2021 秋季本科时间序列

第 6 次作业答案

11 月 18 日

1. (a)

$$\boldsymbol{\Sigma} = \mathbb{E}(\boldsymbol{X} - \boldsymbol{\mu}_X)(\boldsymbol{X} - \boldsymbol{\mu}_X)^\intercal$$

$$\boldsymbol{\Sigma}^\intercal = \mathbb{E}[(\boldsymbol{X} - \boldsymbol{\mu}_X)(\boldsymbol{X} - \boldsymbol{\mu}_X)^\intercal]^\intercal = \mathbb{E}[(\boldsymbol{X} - \boldsymbol{\mu}_X)^\intercal]^\intercal(\boldsymbol{X} - \boldsymbol{\mu}_X)^\intercal = \mathbb{E}(\boldsymbol{X} - \boldsymbol{\mu}_X)(\boldsymbol{X} - \boldsymbol{\mu}_X)^\intercal$$

故得证 $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}^\intercal$, 即 $\boldsymbol{\Sigma}$ 是对称矩阵

(b) 已知 $Z \equiv \boldsymbol{w}^\intercal\boldsymbol{X}$，令 $\boldsymbol{\mu}_Z = \boldsymbol{w}^\intercal\boldsymbol{\mu}_X$

$$\begin{aligned}
\boldsymbol{\Sigma}_Z &= \mathbb{E}(\boldsymbol{Z} - \boldsymbol{\mu}_Z)(\boldsymbol{Z} - \boldsymbol{\mu}_Z)^\intercal \\
&= \mathbb{E}(\boldsymbol{w}^\intercal\boldsymbol{X} - \boldsymbol{w}^\intercal\boldsymbol{\mu}_X)(\boldsymbol{w}^\intercal\boldsymbol{X} - \boldsymbol{w}^\intercal\boldsymbol{\mu}_X)^\intercal \\
&= \mathbb{E}\boldsymbol{w}^\intercal(\boldsymbol{X} - \boldsymbol{\mu}_X)(\boldsymbol{X} - \boldsymbol{\mu}_X)^\intercal\boldsymbol{w} \\
&= \boldsymbol{w}^\intercal\mathbb{E}(\boldsymbol{X} - \boldsymbol{\mu}_X)(\boldsymbol{X} - \boldsymbol{\mu}_X)^\intercal\boldsymbol{w} \\
&= \boldsymbol{w}^\intercal\boldsymbol{\Sigma}\boldsymbol{w}
\end{aligned}$$

因为 $\boldsymbol{\Sigma}_Z \geq 0$

故得证 $\boldsymbol{w}^\intercal\boldsymbol{\Sigma}\boldsymbol{w} \geq 0$，即 $\boldsymbol{\Sigma}$ 是一个半正定矩阵

(c) $\boldsymbol{X}$ 各分量线性独立时，$\boldsymbol{\Sigma}_Z \neq 0$

由 (b) 可知，此时 $\boldsymbol{w}^\intercal\boldsymbol{\Sigma}\boldsymbol{w} > 0$

故得证 $\boldsymbol{\Sigma}$ 为正定矩阵当且仅当 $\boldsymbol{X}$ 各分量线性独立

2. (a) i.

$$\begin{aligned}
\boldsymbol{P}_X\boldsymbol{\xi} &= \boldsymbol{X}(\boldsymbol{X}^\intercal\boldsymbol{X})^{-1}\boldsymbol{X}^\intercal\boldsymbol{X}\boldsymbol{a} \\
&= \boldsymbol{X}\boldsymbol{I}\boldsymbol{a} \\
&= \boldsymbol{X}\boldsymbol{a} \\
&= \boldsymbol{\xi}
\end{aligned}$$

ii.

$$\begin{aligned}
(\boldsymbol{X}\boldsymbol{a})^\intercal(\boldsymbol{I} - \boldsymbol{P}_X)\boldsymbol{\zeta} &= \boldsymbol{a}^\intercal\boldsymbol{X}^\intercal(\boldsymbol{\zeta} - \boldsymbol{X}(\boldsymbol{X}^\intercal\boldsymbol{X})^{-1}\boldsymbol{X}^\intercal\boldsymbol{\zeta}) \\
&= \boldsymbol{a}^\intercal\boldsymbol{X}^\intercal\boldsymbol{\zeta} - \boldsymbol{a}^\intercal\boldsymbol{X}^\intercal\boldsymbol{X}(\boldsymbol{X}^\intercal\boldsymbol{X})^{-1}\boldsymbol{X}^\intercal\boldsymbol{\zeta} \\
&= \boldsymbol{a}^\intercal\boldsymbol{X}^\intercal\boldsymbol{\zeta} - \boldsymbol{a}^\intercal\boldsymbol{I}\boldsymbol{X}^\intercal\boldsymbol{\zeta} \\
&= \boldsymbol{0}
\end{aligned}$$

(b) i. 由 OLS 估计系数表达式可知，$\boldsymbol{Y}$ 对 $\boldsymbol{W}$ 回归的系数为 $(\boldsymbol{W}^\intercal\boldsymbol{W})^{-1}\boldsymbol{W}^\intercal\boldsymbol{Y}$，而 $\boldsymbol{Z}$ 对 $\boldsymbol{W}$ 回归的系数为 $(\boldsymbol{W}^\intercal\boldsymbol{W})^{-1}\boldsymbol{W}^\intercal\boldsymbol{Z}$，故两组回归对应的残差向量分别为

$$(\boldsymbol{I}-\boldsymbol{W}(\boldsymbol{W}^\intercal\boldsymbol{W})^{-1}\boldsymbol{W}^\intercal)\boldsymbol{Y}=(\boldsymbol{I}-\boldsymbol{P}_W)\boldsymbol{Y}=\tilde{\boldsymbol{Y}}$$
$$(\boldsymbol{I}-\boldsymbol{W}(\boldsymbol{W}^\intercal\boldsymbol{W})^{-1}\boldsymbol{W}^\intercal)\boldsymbol{Z}=(\boldsymbol{I}-\boldsymbol{P}_W)\boldsymbol{Z}=\tilde{\boldsymbol{Z}}$$

ii. 由 $\hat{\boldsymbol{e}}=\boldsymbol{I}-\boldsymbol{X}(\boldsymbol{X}^\intercal\boldsymbol{X})^{-1}\boldsymbol{X}^\intercal\boldsymbol{Y}=(\boldsymbol{I}-\boldsymbol{P}_X)\boldsymbol{Y}$，由 (a) 的结论可得 $\hat{\boldsymbol{e}}$ 与 $\boldsymbol{X}$ 的每个列向量相互垂直，故垂直于 $\boldsymbol{Z}$ 与 $\boldsymbol{W}$。

iii. $\boldsymbol{P}_W\hat{\boldsymbol{e}}=\boldsymbol{W}(\boldsymbol{W}^\intercal\boldsymbol{X})^{-1}\boldsymbol{W}^\intercal\hat{\boldsymbol{e}}$，由 iii 可知 $\boldsymbol{W}^\intercal\boldsymbol{e}=\boldsymbol{0}$，故得所证。

iv. 只需要说明 $\hat{\boldsymbol{\delta}}$ 正好满足回归 $\tilde{\boldsymbol{Y}}=\tilde{\boldsymbol{Z}}\boldsymbol{\delta}+\boldsymbol{u}$ 的 OLS 系数估计所需满足的一阶条件 $\tilde{\boldsymbol{Z}}^\intercal\tilde{Y}-\tilde{\boldsymbol{Z}}^\intercal\tilde{\boldsymbol{Z}}\boldsymbol{\delta}=\boldsymbol{0}$。为此，首先注意到 $\hat{\boldsymbol{e}}=\boldsymbol{Y}-\boldsymbol{X}\hat{\boldsymbol{\beta}}=\boldsymbol{Y}-\boldsymbol{Z}\hat{\boldsymbol{\delta}}-\boldsymbol{W}\hat{\boldsymbol{\theta}}$，两端同时乘以 $\boldsymbol{I}-\boldsymbol{P}_W$ 可得

$$(\boldsymbol{I}-\boldsymbol{P}_W)\hat{\boldsymbol{e}}=(\boldsymbol{I}-\boldsymbol{P}_W)\boldsymbol{Y}-(\boldsymbol{I}-\boldsymbol{P}_W)\boldsymbol{Z}\hat{\boldsymbol{\delta}}-(\boldsymbol{I}-\boldsymbol{P}_W)\boldsymbol{W}\hat{\boldsymbol{\theta}}$$

由 i–iii 结论可知，上式可以进一步写为

$$\hat{\boldsymbol{e}}=\tilde{\boldsymbol{Y}}+\tilde{\boldsymbol{Z}}\hat{\boldsymbol{\delta}}$$

再在两端乘以 $\tilde{\boldsymbol{Z}}^\intercal$ 可得

$$\boldsymbol{0}=\tilde{\boldsymbol{Z}}\hat{\boldsymbol{e}}=\tilde{\boldsymbol{Z}}^\intercal\tilde{\boldsymbol{Y}}+\tilde{\boldsymbol{Z}}^\intercal\tilde{\boldsymbol{Z}}\hat{\boldsymbol{\delta}}$$

即得所证。

3. (a)

$$\hat{\boldsymbol{Y}}=\boldsymbol{Z}\hat{\boldsymbol{\delta}}+\boldsymbol{W}\hat{\boldsymbol{\theta}}=\boldsymbol{X}\hat{\boldsymbol{\beta}}=\boldsymbol{X}(\boldsymbol{X}^\intercal\boldsymbol{X})^{-1}\boldsymbol{X}^\intercal\boldsymbol{Y}=\boldsymbol{P}_X\boldsymbol{Y}$$
$$\bar{\boldsymbol{Y}}=\boldsymbol{Z}\hat{\boldsymbol{\delta}}=\boldsymbol{Z}(\boldsymbol{Z}^\intercal\boldsymbol{Z})^{-1}\boldsymbol{Z}^\intercal\boldsymbol{Y}=\boldsymbol{P}_Z\boldsymbol{Y}$$

(b)

$$\begin{aligned}(\hat{\boldsymbol{Y}}-\bar{\boldsymbol{Y}})^\intercal\boldsymbol{Z}&=[(\boldsymbol{P}_X-\boldsymbol{P}_Z)\boldsymbol{Y}]^\intercal\boldsymbol{Z}\\&=\boldsymbol{Y}^\intercal(\boldsymbol{P}_X-\boldsymbol{P}_Z)^\intercal\boldsymbol{Z}\\&=\boldsymbol{Y}^\intercal[(\boldsymbol{I}-\boldsymbol{P}_Z)-(\boldsymbol{I}-\boldsymbol{P}_X)]^\intercal\boldsymbol{Z}\\&=\boldsymbol{Y}^\intercal(\boldsymbol{I}-\boldsymbol{P}_Z)^\intercal\boldsymbol{Z}-\boldsymbol{Y}^\intercal(\boldsymbol{I}-\boldsymbol{P}_X)^\intercal\boldsymbol{Z}\end{aligned}$$

由第 2 题可知：$(\boldsymbol{I}-\boldsymbol{P}_Z)^\intercal\boldsymbol{Z}=0,[(\boldsymbol{I}-\boldsymbol{P}_X)\boldsymbol{Y}]^\intercal\boldsymbol{Z}=0$
因此 $(\hat{\boldsymbol{Y}}-\bar{\boldsymbol{Y}})^\intercal\boldsymbol{Z}=0$
故得证 $(\hat{\boldsymbol{Y}}-\bar{\boldsymbol{Y}})^\intercal\bar{\boldsymbol{Y}}=(\hat{\boldsymbol{Y}}-\bar{\boldsymbol{Y}})^\intercal\boldsymbol{P}_Z\boldsymbol{Y}=(\hat{\boldsymbol{Y}}-\bar{\boldsymbol{Y}})^\intercal\boldsymbol{Z}(\boldsymbol{Z}^\intercal\boldsymbol{Z})^{-1}\boldsymbol{Z}^\intercal\boldsymbol{Y}=0$

(c) 已知 $(\hat{\boldsymbol{Y}}-\bar{\boldsymbol{Y}})^\intercal\bar{\boldsymbol{Y}}=0$
要证 $\hat{\boldsymbol{Y}}^\intercal\hat{\boldsymbol{Y}}\geq\bar{\boldsymbol{Y}}^\intercal\bar{\boldsymbol{Y}}$ 即证 $\hat{\boldsymbol{Y}}^\intercal\hat{\boldsymbol{Y}}\geq\hat{\boldsymbol{Y}}^\intercal\bar{\boldsymbol{Y}}$

$$\begin{aligned}\hat{\boldsymbol{Y}}^\intercal(\hat{\boldsymbol{Y}}-\bar{\boldsymbol{Y}})&=(\hat{\boldsymbol{Y}}-\bar{\boldsymbol{Y}}+\bar{\boldsymbol{Y}})^\intercal(\hat{\boldsymbol{Y}}-\bar{\boldsymbol{Y}})\\&=(\hat{\boldsymbol{Y}}-\bar{\boldsymbol{Y}})^\intercal(\hat{\boldsymbol{Y}}-\bar{\boldsymbol{Y}})+\bar{\boldsymbol{Y}}^\intercal(\hat{\boldsymbol{Y}}-\bar{\boldsymbol{Y}})\\&=(\hat{\boldsymbol{Y}}-\bar{\boldsymbol{Y}})^\intercal(\hat{\boldsymbol{Y}}-\bar{\boldsymbol{Y}})\geq0\end{aligned}$$

当且仅当 $\hat{\boldsymbol{Y}}-\bar{\boldsymbol{Y}}\neq0$ 时，$\hat{\boldsymbol{Y}}^\intercal\hat{\boldsymbol{Y}}>\bar{\boldsymbol{Y}}^\intercal\bar{\boldsymbol{Y}}$ 严格成立

(d) 由 c 可知，当解释变量只有 $\boldsymbol{Z}$ 时，估计结果为 $\bar{\boldsymbol{Y}}$，当增加解释变量 $\boldsymbol{W}$ 时，估计结果为 $\hat{\boldsymbol{Y}}$

$$\hat{\boldsymbol{R}}^2-\bar{\boldsymbol{R}}^2=\frac{\hat{\boldsymbol{Y}}^\intercal\hat{\boldsymbol{Y}}}{\boldsymbol{Y}^\intercal\boldsymbol{Y}}-\frac{\bar{\boldsymbol{Y}}^\intercal\bar{\boldsymbol{Y}}}{\boldsymbol{Y}^\intercal\boldsymbol{Y}}=\frac{\hat{\boldsymbol{Y}}^\intercal\hat{\boldsymbol{Y}}-\bar{\boldsymbol{Y}}^\intercal\bar{\boldsymbol{Y}}}{\boldsymbol{Y}^\intercal\boldsymbol{Y}}\geq0$$

因此 $\boldsymbol{Y}$ 对 $\boldsymbol{X}=[X_1,\ldots,X_n]$ 回归中增加解释变量 X_{n+1} 时，R^2 单调递增