2022 秋季本科时间序列

第 1 次作业答案

9 月 16 日

1. (a) 解: 首先验证性质 1,$\mu(\varnothing) = 0, \mu(\Omega) = \mu([0,1]) = 1^2 = 1$, 故 μ 满足性质 1

   然后验证性质 2, $\forall \{F_i\}_{i=1}^{\infty} \in \Omega, \bigcup_{i=1}^{\infty} F_i = [a,b], F_i = [K_i, K_{i+1}]$

   且$\forall i, j = 1, \ldots, \infty, F_i \cap F_j = \varnothing$

$$
\begin{aligned}
\mu(\bigcup_{i=1}^{\infty} F_i) &= \mu([a,b]) \\
&= (b-a)^2 = (b + \ldots + K_{i+1} - K_i + \ldots + K_1 - a)^2 \\
&\geq \sum_{i=1}^{\infty} ((K_1 - a)^2 + \ldots + (K_{i+1} - K_i)^2 + \ldots)) \\
&= \sum_{i=1}^{\infty} \mu(F_i)
\end{aligned}
$$

   即 $\mu(\bigcup_{i=1}^{\infty} F_i) \neq \sum_{i=1}^{\infty} \mu(F_i)$, 故 μ 不满足性质 2,μ 不是概率测度。

   (b) 解: 首先验证性质 1,$\varnothing = 0, \mathbb{P}(\Omega) = 1^2 - 0 = 1$, 故 $\mathbb{P}$ 满足性质 1

   然后验证性质 2, $\forall \{F_i\}_{i=1}^{\infty} \in \Omega, \bigcup_{i=1}^{\infty} F_i = [a,b], F_i = [K_i, K_{i+1}]$

   且$\forall i, j = 1, \ldots, \infty, F_i \cap F_j = \varnothing$

$$
\begin{aligned}
\mathbb{P}(\bigcup_{i=1}^{\infty} F_i) &= \mathbb{P}([a,b]) \\
&= b^2 - a^2 = b - \ldots - K_{i+1}^2 + K_{i+1}^2 - K_i^2 + K_i^2 + \ldots + K_1^2 - a^2 \\
&= \sum_{i=1}^{\infty} \mathbb{P}(F_i)
\end{aligned}
$$

   即 $\mathbb{P}(\bigcup_{i=1}^{\infty} F_i) = \sum_{i=1}^{\infty} \mathbb{P}(F_i)$, 故 $\mathbb{P}$ 满足性质 2, 则综上 $\mathbb{P}$ 是概率测度。

   X的CDF :

   $F(x) = \mathbb{P}(\{\omega : X(\omega) \neq x\}) = x^2, x \in [0,1]$

   X的DF :

   $f(x) = F'(x) = 2x, x \in [0,1]$

(c) 证明:

$$\begin{aligned} var(X) &= \mathbb{E}[X-\mathbb{E}X]^2 \\ &= \mathbb{E}[X^2-2X\mathbb{E}X+(\mathbb{E}X)^2] \\ &= \mathbb{E}[X^2]-2\mathbb{E}[X\mathbb{E}X]+(\mathbb{E}X)^2 \\ &= \mathbb{E}[X^2]-2(\mathbb{E}X)^2+(\mathbb{E}X)^2 \\ &= \mathbb{E}[X^2]-(\mathbb{E}X)^2 \end{aligned}$$

(d) i. 两点分布

$\mathbb{E}X = p\times 0+(1-p)\times 1 = 1-p$

$var(X) = \mathbb{E}[X^2]-(\mathbb{E}X)^2 = (1-p)-(1-p)^2 = p(1-p)$

$std(X) = \sqrt{var(X)} = \sqrt{p(1-p)}$

ii. 二项式分布

$\mathbb{E}X = np$

$var(X) = np(1-p)$

$std(X) = \sqrt{var(X)} = \sqrt{np(1-p)}$

iii. Poisson 分布

$$\mathbb{E}X = \sum_{i=0}^{+\infty} x_i P_i = e^{-\lambda}\sum_{i=1}^{+\infty} i\frac{\lambda^i}{i!} = e^{-\lambda}\sum_{i=2}^{+\infty}\frac{\lambda^i}{(i-1)!} = e^{-\lambda}\times\lambda\sum_{i=2}^{+\infty}\frac{\lambda^{i-1}}{(i-1)!} = e^{-\lambda}\times\lambda e^{\lambda} = \lambda$$

$$\begin{aligned} \mathbb{E}(X^2) &= \sum_{i=0}^{+\infty} x_i^2 P_i = e^{-\lambda}\sum_{i=1}^{+\infty} i^2\frac{\lambda^i}{i!} = e^{-\lambda}\sum_{i=2}^{+\infty} i\frac{\lambda^i}{(i-1)!} = e^{-\lambda}\sum_{i=2}^{+\infty}[(i-1)+1]\frac{\lambda^i}{(k-1)!} \\ &= e^{-\lambda}[\sum_{i=3}^{+\infty}\frac{\lambda^i}{(i-2)!}+\sum_{i=2}^{+\infty}\frac{\lambda^i}{(i-1)!}] = e^{-\lambda}[\lambda^2\sum_{i=3}^{+\infty}\frac{\lambda^{i-2}}{(i-2)!}+\lambda\sum_{i=2}^{+\infty}\frac{\lambda^{i-1}}{(i-1)!}] \\ &= e^{-\lambda}(\lambda^2 e^{\lambda}+\lambda e^{\lambda}) = \lambda^2+\lambda \end{aligned}$$

即 $\mathbb{E}X = \lambda$

$var(X) = \mathbb{E}(X^2)-(\mathbb{E}X)^2 = \lambda$

$std(X) = \sqrt{\lambda}$

iv. 均匀分布

$\mathbb{E}X = \frac{a+b}{2}$

$var(X) = \mathbb{E}(X^2)-(\mathbb{E}X)^2 = \frac{a^2+ab+b^2}{3}-\frac{(a+b)^2}{4} = \frac{a^2-2ab+b^2}{12} = \frac{(a-b)^2}{12}$

$std(X) = \frac{(a-b)}{2\sqrt{3}}$

v. 正态分布

$$
\begin{aligned}
\mathbb{E}X &= \int_{-\infty}^{+\infty} xf(x)dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} xe^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma}(x-\mu)e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx + \mu \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \\
&= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} xe^{-\frac{x^2}{2\sigma^2}} dx + \mu \int_{-\infty}^{+\infty} f(x)dx = \mu
\end{aligned}
$$

$$var(X) = \mathbb{E}[(x-\mu)^2] = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma}(x-\mu)^2 e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} x^2 e^{-\frac{x^2}{2\sigma^2}} dx = \sigma^2$$

即 $\mathbb{E}X = \mu$

$var(X) = \sigma^2$

$std(X) = \sigma$

vi. 指数分布

$$
\begin{aligned}
\mathbb{E}X &= \int_{-\infty}^{+\infty} xf(x)dx = \int_{0}^{+\infty} xf(x)dx = \int_{0}^{+\infty} x \times \lambda e^{-\lambda x} dx = \frac{1}{\lambda}\int_{0}^{+\infty} \lambda x e^{-\lambda x} d\lambda x \\
&= \frac{1}{\lambda}\int_{0}^{\infty} ue^{-u}du (令 u = \lambda x) \\
&= \frac{1}{\lambda}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}(X^2) &= \int_{-\infty}^{+\infty} x^2 f(x)dx = \int_{0}^{+\infty} x^2 f(x)dx = \int_{0}^{+\infty} x^2 \times \lambda e^{-\lambda x} dx \\
&= \frac{1}{\lambda^2}\int_{0}^{+\infty} u^2 e^{-u}du (令 u = \lambda x) \\
&= \frac{2}{\lambda^2}
\end{aligned}
$$

即 $\mathbb{E}X = \frac{1}{\lambda}$

$var(X) = \mathbb{E}(X^2) - (\mathbb{E}X)^2 = \frac{1}{\lambda^2}$

$std(X) = \frac{1}{\lambda}$

2. 证明:

$$
\begin{aligned}
\sum_{i=0}^{\infty} \mathbb{P}(X=i) &= \mathbb{P}(X=0) + \mathbb{P}(X=1) + \dots \\
&= \sum_{i=0}^{\infty} \frac{e^{-\lambda}\lambda^i}{i!} \\
&= e^{-\lambda}\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} \\
&= e^{-\lambda} \times e^{\lambda} \\
&= 1
\end{aligned}
$$

3. 解:

$$\begin{aligned}\lim_{n\to\infty}\mathbb{P}(X=i) &= \lim_{n\to\infty}\binom{n}{i}p^i(1-p)^{n-i} \\ &= \lim_{n\to\infty}\frac{n!}{i!(n-i)!}p^i(1-p)^{n-i} \\ &= \lim_{n\to\infty}\frac{n!}{i!(n-i)!}(\frac{\lambda}{n})^i(1-\frac{\lambda}{n})^{n-i} \\ &= \frac{\lambda^i}{i!}\lim_{n\to\infty}\frac{n!}{(n-i)!n^i}(1-\frac{\lambda}{n})^{n-i} \\ &= \frac{\lambda^i}{i!}\times e^{-\lambda} \\ &= \frac{\lambda^i e^{-\lambda}}{i!}\end{aligned}$$

由上述推导可看出当 n 充分大而 p 较小时, 二项分布的概率可由 Poisson 分布近似

4. 证明: 对于凸函数 $g(\cdot)$, $\alpha g(x)+(1-\alpha)g(y)\geq g(\alpha x+(1-\alpha)y)$, 其中$\alpha\in(0,1)$
令点 k 与 $g(\cdot)$ 切于点 x_0, 则对任意点 x_0 有

$$g(x)-g(x_0)\geq k(x-x_0)$$

其中 $x\in\mathbb{R}$, 考虑 $x=X, x_0=\mathbb{E}X$, 其中 X 为随机变量, 则有

$$g(X)-g(\mathbb{E}X)\geq k(X-\mathbb{E}X)$$

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$$\begin{aligned}\mathbb{E}[g(X)]-E[g(\mathbb{E}X)] &\geq k(\mathbb{E}X-\mathbb{E}X) \\ \mathbb{E}[g(X)] &\geq E[g(\mathbb{E}X)]\end{aligned}$$

若 $g(\cdot)$ 严格凸则同理可得 $\mathbb{E}[g(X)]>E[g(\mathbb{E}X)]$

5. 证明: 由 4 有 $\mathbb{E}[g(X)]\geq g(\mathbb{E}X)$
令 $g(x)=x^2$, 则由 $g'(x)=2x, g''(x)=x$, 则 $g(x)$ 为凸函数
则 $\mathbb{E}(X^2)\geq g(\mathbb{E}(X))=(\mathbb{E}X)^2$, 得证