2021 秋季本科时间序列

第 4 次作业答案

11 月 8 日

1.

$$(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})^{\top}\boldsymbol{X}^{\top}\boldsymbol{X}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta}) = \left[\boldsymbol{X}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})\right]^{\top}\boldsymbol{X}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})$$

记

$$\boldsymbol{X}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta}) = \begin{bmatrix} \boldsymbol{X}_1^{\top} \\ \vdots \\ \boldsymbol{X}_T^{\top} \end{bmatrix}_{T\times(k+1)} \cdot \begin{bmatrix} \hat{\beta}_0-\beta_0 \\ \vdots \\ \hat{\beta}_k-\beta_k \end{bmatrix}_{(k+1)\times 1} = \begin{bmatrix} z_1 \\ \vdots \\ z_T \end{bmatrix}_{T\times 1}$$

则

$$(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta})^{\top}\boldsymbol{X}^{\top}\boldsymbol{X}(\hat{\boldsymbol{\beta}}-\boldsymbol{\beta}) = \boldsymbol{Z}^{\top}\cdot\boldsymbol{Z} = \sum_{1}^{T} z_i^2 \geq 0$$

2.

$$\boldsymbol{X} = \begin{bmatrix} \boldsymbol{X}_1^{\top} \\ \vdots \\ \boldsymbol{X}_T^{\top} \end{bmatrix} = \begin{bmatrix} x_{11} & \cdots & x_{1K} \\ \vdots & \ddots & \vdots \\ x_{T1} & \cdots & x_{TK} \end{bmatrix}, \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_k \end{bmatrix}$$

$$\begin{aligned}
\boldsymbol{X}\cdot\boldsymbol{\beta} &= \begin{bmatrix} x_{11} & \cdots & x_{1K} \\ \vdots & \ddots & \vdots \\ x_{T1} & \cdots & x_{TK} \end{bmatrix} \cdot \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_k \end{bmatrix} \\
&= \begin{bmatrix} x_{11}\cdot\beta_1 \\ \vdots \\ x_{T1\cdot\beta_1} \end{bmatrix} + \cdots + \begin{bmatrix} x_{1k}\beta_k \\ \vdots \\ x_{Tk}\beta_k \end{bmatrix} \\
&= \beta_1\boldsymbol{X}_1 + \cdots + \beta_K\boldsymbol{X}_K
\end{aligned}$$

3.

$$\det\begin{bmatrix}\phi_1 & \phi_2-1 & 0\\ \phi_2 & \phi_1 & -1\\ 1 & -\phi_1 & -\phi_2\end{bmatrix} = -\phi_1^2\phi_2-\phi_2+1-\phi_1^2+\phi_2^2(\phi_2-1)$$

$$= (\phi_2+1)(\phi_2+\phi_1-1)(\phi_2-\phi_1-1)$$

由于 AR(2) 过程平稳，故其特征多项式零点位于单位圆之外

即 $A(Z)=1-\phi_1 Z-\phi_2 Z^2=0$ 的零点 $|z_1|>1, |z_2|>1. \phi_1=\frac{1}{z_1}+\frac{1}{z_2}, \phi_2=-\frac{1}{z_1z_2}$

则 $|\phi_2|<1, 1+\phi_2>0, \phi_2+\phi_1-1=-(1-\frac{1}{z_1})(1-\frac{1}{z_2})<0, \phi_2-\phi_1-1=-(1+\frac{1}{z_1})(1+\frac{1}{z_2})<0$

故 $(\phi_2+1)(\phi_2+\phi_1-1)(\phi_2-\phi_1-1)>0$, 得证。

4. (a) $X_t=1+1.3X_{t-1}-0.4X_{t-2}+\varepsilon_t$，取期望得

$\mathbb{E}X_t=1+1.3\mathbb{E}X_{t-1}-0.4\mathbb{E}X_{t-2}=1+1.3\mathbb{E}X_t-0.4\mathbb{E}X_t$, 解得 $\mathbb{E}X_t=10$

设 $Y_t=X_t-\mathbb{E}X_t$, 则 $Y_t=1.3Y_{t-1}-0.4Y_{t-2}+\varepsilon_t$, 两边分别同乘 Y_t, Y_{t-1}, Y_{t-2}, 并取期望得

$$\gamma_0=\phi_1\gamma_1+\phi_2\gamma_2+\sigma_\epsilon^2$$
$$\gamma_1=\phi_1\gamma_0+\phi_2\gamma_1$$
$$\gamma_2=\phi_1\gamma_1+\phi_2\gamma_0$$

矩阵表示为

$$\begin{bmatrix}\phi_1 & \phi_2-1 & 0\\ \phi_2 & \phi_1 & -1\\ 1 & -\phi_1 & -\phi_2\end{bmatrix}\cdot\begin{bmatrix}\gamma_0\\ \gamma_1\\ \gamma_2\end{bmatrix}=\begin{bmatrix}0\\ 0\\ \sigma_\epsilon^2\end{bmatrix}$$

则

$$\begin{bmatrix}\gamma_0\\ \gamma_1\\ \gamma_2\end{bmatrix}=\begin{bmatrix}\phi_1 & \phi_2-1 & 0\\ \phi_2 & \phi_1 & -1\\ 1 & -\phi_1 & -\phi_2\end{bmatrix}^{-1}\cdot\begin{bmatrix}0\\ 0\\ 1\end{bmatrix}=\begin{bmatrix}\frac{700}{81}\\ \frac{650}{81}\\ \frac{565}{81}\end{bmatrix}$$

由 $\gamma_k=\phi_1\gamma_{k-1}+\phi_2\gamma_{k-2}$，得

$$\begin{bmatrix}\gamma_k\\ \gamma_{k-1}\end{bmatrix}=\begin{bmatrix}\phi_1 & \phi_2\\ 1 & 0\end{bmatrix}\cdot\begin{bmatrix}\gamma_{k-1}\\ \gamma_{k-2}\end{bmatrix}$$
$$=\begin{bmatrix}\phi_1 & \phi_2\\ 1 & 0\end{bmatrix}^{k-1}\cdot\begin{bmatrix}\gamma_1\\ \gamma_0\end{bmatrix}$$

记 $\mathbb{A}=\begin{bmatrix}\phi_1 & \phi_2\\ 1 & 0\end{bmatrix}=\begin{bmatrix}1.3 & -0.4\\ 1 & 0\end{bmatrix}$, 将 $\mathbb{A}$ 矩阵对角化得

$$\mathbb{A}=\begin{bmatrix}0.5 & 0.8\\ 1 & 1\end{bmatrix}\cdot\begin{bmatrix}0.5 & 0\\ 0 & 0.8\end{bmatrix}\cdot\begin{bmatrix}0.5 & 0.8\\ 1 & 1\end{bmatrix}^{-1}$$

则

$$\begin{bmatrix}\gamma_k\\ \gamma_{k-1}\end{bmatrix}=\mathbb{A}^{k-1}\cdot\begin{bmatrix}\gamma_1\\ \gamma_0\end{bmatrix}=\begin{bmatrix}0.5 & 0.8\\ 1 & 1\end{bmatrix}\cdot\begin{bmatrix}0.5^{k-1} & 0\\ 0 & 0.8^{k-1}\end{bmatrix}\cdot\begin{bmatrix}0.5 & 0.8\\ 1 & 1\end{bmatrix}^{-1}\cdot\begin{bmatrix}\frac{650}{81}\\ \frac{700}{81}\end{bmatrix}$$

故 $\gamma_k=\frac{10}{3}\times\frac{650}{81}(0.8^k-0.5^k)+\frac{10}{3}\times\frac{700}{81}(0.8\times0.5^k-0.5\times0.8^k)$.

(b) 代码如下：

```
library(tidyverse)
#生成长度为1000的样本X_t
set.seed(048)
et <- rnorm(1002,0,1)
x <- vector("double",1002)
x[1] <- 0
x[2] <- 0
for(i in 3:1002){
  x[i] <- 1+1.3*x[i-1]-0.4*x[i-2]+et[i]
}
#生成不同长度样本，计算自协方差
r<-matrix(ncol=12,nrow=21)
new <- vector("list",10)
for(n in 1:10)
{
  new[[n]] <- x[3:(100*n+2)]
}

for (n in 1:10) {
  r[,n] <- acf(new[[n]],type = "covariance",lag = 20,
  plot = FALSE)[[1]]
}
a1<-4/5
a2<-1/2
for(k in 0:20){
```

```
  r[,11][k+1] <- 1/(a1-a2)*((a1^k-a2^k)*(650/81)+
  (-a1^k*a2+a1*a2^k)*(700/81))
}
r<-data.frame(r)
r[,12]<-c(0:20)
#数据框转换, 便于绘图
r1<-gather(r,key='i',value='X_t','X1','X2','X3','X4','X5',
'X6','X7','X8','X9','X10','X11')
colnames(r1)<-c("K","i","X_t")
#X1–X10样本长度逐渐增加, x11为理论值
ggplot(data=r1,mapping = aes(x=K,y=X_t,color=i))+
  geom_point( )+
  geom_smooth(data = filter(r1,i=='X11'))
```