2021 秋季本科时间序列

第 9 次作业答案

12 月 26 日

1. (a) 若 $\mathbf{\Phi}$ 的特征值模长小于 1，则 $\boldsymbol{X}_t$ 平稳。

$$|\mathbf{\Phi}-\lambda \boldsymbol{I}| = \begin{vmatrix} 0.1-\lambda & 0.5 \\ -0.4 & 1-\lambda \end{vmatrix} = \lambda^2 - 1.1\lambda + 0.3 = 0$$

解得 $\lambda_1 = 0.6, \lambda_2 = 0.5$, 故满足平稳性要求。$\mathbb{E}\boldsymbol{X}_t = \boldsymbol{c} + \mathbf{\Phi}\mathbb{E}\boldsymbol{X}_{t-1}$, 则有

$$\mathbb{E}\boldsymbol{X}_t = (\boldsymbol{I}-\mathbf{\Phi})^{-1}\boldsymbol{c} = \begin{bmatrix} 0 & 2.5 \\ -2 & 4.5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 2.5 \end{bmatrix}$$

(b) 把 $\lambda = 0.6$ 代入得，

$$(\mathbf{\Phi}-\lambda \boldsymbol{I})\boldsymbol{x}_1 = \begin{bmatrix} -0.5 & 0.5 \\ -0.4 & 0.4 \end{bmatrix} \boldsymbol{x}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \boldsymbol{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

同理，当 $\lambda = 0.5$ 时，解得 $\boldsymbol{x}_2 = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$，故 $\boldsymbol{A} = \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 1 & 4 \end{bmatrix}, \boldsymbol{A}^{-1} = \begin{bmatrix} -4 & 5 \\ 1 & -1 \end{bmatrix}$

$$\mathbf{\Phi}^i = \boldsymbol{A}\mathbf{\Lambda}^i\boldsymbol{A}^{-1} = \begin{bmatrix} 1 & 5 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 0.6^i & 0 \\ 0 & 0.5^i \end{bmatrix} \begin{bmatrix} -4 & 5 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -4\times 0.6^i + 5\times 0.5^i & 5\times 0.6^i - 5\times 0.5^i \\ -4\times 0.6^i + 4\times 0.5^i & 5\times 0.6^i - 4\times 0.5^i \end{bmatrix}$$

$\boldsymbol{X}_t$ 的 $MA(\infty)$ 展开为 $\boldsymbol{X}_t = (\boldsymbol{I}+\mathbf{\Phi}+\cdots+\mathbf{\Phi}^n)\boldsymbol{c} + \sum_{i=0}^{n}\mathbf{\Phi}^i\epsilon_{t-i}, n\to\infty$，则

$$\begin{aligned} \mathrm{var}\boldsymbol{X}_t &= \sum_{i=0}^{\infty}\mathbf{\Phi}^i\mathbf{\Omega}\boldsymbol{Phi}^{Ti} \\ &= \sum_{i=0}^{\infty}\boldsymbol{A}\mathbf{\Lambda}^i\boldsymbol{A}^{-1}\mathbf{\Omega}(A^{-1})^T\mathbf{\Lambda}^{Ti}\boldsymbol{A}^T \\ &= \begin{bmatrix} \frac{475}{84} & \frac{515}{84} \\ \frac{515}{84} & \frac{1003}{84} \end{bmatrix} \end{aligned}$$

(c) 设 $\boldsymbol{\Omega} = \boldsymbol{A}\boldsymbol{D}\boldsymbol{A}^T$，其中 $\boldsymbol{A} = \begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}, \boldsymbol{D} = \begin{bmatrix} b & 0 \\ 0 & c \end{bmatrix}$，解得 $a = \frac{1}{2}, b = 2, c = \frac{7}{2}$，令 $\boldsymbol{P} = \boldsymbol{A}\boldsymbol{D}^{\frac{1}{2}}$，则 $\boldsymbol{P}^T = \boldsymbol{D}^{\frac{1}{2}}\boldsymbol{A}^T$，则 $\boldsymbol{\Omega} = \boldsymbol{P}\boldsymbol{P}^T$

$$\boldsymbol{P} = \boldsymbol{A}\boldsymbol{D}^{\frac{1}{2}} = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{\frac{7}{2}} \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{14}}{2} \end{bmatrix}, \boldsymbol{\Omega} = \begin{bmatrix} \sqrt{2} & 0 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{14}}{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{14}}{2} \end{bmatrix}$$

$$\boldsymbol{X}_{t+s} = (\boldsymbol{I} - \boldsymbol{\Phi})^{-1}\boldsymbol{c} + \sum_{i=0}^{\infty} \boldsymbol{\Phi}^i \boldsymbol{P}\boldsymbol{u}_{t+s-i}, \frac{\partial \boldsymbol{X}_{t+s}}{\partial \boldsymbol{u}_{kt}} = \boldsymbol{\Phi}^s \boldsymbol{p}_k$$

则

$$\begin{aligned}
\frac{\partial \boldsymbol{X}_{1t+s}}{\partial \boldsymbol{u}_{1t}} &= \sqrt{2}(-4 \times 0.6^s + 5 \times 0.5^s) + \frac{\sqrt{2}}{2}(5 \times 0.6^s - 5 \times 0.5^s) \\
\frac{\partial \boldsymbol{X}_{2t+s}}{\partial \boldsymbol{u}_{1t}} &= \sqrt{2}(-4 \times 0.6^s + 4 \times 0.5^s) + \frac{\sqrt{2}}{2}(5 \times 0.6^s - 4 \times 0.5^s) \\
\frac{\partial \boldsymbol{X}_{1t+s}}{\partial \boldsymbol{u}_{2t}} &= \frac{\sqrt{14}}{2}(5 \times 0.6^s - 5 \times 0.5^s) \\
\frac{\partial \boldsymbol{X}_{2t+s}}{\partial \boldsymbol{u}_{2t}} &= \frac{\sqrt{14}}{2}(5 \times 0.6^s - 4 \times 0.5^s)
\end{aligned}$$

2. (a) 令 $\boldsymbol{Y} = \begin{bmatrix} X_2 \\ \vdots \\ X_t \end{bmatrix}, \boldsymbol{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_{t-1} \end{bmatrix}, \boldsymbol{u} = \begin{bmatrix} u_2 \\ \vdots \\ u_t \end{bmatrix}$，有 $\boldsymbol{Y} = \boldsymbol{X}\rho + \boldsymbol{u}$，则 OLS 估计量

$$\begin{aligned}
\hat{\rho} &= (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{Y} \\
&= (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T(\boldsymbol{X}\rho + \boldsymbol{u}) \\
&= \rho + (\boldsymbol{X}^T\boldsymbol{X})^{-1}\boldsymbol{X}^T\boldsymbol{u} \\
&= \rho + (\frac{1}{T}\boldsymbol{X}^T\boldsymbol{X})^{-1}\frac{1}{T}\boldsymbol{X}^T\boldsymbol{u} \\
&= \rho + (\frac{1}{T}\sum_{t=1}^{T} X_t^2)^{-1}\frac{1}{T}\sum_{t=1}^{T} X_{t-1}u_t
\end{aligned}$$

又因为 $\mathbb{E}[u_t] = \mathbb{E}\mathbb{E}[u_t|X_{t-1}, X_{t-2}, \cdots, X_1] = \sqrt{\alpha_0 + \alpha_1 u_{t-1}^2}\mathbb{E}[\epsilon_t|\Omega_{t-1}] = 0$，所以 $\frac{1}{T}\sum_{t=1}^{T} X_{t-1}u_t \xrightarrow{a.s} 0$，OLS 估计依然具有一致性，即 $\hat{\rho} \xrightarrow{\text{a.s.}} \rho$

(b)

$$\sqrt{T}(\hat{\rho} - \rho) = (\frac{1}{T}\sum_{t=1}^{T} X_t^2)^{-1}\frac{1}{T}\sum_{t=1}^{T} X_t u_t$$

其中 $\frac{1}{T}\sum_{t=1}^{T} X_t u_t \xrightarrow{d} N(0, \Sigma), \Sigma = \lim_T \frac{1}{T}\sum_t \mathbb{E}[u_t^2 X_t^2]$。设 $\lim_{T\to\infty} \frac{1}{T}\sum_t X_t^2 = M$, 则 $\sqrt{T}(\hat{\rho} - \rho) \xrightarrow{d} N(0, M^{-1}\Sigma M^{-1})$

(c) 不需要知道 α_0, α_1 的参数取值。

(d) 首先利用 $\hat{\rho}$ 得到 $\hat{u}_t, \hat{u}_t = Y_t - \hat{\rho} X_t$，进而计算 $\hat{\Sigma} = \frac{1}{T}\sum_{t=1}^T \hat{u}_t^2 X_t^2$, 则 $\hat{\rho}$ 的异方差渐进标准误是样本渐进协方差 $\frac{1}{T}\hat{M}^{-1}\hat{\Sigma}\hat{M}^{-1}$ 的平方根。

若已知 α_0, α_1 的取值，则

$$\lim_{T\to\infty} \frac{1}{T}\sum_t \mathbb{E}[u_t^2 X_t^2] = \lim_{T\to\infty} \frac{1}{T}\sum_t \mathbb{E}[(\alpha_0 + \alpha_1 u_{t-1}^2)\epsilon_t^2 X_t^2]$$

因为 u_t 为一个平稳 ARCH 过程，故 $\mathbb{E}[u_t^2] = \frac{\alpha_0}{1-\alpha_1}$，故上式为

$$(\alpha_0 + \frac{\alpha_0\alpha_1}{1-\alpha_1}) \lim_{T\to\infty} \frac{1}{T}\sum_t \mathbb{E}[\epsilon_t^2 X_t^2]$$

设 $\frac{1}{T}\sum_{t=1}^T X_t\epsilon_t \xrightarrow{d} N(0, \Sigma')$，有 $\sqrt{T}(\hat{\rho} - \rho) \xrightarrow{d} N(0, \frac{\alpha_0}{1-\alpha_1} M^{-1}\Sigma' M^{-1})$

3. (a)

$$T(\hat{\rho} - 1) = \frac{\frac{1}{T}\sum_{t=1}^T \Delta X_t X_{t-1}}{\frac{1}{T^2}\sum_{t=1}^T X_{t-1}^2} \xrightarrow{d} \frac{\frac{\sigma_\epsilon^2}{2}B(1)^2 - \frac{\sigma_\epsilon^2}{2}}{\sigma_\epsilon^2 \int_0^1 B(s)^2 ds} = \frac{0.5(B(1)^2 - 1}{\int_0^1 B(s)^2 ds}$$

可得

$$T^{1-\delta}(\hat{\rho} - 1) \xrightarrow{d} T^{-\delta}\frac{0.5(B(1)^2 - 1}{\int_0^1 B(s)^2 ds}$$

当 $T \to \infty, T^{-\delta} \overset{\text{a.s.}}{\to} 0$, 故对任意的 $\delta > 0, T^{1-\delta}(\hat{\rho} - 1) \overset{\text{a.s.}}{\to} 0$

(b) $T(\hat{\rho} - 1) \overset{\text{a.s.}}{\to} T(\rho - 1)$，由 $|\rho| < 1$ 有 $\rho - 1 < 0$，故 $T \to \infty$ 时有 $T(\rho - 1) \to -\infty$，即 $T(\hat{\rho} - 1) \overset{\text{a.s.}}{\to} -\infty$。