2020 秋季本科时间序列

第 2 次作业答案

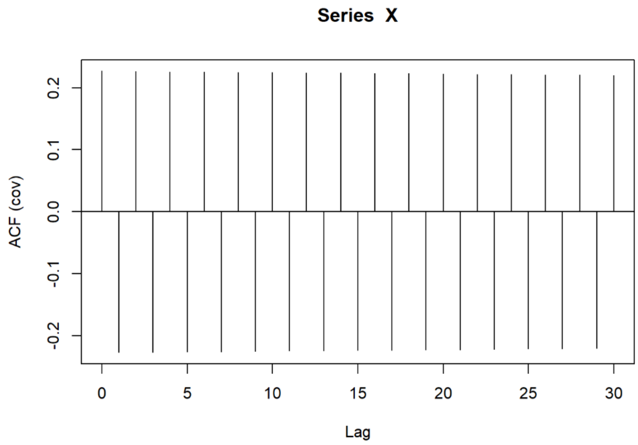
10 月 26 日

1. (a) 由于 $\mathbb{E}X_t = \mathbb{E}\cos(\pi + U) = \int_{-\pi}^{\pi} \frac{1}{2\pi}\cos(u + \pi t)du = 0$，可得：

$$
\begin{aligned}
\sigma_x^2(k) &= \mathrm{cov}(X_{t+k}, X_t) = \mathbb{E}X_{t+k}X_t - \mathbb{E}X_{t+k}\mathbb{E}X_t = \mathbb{E}X_{t+k}X_t \\
&= \mathbb{E}\left[\cos(U + \pi t + \pi k)\cos(U + \pi t)\right] \\
&= \int_{-\pi}^{\pi} \frac{1}{2\pi}\cos(u + \pi t + \pi k)\cos(u + \pi t)du \\
&= \frac{1}{4\pi}\int_{-\pi}^{\pi} \left[\cos(2u + 2\pi t + \pi k) + \cos(\pi k)\right] du \\
&= \frac{1}{2}\cos(\pi k), R \in N
\end{aligned}
$$

(b) 代码如下：

```
library(tidyverse)
library(ggplot2)
library(tseries)
X <- vector("double",1000)
U <- runif(1,-pi,pi)
for (i in 1:1000) {
 X[i] <- cos(pi*i + U)
}
acf(X, type = "covariance")
```



2. 证明如下：

$$
\begin{aligned}
\mathbb{E}\hat{\sigma}_N^2 &= \mathbb{E}\left[\frac{1}{N-1}\sum_{i=1}^{N}\left(x_i-\hat{\mu}_N\right)^2\right]\\
&= \frac{1}{N-1}\mathbb{E}\left[\sum_{i=1}^{N}\left(x_i-\mu+\mu-\hat{\mu}_N\right)^2\right]\\
&= \frac{1}{N-1}\mathbb{E}\left[\sum_{i=1}^{N}\left(x_i-\mu\right)^2+2\sum_{i=1}^{N}\left(x_i-\mu\right)\left(\mu-\hat{\mu}_N\right)+\sum_{i=1}^{N}\left(\mu-\hat{\mu}_N\right)^2\right]\\
&= \frac{1}{N-1}\mathbb{E}\left[\sum_{i=1}^{N}\left(x_i-\mu\right)^2-2N\left(\mu-\hat{\mu}_N\right)^2+N\left(\mu-\hat{\mu}_N\right)^2\right]\\
&= \frac{1}{N-1}\left[\sum_{i=1}^{N}\mathbb{E}\left(x_i-\mu\right)^2-N\mathbb{E}\left(\mu-\hat{\mu}_N\right)^2\right]\\
&= \frac{1}{N-1}(N\sigma^2-N\frac{\sigma^2}{N})\\
&= \sigma^2
\end{aligned}
$$

3. (a) 由 $X_t=\rho X_{t-1}+\varepsilon_t$ 及 $X_0=a\in R$ 可得：

$$X_t=\rho X_{t-1}+\varepsilon_t=\rho^2X_{t-2}+\rho\varepsilon_{t-1}+\varepsilon_t=\cdots=\rho^t a+\sum_{i=0}^{t-1}\rho^i\varepsilon_{t-i},(t\geq 1)$$

由于 ε_t 为独立同分布下的标准正态分布白噪声序列，可得 $X_t\sim N(\rho^t a,\sum_{i=1}^{t}\rho^{2(t-i)})$，即 $X_t\sim N(\rho^t a,\frac{1-\rho^{2t}}{1-\rho^2})$，则 X_t 的分布函数为：

$$F_t(x)=\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi\frac{1-\rho^{2t}}{1-\rho^2}}}e^{-\frac{(x-\rho^t a)^2}{2\frac{1-\rho^{2t}}{1-\rho^2}}}dx,t\geq 1$$

(b) 由于 $|\rho|<1$，当 $t\to\infty$ 时，$\rho^t\to 0$，因此

$$F_\infty(X)=\lim_{t\to\infty}F_t(x)=\int_{-\infty}^{x}\frac{1}{\sqrt{2\pi\frac{1}{1-\rho^2}}}e^{-\frac{x^2}{2\frac{1}{1-\rho^2}}}dx$$

此时 $\lim_{t\to\infty} X_t \sim N(0, \frac{1}{1-\rho^2})$。

若当 $X_0 \sim N(0, \frac{1}{1-\rho^2})$ 且 $X_t \sim N(0, \frac{1}{1-\rho^2})$ 时：

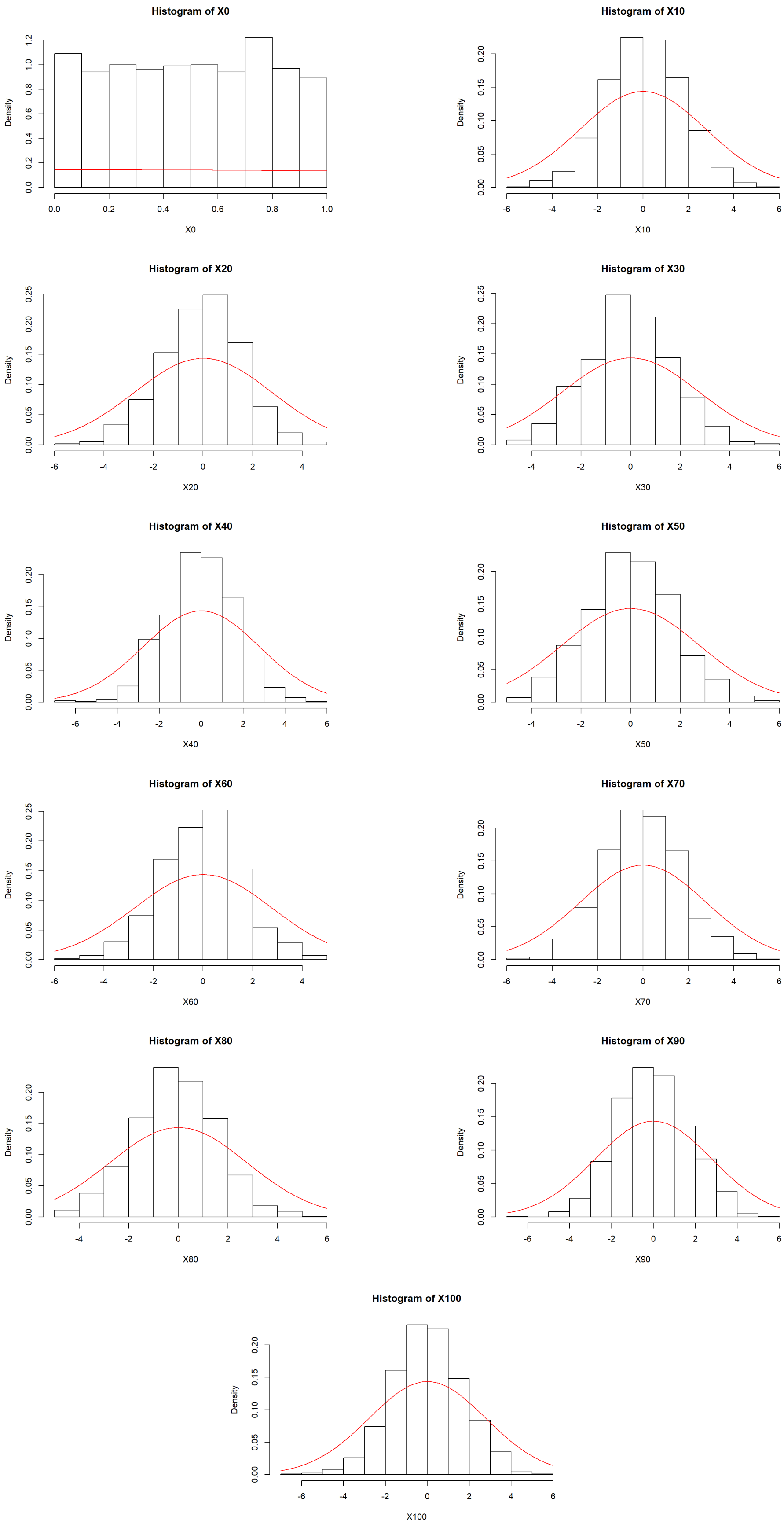
$$\mathbb{E}X_{t+1} = \rho\mathbb{E}X_t + \mathbb{E}\varepsilon_{t+1} = 0$$

$$\text{var}(X_{t+1}) = \text{var}(\rho X_t) + \text{var}(\varepsilon_{t+1}) = \frac{\rho^2}{1-\rho^2} + 1 = \frac{1}{1-\rho^2}$$

即 $X_{t+1} \sim N(0, \frac{1}{1-\rho^2})$，$X_t$ 的分布均为该极限分布，故该分布为 AR(1) 过程的平稳分布。

(c) 代码如下：

```
library(tidyverse)
library(ggplot2)
library(tseries)
X0 <- runif(1000,0,1)
epsilon <- vector("double",100)
Xt <- vector("double",100)
X <- tibble(c(1:101))
for (i in 1:1000) {
 for (j in 2:101) {
  Xt[1] = X0[i]
  epsilon[j] = rnorm(1,0,1)
  Xt[j] = 0.8*Xt[j-1]+epsilon[j]
 }
 X[i] = Xt
}
Xlab <- str_c("X",seq(0,100,10))
for (k in 0:10) {
 hist(t(X[10*k+1,]),freq = FALSE,xlab = Xlab[k+1],
 main = str_c("Histogram of ",Xlab[k+1]))
 curve(dnorm(x,0,100/36),add = T,col = "red")
}
```


Histogram of X0
Density
X0
Histogram of X10
Density
X10
Histogram of X20
Density
X20
Histogram of X30
Density
X30
Histogram of X40
Density
X40
Histogram of X50
Density
X50
Histogram of X60
Density
X60
Histogram of X70
Density
X70
Histogram of X80
Density
X80
Histogram of X90
Density
X90
Histogram of X100
Density
X100