2020 秋季本科时间序列

第 3 次作业答案

11 月 2 日

1. (a) $z\overline{z} = (a+bi)(a-bi)$
$= a^2 - abi + abi - b^2i^2$
$= a^2 + b^2 = \sqrt{a^2+b^2} = |z^2|$

(b) $\because zw = (a+bi)(c+di)$
$= ac + adi + bci + bdi^2$
$= (ac-bd) + (ad+bc)i$
$\therefore |zw| = \sqrt{(ac-bd)^2 + (ad+bc)^2} = \sqrt{(ac)^2 + (bd)^2 + (ad)^2 + (bc)^2}$
又 $|z||w| = \sqrt{a^2+b^2+c^2+d^2} = \sqrt{(ac)^2 + (bd)^2 + (ad)^2 + (bc)^2}$
$\therefore |zw| = |z||w|$
根据数学归纳法
$k = 1$ 时，$|z| = |z|$
假设 $k = n$ 时，$|z^n| = |z|^n$
$k = n+1$ 时，$|z^{n+1}| = |z^n z| = |z^n||z| = |z|^n|z| = |z|^{n+1}$
$\therefore |z^k| = |z|^k$

(c) 令 $S_n = z^1 + z^2 + \ldots + z^n = \frac{1-z^n}{1-z}$
$\because |z| < 1 \therefore \lim_{n\to+\infty} z^n = 0$
$\therefore \lim_{n\to+\infty} S_n = \frac{1}{1-z}$
$\therefore \sum_{i=0}^{\infty} z^i$ 收敛且等于 $\frac{1}{1-z}$

2. $\because (1-\rho\mathscr{L})X_t = Y_t$
$\therefore X_t = \frac{1}{1-\rho\mathscr{L}}Y_t = \sum_{i=0}^{\infty}(\rho^i\mathscr{L}^i)Y_t = \sum_{i=0}^{\infty}\rho^i Y_{t-i}$
$\mu_x = \mathbb{E}(X_t) = \mathbb{E}(\sum_{i=0}^{\infty}\rho^i Y_{t-i}) = \sum_{i=0}^{\infty}\rho^i\mathbb{E}(Y_{t-i}) = \mu_Y\sum_{i=0}^{\infty}\rho^i = \frac{\mu_Y}{1-\rho}$
$\sigma_x^2(k) = \mathrm{cov}(X_t, X_{t-k})$

$= \text{cov}(\sum_{i=0}^{\infty} \rho^i Y_{t-i}, \sum_{j=0}^{\infty} \rho^j Y_{t-k-j})$

$= \sum_{i=0}^{\infty} \rho^i \sum_{j=0}^{\infty} \rho^j \text{cov}(Y_{t-i}, Y_{t-k-j})$

$= \sum_{i=0}^{\infty} \rho^i \sum_{j=0}^{\infty} \rho^j \sigma_y^2(j-i-k)$

由协方差的Cauchy-Schwartz不等式可知,$\because \sum_{i=0}^{\infty} \rho^i \sum_{j=0}^{\infty} \rho^j |\sigma_y^2(j-i-k)| \le \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \rho^{j+i} \sigma_y^2(0) = \sigma_y^2(0) \frac{1}{(1-\rho)^2}$，故原级数收敛

$\because Y_t$ 为平稳过程

$\therefore \mu_Y$，$\sigma_Y^2(k+j-i)$ 均与 t 无关

$\therefore \mu_x$，$\sigma_x^2(k)$ 也与 t 无关，即 X_t 为平稳序列

3. (a) $\because X_t = 1.5X_{t-1} - 0.56X_{t-2} + \varepsilon_t$

   $\therefore (1 - 1.5\mathscr{L} + 0.56\mathscr{L}^2)X_t = \varepsilon_t$

   $\therefore$ 特征多项式为 $A(z) = 1 - 1.5z + 0.56z^2$

   其零点分别为 $z_1 = \frac{5}{4}$，$z_2 = \frac{10}{7}$

   $\because |z_1| > 1$，$|z_2| > 1$，均在单位圆外

   $\therefore AR(2)$ 方程定义的 X_t 为平稳过程

   (b) $X_t = A^{-1}(\mathscr{L})\epsilon_t$

   $= \frac{1}{(1-\frac{4}{5}\mathscr{L})} \frac{1}{(1-\frac{7}{10}\mathscr{L})} \varepsilon_t$

   $= \frac{1}{(1-\frac{4}{5}\mathscr{L})} (\sum_{i=0}^{\infty} (\frac{7}{10}\mathscr{L})^i) \varepsilon_t$

   $= (\sum_{i=0}^{\infty} (\frac{4}{5}\mathscr{L})^i)(\sum_{i=0}^{\infty} (\frac{7}{10}\mathscr{L})^i) \varepsilon_t$

   $= (1 + \frac{4}{5}\mathscr{L} + (\frac{4}{5}\mathscr{L})^2 + \ldots)(\varepsilon_t + \frac{7}{10}\varepsilon_{t-1} + (\frac{7}{10})^2 \varepsilon_{t-2} + \ldots)$

   $\therefore \theta_1 = \frac{7}{10} + \frac{4}{5} = 1.5$

   $\theta_2 = (\frac{7}{10})^2 + (\frac{4}{5})^2 + \frac{7}{10} + \frac{4}{5} = 1.69$

   $\theta_3 = (\frac{7}{10})^3 + (\frac{4}{5})^3 + \frac{4}{5}(\frac{7}{10})^2 + \frac{7}{10}(\frac{4}{5})^2 = 1.695$

   (c) 令 $\theta_1 = 1.5$，$\theta_2 = -0.56$

   $\therefore X_t = \theta_1 X_{t-1} + \theta_2 X_{t-2} + \varepsilon_t$

   等式两边同时乘以 X_{t-1}，X_{t-1}，X_t 得

$$\begin{cases} \sigma_x^2(1) = & \phi_1 \sigma_x^2(0) + \phi_2 \sigma_x^2(1) \\ \sigma_x^2(2) = & \phi_1 \sigma_x^2(1) + \phi_2 \sigma_x^2(0) \\ \sigma_x^2(0) = & \phi_1 \sigma_x^2(1) + \phi_2 \sigma_x^2(2) + \sigma_\varepsilon^2 \end{cases}$$

$$\therefore \begin{bmatrix} \phi_1 & \phi_2 - 1 & 0 \\ \phi_2 & \phi_1 & -1 \\ 1 & -\phi_1 & -\phi_2 \end{bmatrix} \begin{bmatrix} \sigma_x^2(0) \\ \sigma_x^2(1) \\ \sigma_x^2(2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \sigma_\varepsilon^2 \end{bmatrix}$$

解得

$$\begin{bmatrix}\sigma_x^2(0)\\ \sigma_x^2(1)\\ \sigma_x^2(2)\end{bmatrix} = \frac{\sigma_\varepsilon^2}{(\phi_2+1)(\phi_2+\phi_1-1)(\phi_2-\phi_1-1)} \begin{bmatrix}1-\phi_2\\ \phi_1\\ \phi_1^2-\phi_2^2+\phi_2\end{bmatrix} = \begin{bmatrix}19.31\sigma_\varepsilon^2\\ 18.57\sigma_\varepsilon^2\\ 17.04\sigma_\varepsilon^2\end{bmatrix}$$

(d) $X_t = \frac{1}{1-\frac{7}{10}\mathscr{L}}\frac{1}{1-\frac{4}{5}\mathscr{L}}\varepsilon_t$

$= \sum_{j=0}^{\infty}(\frac{7}{10}\mathscr{L})^j \sum_{i=0}^{\infty}(\frac{4}{5})^i \varepsilon_{t-i}$

$\therefore\ \theta_i = \sum_{j=0}^{i}(\frac{7}{10})^j(\frac{4}{5})^{i-j}$

$= \sum_{j=0}^{i}(\frac{7}{10})^j(\frac{5}{4})^j(\frac{4}{5})^i$

$= (\frac{4}{5})^i(8-\frac{7^{i+1}}{8^i})$

$= (\frac{1}{10})^i(8^{i+1}-7^{i+1})$