 + \rho^2 + ... + \rho^{2t} = \frac{\rho^{2t}-1}{\rho^2-1}, \lim\limits_{t\to+\infty} \frac{\rho^{2t}-1}{\rho^2-1} \to \frac{1}{1-\rho^2}$,

即 $X_t \sim N\left(\frac{1}{1-\rho}\mu, \frac{1}{1-\rho^2}\sigma_\epsilon^2\right)$

(c) 解：

$\rho = 1, X_t \sim N(t\mu, t\sigma_\epsilon^2)$

$\rho = -1$, 若 t 为奇数, 则 $X_t \sim N(\mu, t\sigma_\epsilon^2)$, 若 t 为偶数, 则 $X_t \sim N(0, t\sigma_\epsilon^2)$

由于方差均不收敛，故 $\rho = \pm 1$ 时分布不收敛.

(d) 解：

$\left|\rho\right| > 1$ 时, $\lim\limits_{t\to+\infty} \mu \sum_{k=0}^{t-1} \rho^k \to \infty$, 同时 $\lim\limits_{t\to+\infty} \sigma_\epsilon^2 \sum_{k=0}^{t} \rho^{2k} \to \infty$, 显然此时 X_t 分布不收敛.