2020 秋季本科时间序列

第 7 次作业答案

12 月 16 日

1. (a) 令 $Z = (X_i - \mathbb{E}X_i) - \Sigma_{XY,i}\Sigma_{YY}^{-1}(Y - \mathbb{E}Y)$

   易得:

   i. $\mathbb{E}Z = 0$

      证明:

$$\begin{aligned}\mathbb{E}Z &= \mathbb{E}[(X_i - \mathbb{E}X_i) - \Sigma_{XY,i}\Sigma_{YY}^{-1}(Y - \mathbb{E}Y)\\ &= \mathbb{E}X_i - \mathbb{E}X_i - \Sigma_{XY,i}\Sigma_{YY}^{-1}(\mathbb{E}Y - \mathbb{E}Y)\\ &= 0\end{aligned}$$

   ii. $\mathrm{cov}(Z, Y^\intercal) = \vec{0}$

      证明:

$$\begin{aligned}\mathrm{cov}(Z, Y^\intercal) &= \mathbb{E}[Z(Y - \mu_Y)^\intercal]\\ &= \mathbb{E}[(X_i - \mathbb{E}X_i)(Y - \mu_Y)^\intercal - \Sigma_{XY,i}\Sigma_{YY}^{-1}(Y - \mathbb{E}Y)(Y - \mu_Y)^\intercal]\\ &= \Sigma_{XY,i} - \Sigma_{XY,i}\Sigma_{YY}^{-1}\Sigma_{YY}\\ &= \vec{0}\end{aligned}$$

   所以 Z 和 Y 互相独立对于任意关于 X_i 的预测函数 f(Y)

   有

$$\begin{aligned}\mathbb{E}[X_i - f(Y)]^2 &= \mathbb{E}[X_i - \mathbb{E}(X_i|Y) + \mathbb{E}(X_i|Y) - f(Y)]^2\\ &= \mathbb{E}[X_i - \mathbb{E}(X_i|Y)]^2 + 2\mathbb{E}[(X_i - \mathbb{E}(X_i|Y))(\mathbb{E}(X_i|Y) - f(Y))] + \mathbb{E}[\mathbb{E}(X_i|Y) - f(Y)]^2\\ &= \mathbb{E}[X_i - \mathbb{E}(X_i|Y)]^2 + 2\mathbb{E}[\mathbb{E}[(X_i - \mathbb{E}(X_i|Y))(\mathbb{E}(X_i|Y) - f(Y))|Y]]\\ &+ \mathbb{E}[\mathbb{E}(X_i|Y) - f(Y)]^2\\ &= \mathbb{E}[X_i - \mathbb{E}(X_i|Y)]^2 + \mathbb{E}[\mathbb{E}(X_i|Y) - f(Y)]^2\\ &\geq \mathbb{E}[X_i - \mathbb{E}(X_i|Y)]^2 \qquad \text{当且仅当} f(Y) = \mathbb{E}(X_i|Y) \text{时等式成立}\end{aligned}$$

由此可知 $\mathbb{E}(X_i|Y)$ 是 X_i 关于 Y 的最小均方误标准下的最优预测

则 $\mathbb{E}[X_i - Z|Y] = \mathbb{E}(X_i|Y) - \mathbb{E}(Z|Y) = \mathbb{E}(X_i|Y)$

可知 $X_i - Z$ 是 X_i 关于 Y 的最优预测

则 $\mathbb{E}(X_i|Y) = \mathbb{E}[X_i - Z|Y] = \mathbb{E}[\mathbb{E}X_i - \Sigma_{XY,i}\Sigma_{YY}^{-1}(Y-\mu_Y)|Y]$

$= \mathbb{E}X_i + \Sigma_{XY,i}\Sigma_{YY}^{-1}(Y-\mu_Y)$得证

(b) 由 (a) 问可知 $\mathbb{E}(X_i|Y) = \mathbb{E}X_i + \Sigma_{XY,i}\Sigma_{YY}^{-1}(Y-\mu_Y)$

则 $\mathbb{E}(X|Y) = \mathbb{E}\begin{pmatrix} X_1|Y \\ X_2|Y \\ \vdots \\ X_n|Y \end{pmatrix} = \begin{pmatrix} \mathbb{E}(X_1|Y) \\ \mathbb{E}(X_2|Y) \\ \vdots \\ \mathbb{E}(X_n|Y) \end{pmatrix} = \begin{pmatrix} \mathbb{E}X_1 + \Sigma XY,1\Sigma_{YY}^{-1}(Y-\mu_Y) \\ \mathbb{E}X_2 + \Sigma_{XY,2}\Sigma_{YY}^{-1}(Y-\mu_Y) \\ \vdots \\ \mathbb{E}X_n + \Sigma_{XY,n}\Sigma_{YY}^{-1}(Y-\mu_Y) \end{pmatrix}$

$= \begin{pmatrix} \mathbb{E}X_1 \\ \mathbb{E}X_2 \\ \vdots \\ \mathbb{E}X_n \end{pmatrix} + \begin{pmatrix} \Sigma_{XY,1}\Sigma_{YY}^{-1}(Y-\mu_Y) \\ \Sigma_{XY,2}\Sigma_{YY}^{-1}(Y-\mu_Y) \\ \vdots \\ \Sigma_{XY,n}\Sigma_{YY}^{-1}(Y-\mu_Y) \end{pmatrix}$

$= \mu_X + \Sigma_{XY}\Sigma_{YY}^{-1}(Y-\mu_Y)$

(c) 由 MA(2) 过程 $X_t = \varepsilon_t + \varepsilon_{t-1} + \varepsilon_{t-2}$可知$\varepsilon_t \sim N(0,\sigma_\varepsilon^2)$

则 $\mathbb{E}X_t = 0$ 且

$$\sigma_\varepsilon^2(k) = \begin{cases} (3-k)\sigma_\varepsilon^2 & ,k=0,1,2 \\ 0 & ,k>2 \end{cases}$$

定义 $X = (X_{t+1}, X_{t+2}, \cdots)^\intercal, Y = (X_{t-1}, X_{t-2}, \cdots)^\intercal$

由 (a) 问可知 $\mathbb{E}_t X_{t+j} = \mathbb{E}[X_{t+j}|X_t, X_{t-1}, X_{t-2}, \cdots] = \mathbb{E}X_{t+j} + \Sigma_{XY,j}\Sigma_{YY}^{-1}(Y-\mu_Y) = \Sigma_{XY,j}\Sigma_{YY}^{-1}Y$

且 $\Sigma_{YY} = \sigma_\varepsilon^2\begin{pmatrix} 3 & 2 & 1 & 0 & \cdots & 0 \\ 2 & 3 & 2 & 1 & \cdots & 0 \\ 1 & 2 & 3 & 2 & \cdots & 0 \\ 0 & 1 & 2 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \sum_{XY} = \sigma_\varepsilon^2\begin{pmatrix} 2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$

将 $\Sigma_{YY}, \Sigma_{XY}, Y$ 代入即可求得条件期望

$$\mathbb{E}_t X_{t+j} = \begin{cases} \Sigma_{XY,j}\Sigma_{YY}^{-1}Y & ,j \le 2 \\ 0 & ,j>2 \end{cases}$$

则当 j=1 时, X_{t+1} 与 $X_{t-2}, \cdots$ 独立

$\mathbb{E}_t X_{t+1} = \sigma_\varepsilon^2\begin{pmatrix} 2 & 1 \end{pmatrix}\left[\sigma_\varepsilon^2\begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}\right]^{-1}\begin{pmatrix} X_t \\ X_{t-1} \end{pmatrix} = \frac{4}{5}X_t - \frac{1}{5}X_{t-1}$

当 j=2 时,X_{t+2} 与 $X_{t-1},\cdots$ 独立, $\mathbb{E}_t X_{t+2}=\sigma_\varepsilon^2\frac{1}{3\sigma_\varepsilon^2}X_t=\frac{1}{3}X_t$

当 j>2 时, $\mathbb{E}_t X_{t+j}=0$

2. (a) 又题易得, 对于 AR(2) 过程 $X_t=\phi_1X_{t-1}+\phi_2X_{t-2}+\varepsilon_t$

$\hat{X}_{t+1|t}=\phi_1X_t+\phi_2X_{t-1}$

写为矩阵形式可得 $\begin{pmatrix}\hat{X}_{t+1|t}\\ \hat{X}_{t|t}\end{pmatrix}=\begin{pmatrix}\phi_1&\phi_2\\1&0\end{pmatrix}\begin{pmatrix}X_{t|t}\\X_{t-1|t}\end{pmatrix}=\begin{pmatrix}\phi_1&\phi_2\\1&0\end{pmatrix}\begin{pmatrix}X_t\\X_{t-1}\end{pmatrix}$

进而可得 $\begin{pmatrix}\hat{X}_{t+j|t}\\ \hat{X}_{t+j-1|t}\end{pmatrix}=\begin{pmatrix}\phi_1&\phi_2\\1&0\end{pmatrix}^j\begin{pmatrix}X_t\\X_{t-1}\end{pmatrix}$

下求 $\begin{pmatrix}\phi_1&\phi_2\\1&0\end{pmatrix}^j$, 令 $\Phi=\begin{pmatrix}\phi_1&\phi_2\\1&0\end{pmatrix}$

易得 Φ 的两个特征值为 $\lambda_1=\frac{\phi_1+\sqrt{\phi_1^2+4\phi_2}}{2},\lambda_2=\frac{\phi_1-\sqrt{\phi_1^2+4\phi_2}}{2}$ 且有 $\lambda_1+\lambda_2=\phi_1,-\lambda_1\lambda_2=\phi_2$

进而求得 Φ 的 $P\Lambda P^{-1}$ 分解

$\Phi=\frac{-1}{\phi_2\sqrt{\phi_1^2+4\phi_2}}\begin{pmatrix}\phi_2&\phi_2\\ \frac{\sqrt{\phi_1^2+4\phi_2}-\phi_1}{2}&\frac{-\sqrt{\phi_1^2+4\phi_2}-\phi_1}{2}\end{pmatrix}\begin{pmatrix}\frac{\sqrt{\phi_1^2+4\phi_2}+\phi_1}{2}&0\\0&\frac{\phi_1-\sqrt{\phi_1^2+4\phi_2}}{2}\end{pmatrix}\begin{pmatrix}\frac{-\sqrt{\phi_1^2+4\phi_2}-\phi_1}{2}&-\phi_2\\ \frac{\sqrt{\phi_1^2+4\phi_2}-\phi_1}{2}&\phi_2\end{pmatrix}$

其中 $P=\begin{pmatrix}\phi_2&\phi_2\\ \frac{\sqrt{\phi_1^2+4\phi_2}-\phi_1}{2}&\frac{-\sqrt{\phi_1^2+4\phi_2}-\phi_1}{2}\end{pmatrix},\Lambda=\begin{pmatrix}\frac{\sqrt{\phi_1^2+4\phi_2}+\phi_1}{2}&0\\0&\frac{\phi_1-\sqrt{\phi_1^2+4\phi_2}}{2}\end{pmatrix}$,

$P^{-1}=\frac{-1}{\phi_2\sqrt{\phi_1^2+4\phi_2}}\begin{pmatrix}\frac{-\sqrt{\phi_1^2+4\phi_2}-\phi_1}{2}&-\phi_2\\ \frac{\sqrt{\phi_1^2+4\phi_2}-\phi_1}{2}&\phi_2\end{pmatrix}$

进而 $\Phi^j=P\Lambda^jP^{-1}=\frac{1}{\lambda_2-\lambda_1}\begin{pmatrix}-\lambda_1^{j+1}+\lambda_2^{j+1}&\lambda_1^{j+1}\lambda_2-\lambda_1\lambda_2^{j+1}\\-\lambda_1^j+\lambda_2^j&\lambda_1^j\lambda_2-\lambda_1\lambda_2^j\end{pmatrix}$(省略计算过程)

因此, 可得 $\hat{X}_{t+j|t}=\frac{-\lambda_1^{j+1}+\lambda_2^{j+1}}{\lambda_2-\lambda_1}X_t+\frac{\lambda_1^{j+1}\lambda_2-\lambda_1\lambda_2^{j+1}}{\lambda_2-\lambda_1}X_{t-1}$

(b) 见 (a)

(c) 令 A $=\begin{pmatrix}\phi_1&\phi_2\\1&0\end{pmatrix}$

可知其特征多项式为 $\lambda^2-\theta_1\lambda-\theta_2=0$, 且根为 $\lambda_1=\frac{1}{Z_1},\lambda_2=\frac{1}{Z_2}$

由于 X_t 是平稳时间序列，则根位于单位圆之外，即 $|\lambda_1|$ 和 $|\lambda_2|$ 均小于 1

将 A 特征分解可得 $A=C\Lambda C^{-1}$ 且 $\Lambda=\begin{pmatrix}\lambda_1&\\&\lambda_2\end{pmatrix}$

则 $A^j=(C\Lambda C^{-1})^j=C\Lambda^jC^{-1}$

则 $\lim_{j\to\infty}A^j=\lim_{j\to\infty}C\begin{pmatrix}\lambda_1^j&\\&\lambda_2^j\end{pmatrix}C^{-1}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$

此时有 $\lim_{j\to\infty}\begin{pmatrix}\hat{X}_{t+j|t}\\ \hat{X}_{t+j-1|t}\end{pmatrix}=\lim_{j\to\infty}\begin{pmatrix}\phi_1 & \phi_2\\ 1 & 0\end{pmatrix}^j\begin{pmatrix}X_t\\ X_{t-1}\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$

则 $\lim_{j\to\infty}\hat{X}_{t+j|t}=0$

(d) 当 $\phi_1=1.7,\phi_2=-0.72$ 时,$A=\begin{pmatrix}1.7 & -0.72\\ 1 & 0\end{pmatrix}$

此时 A 的特征多项式为 $\lambda^2-1.7\lambda+0.72=0$, 且根为 $\lambda_1=0.8,\lambda_2=0.9$

由此可求出 λ_1,λ_2 的特征向量分别为 $T_1=\begin{pmatrix}0.8\\ 1\end{pmatrix},T_2=\begin{pmatrix}0.8\\ 1\end{pmatrix}$

则 $C=\begin{pmatrix}0.8 & 0.9\\ 1 & 1\end{pmatrix},\Lambda=\begin{pmatrix}0.8 & \\ & 0.9\end{pmatrix}$

则 $A=C\Lambda C^{-1}=\begin{pmatrix}0.8 & 0.9\\ 1 & 1\end{pmatrix}\begin{pmatrix}0.8 & \\ & 0.9\end{pmatrix}\begin{pmatrix}0.8 & 0.9\\ 1 & 1\end{pmatrix}^{-1}$

且

$$\begin{aligned}\begin{pmatrix}\hat{X}_{t+j|t}\\ \hat{X}_{t+j-1|t}\end{pmatrix}&=A^j\begin{pmatrix}X_t\\ X_{t-1}\end{pmatrix}\\ &=C\Lambda^jC^{-1}\begin{pmatrix}X_t\\ X_{t-1}\end{pmatrix}\\ &=\begin{pmatrix}0.8 & 0.9\\ 1 & 1\end{pmatrix}\begin{pmatrix}0.8^j & \\ & 0.9^j\end{pmatrix}\begin{pmatrix}0.8 & 0.9\\ 1 & 1\end{pmatrix}^{-1}\begin{pmatrix}X_t\\ X_{t-1}\end{pmatrix}\end{aligned}$$

故 $\hat{X}_{t+j|t}=10(0.9^{j+1}-0.8^{j+1})X_t+(9\cdot 0.8^{j+1}-8\cdot 0.9^{j+1})X_{t-1}$