2021 秋季本科时间序列

第 1 次作业答案

10 月 18 日

1. (a) 由于 X,Y 相互独立，故联合分布密度函数为 $f(x,y)=f(x)f(y)=\frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}$

$$\begin{aligned}
F(z) &= P(\tfrac{X}{Y}\le z)=P(X\le zY, Y>0)+P(X\ge zY, Y<0)\\
&= \iint_{\substack{x\le zy\\ y>0}} f(x,y)dxdy+\iint_{\substack{x\ge zy\\ y<0}} f(x,y)dxdy\\
&= \int_0^{\infty}\left(\int_{-\infty}^{zy} f(x,y)dx\right)dy+\int_{-\infty}^{0}\left(\int_{zy}^{+\infty} f(x,y)dx\right)dy\\
&\overset{t=\frac{x}{y}}{=} \int_{-\infty}^{z}\left(\int_0^{\infty} yf(ty,y)dy\right)dt+\int_z^{\infty}\left(\int_{-\infty}^{0} yf(ty,y)dy\right)dt\\
&= \int_{-\infty}^{z}\left(\int_{-\infty}^{\infty}|y|f(ty,y)dy\right)dt
\end{aligned}$$

故

$$\begin{aligned}
f(z) &= \int_{-\infty}^{\infty}|y|f(zy,y)dy\\
&= \int_{-\infty}^{\infty}|y|\frac{1}{2\pi}e^{-\frac{z^2y^2+y^2}{2}}dy\\
&= -\frac{1}{2\pi(z^2+1)}e^{-\frac{z^2y^2+y^2}{2}}\Bigg|_0^{\infty}+\frac{1}{2\pi(z^2+1)}e^{-\frac{z^2y^2+y^2}{2}}\Bigg|_{-\infty}^{0}\\
&= \frac{1}{\pi(z^2+1)}
\end{aligned}$$

故柯西分布 Z 的概率密度函数为 $f(z)=\frac{1}{\pi(z^2+1)}$

(b) 下面证 (a) 中柯西分布 Z 的期望等于无穷, 故期望不存在。

$$\begin{aligned}
\mathbb{E}(Z) &= \int_{-\infty}^{\infty}|z|\frac{1}{\pi(z^2+1)}dz\\
&= \frac{2}{\pi}\int_0^{+\infty}\frac{z}{z^2+1}\\
&= \frac{1}{\pi}ln(z^2+1)\Bigg|_0^{+\infty}\\
&= +\infty
\end{aligned}$$

2. (a) 由于 $\mathbb{E}X_t = \mathbb{E}\cos(\frac{\pi}{n}t + U) = \int_{-\pi}^{\pi} \frac{1}{2\pi}\cos(\frac{\pi}{n}t + U)du = 0$，可得：

$$\begin{aligned}\sigma_x^2(k) &= \operatorname{cov}(X_{t+k}, X_t) = \mathbb{E}X_{t+k}X_t - \mathbb{E}X_{t+k}\mathbb{E}X_t = \mathbb{E}X_{t+k}X_t \\ &= \int_{-\pi}^{\pi} \frac{1}{2\pi}\cos(\frac{\pi}{n}t + U)\cos(\frac{\pi}{n}(t+k) + U)du \\ &= \frac{1}{4\pi}\int_{-\pi}^{\pi}\left[\cos(2u + \frac{2\pi}{n}t + \frac{k\pi}{n}) + \cos(\frac{k\pi}{n})\right]du \\ &= \frac{1}{2}\cos(\frac{k\pi}{n}), k \in N\end{aligned}$$

(b) 代码如下：

```
library(tidyverse)
library(ggplot2)
library(tseries)
X <- vector("double" ,1000)
U <- runif(1,-pi ,pi)
X<-matrix(nrow=1000,ncol=5)
for(n in 1:5)
{
  y<-c()
  for(t in 1:1000){
    x<-cos(pi/n*t+U)
    y<-c(y,x)
  }
  X[,n]<-y
}
X<-data.frame(X[1:100,])
X%>%ggplot(mapping = aes(x,y))+
  geom_point(mapping = aes(x=(1:100),y=X[,1]),color="black")+
  geom_line(mapping = aes(x=(1:100),y=X[,1]),color="blue")
X%>%ggplot(mapping = aes(x,y))+
  geom_point(mapping = aes(x=(1:100),y=X[,2]),color="black")+
  geom_line(mapping = aes(x=(1:100),y=X[,2]),color="red")
X%>%ggplot(mapping = aes(x,y))+
  geom_point(mapping = aes(x=(1:100),y=X[,3]),color="black")+
  geom_line(mapping = aes(x=(1:100),y=X[,3]),color="yellow")
X%>%ggplot(mapping = aes(x,y))+
  geom_point(mapping = aes(x=(1:100),y=X[,4]),color="black")+
```

```
  geom_line(mapping = aes(x=(1:100),y=X[,4]),color="pink")
X%>%ggplot(mapping = aes(x,y))+
  geom_point(mapping = aes(x=(1:100),y=X[,5]),color="black")+
  geom_line(mapping = aes(x=(1:100),y=X[,5]),color="green")
#n的取值影响了X_t的周期，n值越大，X_t的周期越长
#ii.
acf(X[,1], type = "covariance")
acf(X[,2], type = "covariance")
acf(X[,3], type = "covariance")
acf(X[,4], type = "covariance")
acf(X[,5], type = "covariance")
```

3. (a) 证明如下：

$$
\begin{aligned}
\mathbb{E}\hat{\sigma}_N^2 &= \mathbb{E}\left[\frac{1}{N-1}\sum_{i=1}^{N}\left(x_i-\hat{\mu}_N\right)^2\right] \\
&= \frac{1}{N-1}\mathbb{E}\left[\sum_{i=1}^{N}\left(x_i-\mu+\mu-\hat{\mu}_N\right)^2\right] \\
&= \frac{1}{N-1}\mathbb{E}\left[\sum_{i=1}^{N}\left(x_i-\mu\right)^2+2\sum_{i=1}^{N}\left(x_i-\mu\right)\left(\mu-\hat{\mu}_N\right)+\sum_{i=1}^{N}\left(\mu-\hat{\mu}_N\right)^2\right] \\
&= \frac{1}{N-1}\mathbb{E}\left[\sum_{i=1}^{N}\left(x_i-\mu\right)^2-2N\left(\mu-\hat{\mu}_N\right)^2+N\left(\mu-\hat{\mu}_N\right)^2\right] \\
&= \frac{1}{N-1}\left[\sum_{i=1}^{N}\mathbb{E}\left(x_i-\mu\right)^2-N\mathbb{E}\left(\mu-\hat{\mu}_N\right)^2\right] \\
&= \frac{1}{N-1}(N\sigma^2-N\frac{\sigma^2}{N}) \\
&= \sigma^2
\end{aligned}
$$

(b) $\hat{\sigma}_N^2=\frac{N}{N-1}\left(\frac{1}{N}\sum_{i=1}^{N}x_i^2-N\hat{\mu}_N^2\right),\quad \mathbb{E}(x_i^2)=\mathrm{var}(X)+\mathbb{E}(X_i)^2=\sigma^2+\mu^2$

对独立同分布的随机变量序列 X_t，由 LLN 得 $\frac{1}{N}\sum_{i=1}^{N}x_i^2\xrightarrow{a.s.}\mathbb{E}(X_i^2)=\sigma^2+\mu^2$

故 $\hat{\sigma}_N^2\xrightarrow{a.s.}\frac{N}{N-1}\sigma^2=\sigma^2\quad(N\to+\infty)$

4. (a)

$$
\begin{aligned}
\mathrm{cov}(x,k_1y_1+k_2y_2) &= \mathbb{E}\left[x(k_1y_1+k_2y_2)\right]-\mathbb{E}(x)\mathbb{E}(k_1y_1+k_2y_2) \\
&= k_1\left(\mathbb{E}(xy_1)-\mathbb{E}(x_1\mathbb{E}(y_1)\right)+k_2\left(\mathbb{E}(xy_2)-\mathbb{E}(x_1\mathbb{E}(y_2)\right) \\
&= k_1\mathrm{cov}(x,y_1)+k_2\mathrm{cov}(x,y_2)
\end{aligned}
$$

同理

$$
\mathrm{cov}(k_1x_1+k_2x_2,y)=k_1\mathrm{cov}(x_1,y)+k_2\mathrm{cov}(x_2,y)
$$

$$\forall k_1, k_2 \in \mathbb{R}, \mathrm{cov}(x, y) = \mathbb{E}(xy) - \mathbb{E}(x)\mathbb{E}(y) = \mathrm{cov}(y, x)$$

故 cov(,) 是对称双线性函数。

(b)

$$\begin{aligned} \mathrm{var}(aX + Y) &= \mathrm{cov}(aX + Y, aX + Y) \\ &= a^2\mathrm{cov}(X, X) + 2a\mathrm{cov}(X, Y) + cov(Y.Y) \geq 0 \\ \Delta &= 4\mathrm{cov}(X, Y)^2 - 4\mathrm{var}(X)\mathrm{var}(Y) \leq 0 \\ &\therefore |\mathrm{cov}(X, Y)|^2 \leq \sigma_X^2\sigma_Y^2 \end{aligned}$$

5. (a) 对称性：$f(x, y) = \sum_{i=1}^n x_i y_i = \sum_{i=1}^n y_i x_i = f(y, x)$

双线性性：

$$\begin{aligned} f(x, k_1y_1 + k_2y_2) &= \sum_{i=1}^n x_i(k_1y_{1i} + k_2y_{2i}) \\ &= k_1\sum_{i=1}^n x_iy_{1i} + k_2\sum_{i=1}^n x_iy_{2i} \\ &= k_1f(x, y_1) + k_2f(x, y_2) \end{aligned}$$

同理，$f(k_1x_1 + k_2x_2, y) = k_1f(x_1, y) + k_2f(x_2, y)$

(b)

$$\begin{aligned} f(ax + y, ax + y) &= \sum_{i=1}^n (ax_i + y_i)^2 \\ &= \sum_{i=1}^n a^2x_i^2 + 2a\sum_{i=1}^n x_iy_i + \sum_{i=1}^n y_i^2 \geq 0 \\ \Delta &= 4\sum_{i=1}^n (x_iy_i)^2 - 4\sum_{i=1}^n x_i^2\sum_{i=1}^n y_i^2 \leq 0 \\ \therefore |x_1y_1 + \cdots + x_ny_n|^2 &\leq (x_1^2 + \cdots + x_n^2)(y_1^2 + \cdots + y_n^2) \end{aligned}$$

(c) 由 (b) 知

$$\begin{aligned} \hat{\sigma}_{XY}^2 &= \frac{1}{n}\sum_i (X_i - \hat{\mu}_X)(Y_i - \hat{\mu}_Y) \\ &\leq \frac{1}{n}\sqrt{\sum_i (X_i - \hat{\mu}_X)^2 \sum_i (Y_i - \hat{\mu}_Y)^2} \\ &= \frac{1}{n}\sqrt{n\hat{\sigma}_X^2 \cdot n\hat{\sigma}_Y^2} \\ &= \hat{\sigma}_X\hat{\sigma}_Y \text{得证} \end{aligned}$$

6. 证明如下：$\{\phi_i\}_{i=0}^\infty$ 绝对可和，即 $\sum_{i=0}^\infty |\phi_i| < \infty, \sum_{i=0}^\infty |\phi_i|^2 < \infty$

易知仅有有限项 $|\phi_i| \geq 1$，假设有 t 项，$k \geq 0$

$$\begin{cases} |\phi_i| \geq 1 & i = 1, 2, \cdots, t \\ |\phi_i| < 1 & i \geq t \end{cases}$$

$\sum_{i=k}^{\infty}|\phi_i\phi_{i-k}| = \sum_{i=k}^{t+k}|\phi_i\phi_{i-k}| + \sum_{i=t+k+1}^{\infty}|\phi_i\phi_{i-k}|$

记$S_1 = \sum_{i=k}^{t+k}|\phi_i\phi_{i-k}|, S_2 = \sum_{i=t+k+1}^{\infty}|\phi_i\phi_{i-k}|$

S_1为有限项，故$S_1 < \infty$

$S_2 < \sum_{i=t+k+1}^{\infty}|\phi_i| < \infty$

故$\sum_{i=k}^{\infty}|\phi_i\phi_{i-k}| < \infty$