2021 秋季本科时间序列

第 2 次作业答案

10 月 23 日

1. (a) $z\bar{z} = (a + bi)(a - bi)$
   $= a^2 - abi + abi - b^2i^2$
   $= a^2 + b^2$
   $= (\sqrt{a^2 + b^2})^2$
   $= |z^2|$

   (b) $\because zw = (a + bi)(c + di)$
   $= ac + adi + bci + bdi^2$
   $= (ac - bd) + (ad + bc)i$
   $\therefore |zw| = \sqrt{(ac - bd)^2 + (ad + bc)^2} = \sqrt{(ac)^2 + (bd)^2 + (ad)^2 + (bc)^2}$
   又 $|z||w| = \sqrt{a^2 + b^2 + c^2 + d^2} = \sqrt{(ac)^2 + (bd)^2 + (ad)^2 + (bc)^2}$
   $\therefore |zw| = |z||w|$
   根据数学归纳法
   $k = 1$ 时，$|z| = |z|$
   假设 $k = n$ 时，$|z^n| = |z|^n$
   $k = n + 1$ 时，$|z^{n+1}| = |z^n z| = |z^n||z| = |z|^n|z| = |z|^{n+1}$
   $\therefore |z^k| = |z|^k$

   (c) 令 $S_n = z^1 + z^2 + ... + z^n = \frac{1-z^n}{1-z}$
   $\because |z| < 1 \therefore \lim_{n\to+\infty} z^n = 0$
   $\therefore \lim_{n\to+\infty} S_n = \frac{1}{1-z}$
   $\therefore \sum_{i=0}^{\infty} z^i$ 收敛且等于 $\frac{1}{1-z}$

2. (a) 根据欧拉公式 $e^{\mathrm{i}\theta} = \cos\theta + i\sin\theta$
   $z = |z|e^{\mathrm{i}\theta}$
   $= \sqrt{a^2 + b^2}(\cos\theta + i\sin\theta)$
   $= \sqrt{a^2 + b^2}\cos\theta + i\sqrt{a^2 + b^2}\sin\theta$
   $\therefore \bar{z} = \sqrt{a^2 + b^2}\cos\theta - i\sqrt{a^2 + b^2}\sin\theta$
   $= \sqrt{a^2 + b^2}(\cos\theta - i\sin\theta)$
   $= |z|e^{-\mathrm{i}\theta}$
   则 $z\bar{z} = |z|e^{\mathrm{i}\theta}|z|e^{-\mathrm{i}\theta}$
   $= |z|^2 e^{\mathrm{i}\theta - \mathrm{i}\theta}$

$= |z|^2$

(b) 已知 $e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi}$
$\because e^{i(\theta+\phi)} = \cos(\theta+\phi) + i\sin(\theta+\phi)$
且 $e^{i\theta}e^{i\phi} = (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)$
$= \cos\theta\cos\phi - \sin\theta\sin\phi + i\cos\theta\sin\phi + i\sin\theta\cos\phi$
$\therefore \cos(\theta+\phi) + i\sin(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi + i\cos\theta\sin\phi + i\sin\theta\cos\phi$
由复数相等性质可知：
$\cos(\theta+\phi) = \cos\theta\cos\phi - \sin\theta\sin\phi, \quad \sin(\theta+\phi) = \cos\theta\sin\phi + \sin\theta\cos\phi.$

3. (a) $\because (a-b)^2 = a^2 + b^2 - 2ab \geq 0$
$\therefore \frac{a^2+b^2}{2} \geq ab$
又 $\because (a+b)^2 = a^2 + b^2 + 2ab \geq 0$
$\therefore \frac{a^2+b^2}{2} \geq -ab$
故得证 $\frac{a^2+b^2}{2} \geq |ab|$

(b) $\because \{\varepsilon_t\}$ 为白噪声过程
$\therefore \mathbb{E}X_t = \sum_{i=0}^{\infty}\phi_i\varepsilon_{t-i} = 0$
$\therefore cov(X_t, X_{t-k})$
$= \mathbb{E}(X_tX_{t-k}) - \mathbb{E}X_t\mathbb{E}X_{t-k}$
$= \mathbb{E}(X_tX_{t-k})$
$= \mathbb{E}\sum_{i=0}^{\infty}\phi_i\varepsilon_{t-i}\sum_{i=0}^{\infty}\phi_i\varepsilon_{t-k-i}$
$= \mathbb{E}\sum_{i=0}^{\infty}\phi_i\varepsilon_{t-i}\sum_{i=k}^{\infty}\phi_{i-k}\varepsilon_{t-i}$
$= \sum_{i=k}^{\infty}\phi_i\phi_{i-k}\sigma_{\varepsilon}^2$
又 $\because \frac{\phi_i^2+\phi_{i-k}^2}{2} \geq |\phi_i\phi_{i-k}|$ 且 $\{\phi_i\}$ 为均方可和序列
$\therefore \sum_{i=k}^{\infty}\phi_i\phi_{i-k}$ 收敛且与 t 不相关
故得证 $\forall k \in \mathbb{Z}$，X_t 的 k-阶自协方差存在且不随时间变化