2020 秋季本科时间序列

第 5 次作业答案

11 月 25 日

1.

$$
\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \end{bmatrix}\begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{D} \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1l} & b_{11} & \cdots & b_{1m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kl} & b_{k1} & \cdots & b_{km} \end{bmatrix}\begin{bmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{l1} & \cdots & c_{ln} \\ d_{11} & \cdots & d_{1n} \\ \vdots & \ddots & \vdots \\ d_{m1} & \cdots & d_{mn} \end{bmatrix}
$$

$$
= \begin{bmatrix} \sum_{j=1}^{l} a_{1j}c_{j1} + \sum_{j=1}^{m} b_{1j}d_{j1} & \cdots & \sum_{j=1}^{l} a_{1j}c_{jn} + \sum_{j=1}^{m} b_{1j}d_{jn} \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^{l} a_{kj}c_{j1} + \sum_{j=1}^{m} b_{kj}d_{j1} & \cdots & \sum_{j=1}^{l} a_{kj}c_{jn} + \sum_{j=1}^{m} b_{kj}d_{jn} \end{bmatrix}
$$

$$
= \begin{bmatrix} \sum_{j=1}^{l} a_{1j}c_{j1} & \cdots & \sum_{j=1}^{l} a_{1j}c_{jn} \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^{l} a_{kj}c_{j1} & \cdots & \sum_{j=1}^{l} a_{kj}c_{jn} \end{bmatrix} + \begin{bmatrix} \sum_{j=1}^{m} b_{1j}d_{j1} & \cdots & \sum_{j=1}^{m} b_{1j}d_{jn} \\ \vdots & \ddots & \vdots \\ \sum_{j=1}^{m} b_{kj}d_{j1} & \cdots & \sum_{j=1}^{m} b_{kj}d_{jn} \end{bmatrix}
$$

$$
= \boldsymbol{AC} + \boldsymbol{BD}
$$

2. (a)

$$
\boldsymbol{Y} = \begin{bmatrix} X_{p+1} \\ \vdots \\ X_T \end{bmatrix}, \boldsymbol{X} = \begin{bmatrix} X_p & \cdots & X_1 \\ \vdots & \ddots & \vdots \\ X_{T-1} & \cdots & X_{T-p} \end{bmatrix}
$$

则：

$$
\frac{1}{T}\boldsymbol{X}^{\intercal}\boldsymbol{X} = \frac{1}{T}\begin{bmatrix} X_p & \cdots & X_{T-1} \\ \vdots & \ddots & \vdots \\ X_1 & \cdots & X_{T-p} \end{bmatrix}\begin{bmatrix} X_p & \cdots & X_1 \\ \vdots & \ddots & \vdots \\ X_{T-1} & \cdots & X_{T-p} \end{bmatrix}
$$

$$= \frac{1}{T}\begin{bmatrix} \sum_{t=p}^{T-1} X_t X_t & \cdots & \sum_{t=1}^{T-p} X_{t+p-1} X_t \\ \vdots & \ddots & \vdots \\ \sum_{t=1}^{T-p} X_t X_{t+p-1} & \cdots & \sum_{t=1}^{T-p} X_t X_t \end{bmatrix}$$

由于 $\boldsymbol{X}^\intercal$ 是平稳的，且 $\mathbb{E}(X_t) = 0$，根据大数定律

$$\frac{1}{T}\sum_{t=1}^{T} X_t X_{t-k} \xrightarrow{a.s.} \sigma^2(k)$$

因此

$$\frac{1}{T}\boldsymbol{X}^\intercal\mathbf{X} \xrightarrow{a.s.} \begin{bmatrix} \sigma_X^2(0) & \cdots & \sigma_X^2(p-1) \\ \vdots & \ddots & \vdots \\ \sigma_X^2(p-1) & \cdots & \sigma_X^2(0) \end{bmatrix}$$

同理

$$\frac{1}{T}\boldsymbol{X}^\intercal\mathbf{Y} = \frac{1}{T}\begin{bmatrix} X_p & \cdots & X_{T-1} \\ \vdots & \ddots & \vdots \\ X_1 & \cdots & X_{T-p} \end{bmatrix}\begin{bmatrix} X_{p+1} \\ \vdots \\ X_T \end{bmatrix}$$

$$= \frac{1}{T}\begin{bmatrix} \sum_{t=1}^{T-p} X_{t+p-1} X_{t+p} \\ \vdots \\ \sum_{t=1}^{T-p} X_t X_{t+p} \end{bmatrix} \xrightarrow{a.s.} \begin{bmatrix} \sigma_X^2(1) \\ \vdots \\ \sigma_X^2(p) \end{bmatrix}$$

故 AR(p) 平稳自回归过程的 OLS 估计

$$\hat{\beta} = (\boldsymbol{X}^\intercal\mathbf{X})^{-1}\boldsymbol{X}^\intercal\mathbf{Y} = (\frac{1}{T}\boldsymbol{X}^\intercal\mathbf{X})^{-1}\frac{1}{T}\boldsymbol{X}^\intercal\mathbf{Y}$$

$$\xrightarrow{a.s.} \begin{bmatrix} \sigma_X^2(0) & \cdots & \sigma_X^2(p-1) \\ \vdots & \ddots & \vdots \\ \sigma_X^2(p-1) & \cdots & \sigma_X^2(0) \end{bmatrix}\begin{bmatrix} \sigma_X^2(1) \\ \vdots \\ \sigma_X^2(p) \end{bmatrix}$$

(b)

$$X_t = \phi_0 + \phi_1 X_{t-1} + \epsilon_t \phi_0 \neq 0 p = 1$$

此时有

$$X = \begin{bmatrix} 1 & X_1 \\ \vdots & \vdots \\ 1 & X_{T-1} \end{bmatrix} \Phi = \begin{bmatrix} \phi_2 \\ \phi_1 \end{bmatrix}, Y = \begin{bmatrix} X_2 \\ \vdots \\ X_T \end{bmatrix}$$

故

$$\hat{\phi} = (\boldsymbol{X}^\intercal\mathbf{X})^{-1}\boldsymbol{X}^\intercal\mathbf{Y}$$

$$= \left(\begin{bmatrix} 1 & X_1 \\ \vdots & \vdots \\ 1 & X_{T-1} \end{bmatrix}^\intercal \begin{bmatrix} 1 & X_1 \\ \vdots & \vdots \\ 1 & X_{T-1} \end{bmatrix}\right)^{-1} \begin{bmatrix} 1 & X_1 \\ \vdots & \vdots \\ 1 & X_{T-1} \end{bmatrix}^\intercal \begin{bmatrix} X_2 \\ \vdots \\ X_T \end{bmatrix}$$

$$=\left(\frac{1}{T}\begin{bmatrix} T-1 & \sum_{t=1}^{T-1} X_t \\ \sum_{t=1}^{T-1} X_t & \sum_{t=1}^{T-1} X_t X_t \end{bmatrix}\right)^{-1} \frac{1}{T}\begin{bmatrix} \sum_{t=2}^{T} X_t \\ \sum_{t=1}^{T-1} X_t X_{t+1} \end{bmatrix}$$

$$\xrightarrow{a.s.} \begin{bmatrix} 1 & \mu_X \\ \mu_X & \sigma_X^2(0)+\mu_X^2 \end{bmatrix}^{-1} \begin{bmatrix} \mu_X \\ \sigma_X^2(1)+\mu_X^2 \end{bmatrix}$$

此时，$\hat{\phi}$ 的极限不具有 Yule-Walker 方程形式

(c)

$$X_t = \phi_0 + \sum_{i=1}^{p} \phi_i X_{t-i} + \epsilon_t \, |\phi_1| < 1$$

等式两边同时乘以 X_{t-1} 并取期望可得

$$\mathbb{E}(X_t X_{t-1}) = \mathbb{E}(\phi_0 X_{t-1}) + \sum_{i=1}^{p} \mathbb{E}(\phi_i X_{t-i} X_t)$$

由于 $\mathbb{E}X_t \neq 0 \mathbb{E}X_t = \frac{\phi_0^2}{1-\sum \phi_i}$ 知

$$\sigma^2(1) + \frac{\phi_0^2}{(1-\sum \phi_i)^2} = \frac{\phi_0^2}{1-\sum \phi_i} + \sum_{i=1}^{p} \phi_i \left[\sigma^2(i-1) + \frac{\phi_0^2}{1-\sum \phi_i^2}\right]$$

化简得

$$\sigma^2(1) = \sum_{i=1}^{p} \phi_i \sigma^2(i-1)$$

故同理可得出

$$\begin{bmatrix} \sigma_X^2(1) \\ \vdots \\ \sigma_X^2(p) \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_p \end{bmatrix} \begin{bmatrix} \sigma_X^2(0) & \cdots & \sigma_X^2(p-1) \\ \vdots & \ddots & \vdots \\ \sigma_X^2(p-1) & \cdots & \sigma_X^2(0) \end{bmatrix}$$

(d) 由 (b) 得

$$\hat{\phi} = (\boldsymbol{X}^\intercal \mathbf{X})^{-1} \boldsymbol{X}^\intercal \mathbf{Y}$$

其中

$$\boldsymbol{X} = \begin{bmatrix} 1 & X_p & \cdots & X_1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & X_{T-1} \cdots & X_{T-p} & \end{bmatrix}_{(T-p)\times(p+1)} \quad \boldsymbol{Y} = \begin{bmatrix} X_{p+1} \\ \vdots \\ X_T \end{bmatrix}_{(T-p)\times 1}$$

可知

$$\frac{1}{T}\boldsymbol{X}^\intercal \mathbf{X} = \frac{1}{T}\begin{bmatrix} 1 & X_p & \cdots & X_1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & X_{T-1} & \cdots & X_{T-p} \end{bmatrix}^\intercal \begin{bmatrix} 1 & X_p & \cdots & X_1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & X_{T-1} & \cdots & X_{T-p} \end{bmatrix}$$

$$
= \frac{1}{T}\begin{bmatrix} T-p & \sum_{t=p}^{T-1} X_t & \cdots & \sum_{t=1}^{T-p} X_t \\ \sum_{t=p}^{T-1} X_t & \sum_{t=p}^{T-1} X_t X_t & \cdots & \sum_{t=1}^{T-p} X_{t+p-1} X_t \\ \vdots & \ddots & \vdots & \\ \sum_{t=1}^{T-p} X_t & \sum_{t=1}^{T-p} X_t X_{t+p-1} & \cdots & \sum_{t=1}^{T-p} X_t X_t \end{bmatrix}
$$

$$
\xrightarrow{a.s.} \begin{bmatrix} 1 & \mu_X & \cdots & \mu_X \\ \mu_X & \sigma_X^2(0) + \mu_X^2 & \cdots & \sigma_X^2(p-1) + \mu_X^2 \\ \vdots & \vdots & \cdots & \vdots \\ \mu_X & \sigma_X^2(p-1) + \mu_X^2 & \cdots & \sigma_X^2(0) + \mu_X^2 \end{bmatrix}
$$

同理

$$
\frac{1}{T}\boldsymbol{X}^\intercal \mathbf{Y} = \frac{1}{T}\begin{bmatrix} 1 & X_p & \cdots & X_1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & X_{T-1} & \cdots & X_{T-p} \end{bmatrix}^\intercal \begin{bmatrix} X_{p+1} \\ \vdots \\ X_T \end{bmatrix}
$$

$$
= \frac{1}{T}\begin{bmatrix} \sum_{t=p+1}^{T} X_t \\ \sum_{t=1}^{T-p} X_{t+p-1} X_{t+p} \\ \vdots \\ \sum_{t=1}^{T-p} X_t X_{t+p} \end{bmatrix} \xrightarrow{a.s.} \begin{bmatrix} \mu_X \\ \sigma_X^2(1) + \mu_X^2 \\ \vdots \\ \sigma_X^2(p) + \mu_X^2 \end{bmatrix}
$$

故

$$
\hat{\phi} = (\boldsymbol{X}^\intercal \mathbf{X})^{-1} \boldsymbol{X}^\intercal \mathbf{Y}
$$

$$
\xrightarrow{a.s.} \begin{bmatrix} 1 & \mu_X & \cdots & \mu_X \\ \mu_X & \sigma_X^2(0) + \mu_X^2 & \cdots & \sigma_X^2(p-1) + \mu_X^2 \\ \vdots & \vdots & \cdots & \vdots \\ \mu_X & \sigma_X^2(p-1) + \mu_X^2 & \cdots & \sigma_X^2(0) + \mu_X^2 \end{bmatrix}^{-1} \begin{bmatrix} \mu_X \\ \sigma_X^2(1) + \mu_X^2 \\ \vdots \\ \sigma_X^2(p) + \mu_X^2 \end{bmatrix}
$$

此时回归系数向量 ϕ 的 OLS 估计值不等于 (c) 中的 Yule-Walker 方程

3. (a) i.

$$
\begin{aligned}
\boldsymbol{P}_X \boldsymbol{\xi} &= \boldsymbol{X}(\boldsymbol{X}^\intercal \boldsymbol{X})^{-1} \boldsymbol{X}^\intercal \boldsymbol{X} \boldsymbol{a} \\
&= \boldsymbol{X} \boldsymbol{I} \boldsymbol{a} \\
&= \boldsymbol{X} \boldsymbol{a} \\
&= \boldsymbol{\xi}
\end{aligned}
$$

ii.

$$
\begin{aligned}
(\boldsymbol{X}\boldsymbol{a})^\intercal(\boldsymbol{I}-\boldsymbol{P}_X)\boldsymbol{\zeta} &= \boldsymbol{a}^\intercal\boldsymbol{X}^\intercal(\boldsymbol{\zeta}-\boldsymbol{X}(\boldsymbol{X}^\intercal\boldsymbol{X})^{-1}\boldsymbol{X}^\intercal\boldsymbol{\zeta}) \\
&= \boldsymbol{a}^\intercal\boldsymbol{X}^\intercal\boldsymbol{\zeta}-\boldsymbol{a}^\intercal\boldsymbol{X}^\intercal\boldsymbol{X}(\boldsymbol{X}^\intercal\boldsymbol{X})^{-1}\boldsymbol{X}^\intercal\boldsymbol{\zeta} \\
&= \boldsymbol{a}^\intercal\boldsymbol{X}^\intercal\boldsymbol{\zeta}-\boldsymbol{a}^\intercal\boldsymbol{X}^\intercal\boldsymbol{I}\boldsymbol{\zeta} \\
&= \boldsymbol{0}
\end{aligned}
$$

(b) i. 由 OLS 估计系数表达式可知，$\boldsymbol{Y}$ 对 $\boldsymbol{W}$ 回归的系数为 $\boldsymbol{W}(\boldsymbol{W}^\intercal\boldsymbol{W})^{-1}\boldsymbol{W}^\intercal\boldsymbol{Y}$，而 $\boldsymbol{Z}$ 对 $\boldsymbol{W}$ 回归的系数为 $\boldsymbol{W}(\boldsymbol{W}^\intercal\boldsymbol{W})^{-1}\boldsymbol{W}^\intercal\boldsymbol{Z}$，故两组回归对应的残差向量分别为

$$
\begin{aligned}
(\boldsymbol{I}-\boldsymbol{W}(\boldsymbol{W}^\intercal\boldsymbol{W})^{-1}\boldsymbol{W}^\intercal)\boldsymbol{Y} = (\boldsymbol{I}-\boldsymbol{P}_W)\boldsymbol{Y} = \tilde{\boldsymbol{Y}} \\
(\boldsymbol{I}-\boldsymbol{W}(\boldsymbol{W}^\intercal\boldsymbol{W})^{-1}\boldsymbol{W}^\intercal)\boldsymbol{Z} = (\boldsymbol{I}-\boldsymbol{P}_W)\boldsymbol{Z} = \tilde{\boldsymbol{Z}}
\end{aligned}
$$

ii. 由 $\hat{\boldsymbol{e}} = \boldsymbol{I}-\boldsymbol{X}(\boldsymbol{X}^\intercal\boldsymbol{X})^{-1}\boldsymbol{X}^\intercal\boldsymbol{Y} = (\boldsymbol{I}-\boldsymbol{P}_X)\boldsymbol{Y}$，由 (a) 的结论可得 $\hat{\boldsymbol{e}}$ 与 $\boldsymbol{X}$ 的每个列向量相互垂直，故垂直于 $\boldsymbol{Z}$ 与 $\boldsymbol{W}$。

iii. $\boldsymbol{P}_W\hat{\boldsymbol{e}} = \boldsymbol{W}(\boldsymbol{W}^\intercal\boldsymbol{X})^{-1}\boldsymbol{W}^\intercal\hat{\boldsymbol{e}}$，由 iii 可知 $\boldsymbol{W}^\intercal\boldsymbol{e} = \boldsymbol{0}$，故得所证。

iv. 证法 1：只需要说明 $\hat{\boldsymbol{\delta}}$ 正好满足回归 $\tilde{\boldsymbol{Y}} = \tilde{\boldsymbol{Z}}\boldsymbol{\delta}+\boldsymbol{u}$ 的 OLS 系数估计所需满足的一阶条件 $\tilde{\boldsymbol{Z}}^\intercal\tilde{Y}-\tilde{\boldsymbol{Z}}^\intercal\tilde{\boldsymbol{Z}}\boldsymbol{\delta} = \boldsymbol{0}$。为此，首先注意到 $\hat{\boldsymbol{e}} = \boldsymbol{Y}-\boldsymbol{X}\hat{\boldsymbol{\beta}} = \boldsymbol{Y}-\boldsymbol{Z}\hat{\boldsymbol{\delta}}-\boldsymbol{W}\hat{\boldsymbol{\theta}}$，两端同时乘以 $\boldsymbol{I}-\boldsymbol{P}_W$ 可得

$$
(\boldsymbol{I}-\boldsymbol{P}_W)\hat{\boldsymbol{e}} = (\boldsymbol{I}-\boldsymbol{P}_W)\boldsymbol{Y}-(\boldsymbol{I}-\boldsymbol{P}_W)\boldsymbol{Z}\hat{\boldsymbol{\delta}}-(\boldsymbol{I}-\boldsymbol{P}_W)\boldsymbol{W}\hat{\boldsymbol{\theta}}
$$

由 i–iii 结论可知，上式可以进一步写为

$$
\hat{\boldsymbol{e}} = \tilde{\boldsymbol{Y}}+\tilde{\boldsymbol{Z}}\hat{\boldsymbol{\delta}}
$$

再在两端乘以 $\tilde{\boldsymbol{Z}}^\intercal$ 可得

$$
\boldsymbol{0} = \tilde{\boldsymbol{Z}}\hat{\boldsymbol{e}} = \tilde{\boldsymbol{Z}}^\intercal\tilde{\boldsymbol{Y}}+\tilde{\boldsymbol{Z}}^\intercal\tilde{\boldsymbol{Z}}\hat{\boldsymbol{\delta}}
$$

即得所证。

证法 2：模型 (1) 的 OLS 估计

$$
\begin{aligned}
\hat{\boldsymbol{\beta}} &= (\boldsymbol{X}^\intercal\mathbf{X})^{-1}\boldsymbol{X}^\intercal\mathbf{Y} \\
&= \begin{bmatrix} \boldsymbol{Z}^\intercal\boldsymbol{Z} & \boldsymbol{Z}^\intercal\boldsymbol{W} \\ \boldsymbol{W}^\intercal\boldsymbol{Z} & \boldsymbol{W}^\intercal\boldsymbol{W} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{Z}^\intercal\boldsymbol{Y} \\ \boldsymbol{W}^\intercal\boldsymbol{Y} \end{bmatrix}
\end{aligned}
$$

由分块矩阵的求逆公式

$$
\begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}^{-1} = \begin{bmatrix} (\boldsymbol{A}-\boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C})^{-1} & (\boldsymbol{A}-\boldsymbol{B}\boldsymbol{D}^{-1}\boldsymbol{C})^{-1}\boldsymbol{B}\boldsymbol{D}^{-1} \\ \cdots & \cdots \end{bmatrix}
$$

得

$$
\begin{aligned}
\hat{\delta} &= (\boldsymbol{Z}^{\top}\boldsymbol{Z} - \boldsymbol{Z}^{\top}\boldsymbol{W}(\boldsymbol{W}^{\top}\boldsymbol{W})^{-1}\boldsymbol{W}^{\top}\boldsymbol{Z})^{-1}\boldsymbol{Z}^{\top}\boldsymbol{Y} - (\boldsymbol{Z}^{\top}\boldsymbol{Z} \\
&\quad - \boldsymbol{Z}^{\top}\boldsymbol{W}(\boldsymbol{W}^{\top}\boldsymbol{W})^{-1}\boldsymbol{W}^{\top}\boldsymbol{Z})^{-1}\boldsymbol{Z}^{\top}\boldsymbol{W}(\boldsymbol{W}^{\top}\boldsymbol{W})^{-1}\boldsymbol{W}^{\top}\boldsymbol{Y} \\
&= (\boldsymbol{Z}^{\top}(\boldsymbol{I} - \boldsymbol{P}_W)\boldsymbol{Z})^{-1}\boldsymbol{Z}^{\top}(\boldsymbol{I} - \boldsymbol{P}_W)\boldsymbol{Y}
\end{aligned}
$$

考虑 $\tilde{\boldsymbol{Y}}$ 对 $\tilde{\boldsymbol{Z}}$ 的回归 $\tilde{\boldsymbol{Y}} = \tilde{\boldsymbol{Z}}\boldsymbol{\delta} + \boldsymbol{u}$，系数的 OLS 估计为

$$
\begin{aligned}
\tilde{\delta} &= (\tilde{\boldsymbol{Z}}^{\top}\tilde{\boldsymbol{Z}})^{-1}\tilde{\boldsymbol{Z}}\tilde{\boldsymbol{Y}} \\
&= \boldsymbol{Z}^{\top}(\boldsymbol{I} - \boldsymbol{P}_W)^{\top}(\boldsymbol{I} - \boldsymbol{P}_W)\boldsymbol{Z})^{-1}\boldsymbol{Z}^{\top}(\boldsymbol{I} - \boldsymbol{P}_W)^{\top}(\boldsymbol{I} - \boldsymbol{P}_W)\boldsymbol{Y} \\
&= (\boldsymbol{Z}^{\top}(\boldsymbol{I} - \boldsymbol{P}_W)\boldsymbol{Z})^{-1}\boldsymbol{Z}^{\top}(\boldsymbol{I} - \boldsymbol{P}_W)\boldsymbol{Y} \\
&= \hat{\delta}
\end{aligned}
$$