2020 秋季本科时间序列

第 4 次作业答案

11 月 8 日

1. (a) 由给定的 AR(2) 过程形式可得：

$$X_t - 2\phi X_{t-1} + \phi^2 X_{t-2} = \varepsilon_t$$

即：

$$\left(1 - 2\phi\mathcal{L} + \phi^2\mathcal{L}^2\right) X_t = \varepsilon_t$$

可得其特征多项式 $A(z) = \phi^2 z^2 - 2\phi z + 1$。

令 $A(z) = 0$，解得 $z_1 = z_2 = \frac{1}{\phi}$。由于 $|\phi| < 1$，得 $|z_1| = |z_2| = \left|\frac{1}{\phi}\right| > 1$，即特征多项式零点位于单位圆之外，故 X_t 为平稳过程。

(b) 对于 $X_t = 2\phi X_{t-1} - \phi^2 X_{t-2} + \varepsilon_t$，由 $\operatorname{cov}(X_t, \varepsilon_t) = \sigma_\varepsilon^2$，对上式两边分别同时乘 X_t, X_{t-1}, X_{t-2}，可得：

$$\sigma_X^2(0) = 2\phi\sigma_X^2(1) - \phi^2\sigma_X^2(2) + \sigma_\varepsilon^2$$
$$\sigma_X^2(1) = 2\phi\sigma_X^2(0) - \phi^2\sigma_X^2(1)$$
$$\sigma_X^2(2) = 2\phi\sigma_X^2(1) - \phi^2\sigma_X^2(0)$$

进而可得关于 $\sigma_X^2(0), \sigma_X^2(1), \sigma_X^2(2)$ 的线性方程组：

$$\begin{bmatrix} 1 & -2\phi & \phi^2 \\ -2\phi & 1+\phi^2 & 0 \\ \phi^2 & -2\phi & 1 \end{bmatrix} \begin{bmatrix} \sigma_X^2(0) \\ \sigma_X^2(1) \\ \sigma_X^2(2) \end{bmatrix} = \begin{bmatrix} \sigma_\varepsilon^2 \\ 0 \\ 0 \end{bmatrix}$$

解得：

$$\begin{bmatrix} \sigma_X^2(0) \\ \sigma_X^2(1) \\ \sigma_X^2(2) \end{bmatrix} = \begin{bmatrix} \frac{\left(1+\phi^2\right)\sigma_\varepsilon^2}{1-3\phi^2+3\phi^4-\phi^6} \\ \frac{2\phi\sigma_\varepsilon^2}{1-3\phi^2+3\phi^4-\phi^6} \\ \frac{\left(3\phi^2-\phi^4\right)\sigma_\varepsilon^2}{1-3\phi^2+3\phi^4-\phi^6} \end{bmatrix}$$

(c) $\mathbf{A}$ 的特征多项式为 $|\lambda\mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda - 2\phi & \phi^2 \\ -1 & \lambda \end{vmatrix} = \boldsymbol{\lambda^2 - 2\phi\lambda + \phi^2 = (\lambda - \phi)^2}$

令其等于 0，得：$\lambda_1=\lambda_2=\phi$。此时计算 $\mathbf{A}$ 的 Jordan 分解 $\mathbf{A}=\mathbf{C}\begin{bmatrix}\lambda & 1\\ 0 & \lambda\end{bmatrix}\mathbf{C^{-1}}$。两边同时右乘 $\mathbf{C}$，得 $\mathbf{AC}=\mathbf{C}\begin{bmatrix}\phi & 1\\ 0 & \phi\end{bmatrix}$。记 $\mathbf{C}=[\mathbf{C_1},\mathbf{C_2}]$，得 $[\mathbf{AC_1},\mathbf{AC_2}]=\begin{bmatrix}\mathbf{C_1},\mathbf{C_2}\end{bmatrix}\begin{bmatrix}\phi & 1\\ 0 & \phi\end{bmatrix}$

比较两边系数，得:

$$\mathbf{AC_1}=\phi\mathbf{C_1}$$
$$\mathbf{AC_2}=\mathbf{C_1}+\phi\mathbf{C_2}$$

解得 $\mathbf{C_1}$ 的通解为: $\mathbf{C_1}=\mathbf{k_1}\begin{bmatrix}\phi\\ 1\end{bmatrix}$，取 $\mathbf{C_1}=\begin{bmatrix}\phi\\ 1\end{bmatrix}$，进而解得方程的一个特解：$\mathbf{C_2}=\begin{bmatrix}1\\ 0\end{bmatrix}$。故 $\mathbf{C}=\begin{bmatrix}\phi & 1\\ 1 & 0\end{bmatrix}$，$\mathbf{A}$ 的 Jordan 分解为 $\mathbf{A}=\begin{bmatrix}\phi & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\phi & 1\\ 0 & \phi\end{bmatrix}\begin{bmatrix}\phi & 1\\ 1 & 0\end{bmatrix}^{\mathbf{-1}}$。

(d) 由于 $\begin{bmatrix}\phi & 1\\ 0 & \phi\end{bmatrix}^2=\begin{bmatrix}\phi^2 & 2\phi\\ 0 & \phi^2\end{bmatrix}$，$\begin{bmatrix}\phi & 1\\ 0 & \phi\end{bmatrix}^3=\begin{bmatrix}\phi^3 & 3\phi^2\\ 0 & \phi^3\end{bmatrix}$

通过归纳法可得：$\begin{bmatrix}\phi & 1\\ 0 & \phi\end{bmatrix}^k=\begin{bmatrix}\phi^k & k\phi^{k-1}\\ 0 & \phi^k\end{bmatrix}$

故：

$$\begin{aligned}\mathbf{A^k}&=\left(\mathbf{C}\begin{bmatrix}\phi & 1\\ 0 & \phi\end{bmatrix}\mathbf{C^{-1}}\right)^k=\mathbf{C}\begin{bmatrix}\phi & 1\\ 0 & \phi\end{bmatrix}^{\mathbf{k}}\mathbf{C^{-1}}\\ &=\begin{bmatrix}\phi & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\phi^k & k\phi^{k-1}\\ 0 & \phi^k\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & -\phi\end{bmatrix}\\ &=\begin{bmatrix}(k+1)\phi^k & -k\phi^{k+1}\\ k\phi^{k-1} & (1-k)\phi^k\end{bmatrix}\end{aligned}$$

由于：

$$\begin{aligned}\begin{bmatrix}\sigma_X^2(k)\\ \sigma_X^2(k-1)\end{bmatrix}&=\mathbf{A}\begin{bmatrix}\sigma_X^2(k-1)\\ \sigma_X^2(k-2)\end{bmatrix}=\mathbf{A^{k-1}}\begin{bmatrix}\sigma_X^2(1)\\ \sigma_X^2(0)\end{bmatrix}\\ &=\phi^{k-1}\begin{bmatrix}k & -(k-1)\phi^1\\ (k-1)\phi^{-1} & 2-k\end{bmatrix}\begin{bmatrix}\sigma_X^2(1)\\ \sigma_X^2(0)\end{bmatrix}\end{aligned}$$

可得：

$$\begin{aligned}\sigma_X^2(k)&=\phi^{k-1}\left[k\sigma_X^2(1)-(k-1)\phi\sigma_X^2(0)\right]\\ &=\phi^{k-1}\left[k\frac{2\phi\sigma_\varepsilon^2}{1-3\phi^2+3\phi^4-\phi^6}-(k-1)\phi\frac{\left(1+\phi^2\right)\sigma_\varepsilon^2}{1-3\phi^2+3\phi^4-\phi^6}\right]\\ &=\frac{(1-k)\phi^2+k+1}{1-3\phi^2+3\phi^4-\phi^6}\phi^k\sigma_\varepsilon^2,\quad k\in Z\end{aligned}$$

2. (a) 由于：

$$
\begin{aligned}
f(\boldsymbol{\beta}) = \boldsymbol{e}^{\mathrm{T}}\boldsymbol{e} &= (\mathbf{Y}-\mathbf{X}\boldsymbol{\beta})^{\mathrm{T}}(\mathbf{Y}-\mathbf{X}\boldsymbol{\beta}) = \mathbf{Y}^{\mathrm{T}}\mathbf{Y} - \mathbf{2}\boldsymbol{\beta}^{\mathrm{T}}\mathbf{X}^{\mathrm{T}}\mathbf{Y} + \boldsymbol{\beta}^{\mathrm{T}}\mathbf{X}^{\mathrm{T}}\mathbf{X}\boldsymbol{\beta} \\
&= \sum_{t=1}^{T} Y_t^2 - 2\sum_{t=1}^{T}\left(\sum_{i=1}^{K}\beta_i X_{ti}\right)Y_t + \sum_{t=1}^{T}\left(\sum_{i=1}^{K}\beta_i X_{ti}\right)^2
\end{aligned}
$$

可得：

$$
\begin{aligned}
\frac{\partial f}{\partial \beta_m} &= -2\sum_{t=1}^{T} X_{tm}Y_t + 2\sum_{t=1}^{T}\sum_{i=1}^{K}\beta_i X_{ti}X_{tm} \\
&= -2\sum_{t=1}^{T} X_{tm}\left(Y_t - \sum_{i=1}^{K}\beta_i X_{ti}\right)
\end{aligned}
$$

其中 $m = 1, 2, \ldots, K$，故

$$
\begin{aligned}
\frac{\partial f}{\partial \beta} = \left[\frac{\partial f}{\partial \beta_1}, \ldots, \frac{\partial f}{\partial \beta_K}\right]^{\mathrm{T}} &= -2\begin{bmatrix} \sum_{t=1}^{T} X_{t1}\left(Y_t - \sum_{i=1}^{K}\beta_i X_{ti}\right) \\ \vdots \\ \sum_{t=1}^{T} X_{tK}\left(Y_t - \sum_{i=1}^{K}\beta_i X_{ti}\right) \end{bmatrix} = -2\begin{bmatrix} \mathbf{X}_{\cdot 1}^{\mathrm{T}}\left(\mathbf{Y}-\mathbf{X}^{\mathrm{T}}\boldsymbol{\beta}\right) \\ \vdots \\ \mathbf{X}_{\cdot K}^{\mathrm{T}}\left(\mathbf{Y}-\mathbf{X}^{\mathrm{T}}\boldsymbol{\beta}\right) \end{bmatrix} \\
&= -2\left[\mathbf{X}_{\cdot 1}, \mathbf{X}_{\cdot 2}, \ldots, \mathbf{X}_{\cdot K}\right]\left(\mathbf{Y}-\mathbf{X}^{\mathrm{T}}\boldsymbol{\beta}\right) \\
&= -2\mathbf{X}^{\mathrm{T}}\left(\mathbf{Y}-\mathbf{X}^{\mathrm{T}}\boldsymbol{\beta}\right)
\end{aligned}
$$

(b) 由 (a) 得

$$
\frac{\partial^2 f}{\partial \beta_u \partial \beta_v} = 2\sum_{t=1}^{T} X_{tu}X_{tv} = \mathbf{X}_{\cdot u}^{\mathrm{T}}\mathbf{X}_{\cdot v}
$$

其中 $u, v = 1, 2, \ldots, K$。所以

$$
\begin{aligned}
H(f) &= \begin{bmatrix} \frac{\partial^2 f}{\partial^2 \beta_1} & \cdots & \frac{\partial^2 f}{\partial \beta_1 \partial \beta_K} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial \beta_K \partial \beta_1} & \cdots & \frac{\partial^2 f}{\partial^2 \beta_K} \end{bmatrix} = 2\begin{bmatrix} \sum_{t=1}^{T} X_{t1}X_{t1} & \cdots & \sum_{t=1}^{T} X_{t1}X_{tK} \\ \vdots & \ddots & \vdots \\ \sum_{t=1}^{T} X_{tK}X_{t1} & \cdots & \sum_{t=1}^{T} X_{tK}X_{tK} \end{bmatrix} \\
&= 2\begin{bmatrix} \mathbf{X}_{\cdot 1}^{\mathrm{T}}\mathbf{X}_{\cdot 1} & \cdots & \mathbf{X}_{\cdot 1}^{\mathrm{T}}\mathbf{K}_{\cdot K} \\ \vdots & \ddots & \vdots \\ \mathbf{X}_{\cdot K}^{\mathrm{T}}\mathbf{X}_{\cdot 1} & \cdots & \mathbf{X}_{\cdot K}^{\mathrm{T}}\mathbf{X}_{\cdot K} \end{bmatrix} = 2\mathbf{X}^{\mathrm{T}}\mathbf{X}
\end{aligned}
$$

对于任意非零 K 阶列向量 **a**，均有：

$$
\mathbf{a}^{\mathrm{T}}H(f)\mathbf{a} = \mathbf{2}(\mathbf{Xa})^{\mathrm{T}}(\mathbf{Xa}) = \mathbf{2}\|\mathbf{Xa}\|_{\mathbf{2}}^{\mathbf{2}} \geq \mathbf{0}
$$

当且仅当 $\mathbf{Xa} = \mathbf{0}$ 时取等，故 $H(f)$ 为半正定矩阵。

(c) 若要使 $H(f)$ 为正定矩阵，则需要有 $\mathbf{a}^{\mathrm{T}}H(f)\mathbf{a} = \mathbf{2}(\mathbf{Xa})^{\mathrm{T}}(\mathbf{Xa}) > 0$ 成立。故 $\mathbf{Xa} \neq \mathbf{0}$，即 $\mathbf{Xa} = \mathbf{0}$ 只存在零解，此时矩阵 $\mathbf{X}$（列）满秩即矩阵 $\mathbf{X}^{\mathrm{T}}\mathbf{X}$ 满秩。