Making Sovereign Debt Safe with a Financial Stability Fund*

Yan Liu† Ramon Marimon‡ Adrien Wicht§

September 27, 2020

Abstract

This paper further advances the design of an optimal Financial Stability Fund (Fund) of Ábrahám et al. (2019) by not having the Fund absorbing all the sovereign debt of a country. The Fund's long-term contracts are subject to two-sided limited enforcement constraints: at any point in time the borrowing country may breach the contract and exit, while the Fund cannot have expected losses. The country's constraint therefore represents a sovereignty constraint, whereas the lenders' constraint can be interpreted as a debt sustainability analysis (DSA). The country can borrow long-term defaultable bonds on the private international market, while having a state-contingent contract with the Fund, which provides insurance and, possibly, credit. The Fund contract has no seniority with respect to the privately held sovereign debt and, therefore, takes this external debt into account. The share of debt held by the Fund might be indeterminate; nevertheless, there is one contract that minimizes the debt absorbed by the Fund. In equilibrium, the Fund contract prevents the country from defaulting on its entire debt position. As a result, the debt in the private international market becomes risk-free, although it is constrained when the Fund's limited enforcement constraint binds. The latter therefore internalizes a pecuniary externality that competitive private lenders usually do not: the fact that marginal lending can result in debt becoming unsustainable. In light of this, our model provides an appealing theoretical and quantitative framework to address sovereign debt-overhang problems and, in doing so, increasing the supply of 'safe assets', in the Euro Area and elsewhere.

Keywords: Recursive contracts, limited enforcement, debt, debt overhang, sovereign funds
JEL Classification: E43, E44, E47, E62, F34, F36, F37

*Preliminary and incomplete, do not quote without authors' permission. We would like to thank Aitor Erce for helpful comments. We acknowledge the financial support of the Max Weber Chair programme of the EUI and of the European Stability Mechanism. The views expressed in this study are the authors' and do not necessarily reflect those of the European Stability Mechanism. All remaining errors are our own.

†Wuhan University

‡European University Institute, UPF – Barcelona GSE, CEPR and NBER

§European University Institute

1 Introduction

The global financial crisis of 2007 and 2008, followed by the European sovereign debt crisis of 2010, had a tremendous impact on countries' sovereign indebtedness. This led to precarious debt management, restricted access to safe assets and, most importantly, the possibility of default. Those issues remain largely unresolved as we speak and are most likely to further deteriorate given the current sanitary crisis. The question of how one can stabilize the sovereign debt of a country is therefore of utmost importance.

In light of this, we develop the optimal design of a Financial Stability Fund (Fund) in the form of a long-term contingent contract subject to limited enforcement in the spirit of Ábrahám et al. (2019). While the latter focus on the borrower's perspective, we put emphasis on the lender's side of the contract and derive the optimal relationship between the private competitive lenders and the Fund. More precisely, we assume that sovereign countries can raise debt in the private international market and in the Fund.[1] While private international lenders solely offer credit (i.e. long-term defaultable bonds), the Fund proposes both credit and insurance (i.e. Arrow securities). Neither the Fund nor the private lenders have any kind of exclusivity rights when one of the contracting countries decides to default. In other words, both liabilities are *pari-passu* and when a country defaults, it does so on its entire debt position. The Fund's objective is therefore to stabilize the countries' indebtedness. Most importantly, it ought to prevent debt repudiations of any sort. In view of this, it is ready to provide substantial resources to stabilize the sovereign debt's spread and resolve the scarcity of safe assets, therefore averting potential defaults. However, countries cannot obtain assistance of the Fund *ad libitum*. The Fund's intervention is conditional on a strict *debt sustainability analysis* (DSA), which we identify in our framework as an evaluation of the present value of the country's future surpluses (net savings) from any contingency onwards. In any period and history, the country has to pass this DSA if it wants to continue to receive transfers from the Fund.

The Fund is shaped by two main aspects. First, it is a tripartite *pari-passu* contract involving the borrowing country, the private international lenders and the Fund itself. As the Fund has no seniority with respect to privately held debt, it has to account for the country's indebtedness in both the Fund and the private international market. However, once the Fund contract is specified and accepted by the contracting parties, the Fund perceives the working of the private bond market as standalone, and will not contemplate any direct impact of its own operations upon the choices made in the private bond market, except for the fact that the Fund contract will affect the future state of the country and hence the equilibrium outcomes in the market. Second the Fund is a two-sided limited enforcements contract. On the one hand, the borrowing country is sovereign and, therefore, can default at will. On the other hand, the Fund has a free access to the international financial market and can withdraw whenever additional lending entails expected losses (i.e. when the liabilities of the country become, with positive probability, unsustainable). We interpret this

[1] The adjective "private" is used to distinguish lenders on the international market relative to the Fund. It is not used to differentiate private from public matters.

second constraint as the aforementioned DSA.

The DSA is the main focus of our analysis. It represents the lender's joint participation constraint and states that the net present value of the lenders' entire surplus should at least cover the country's liabilities in the private international market. Hence, the DSA internalizes a pecuniary externality that competitive private lenders usually do not: the fact that marginal lending can result in debt becoming unsustainable. In other words, it monitors whether additional borrowings entails expected losses. The DSA being binding results in a negative spread for the Fund-provided asset. Instead, for the private lenders it can be a run on the debt, unless they follow the DSA of the Fund as a stopper to their lending. Hence, integrating the DSA together with the sovereignty constraint, our Fund prevents the possibility of default triggered by the borrower, but also accounts for the possibility that the lender withdraws before incurring any expected loss. The literature on sovereign debt has focused in the former, we emphasize the importance of the latter. To the best of our knowledge, we are the first to properly analyze and exploit this pecuniary externality.

The Fund contract involves DSA not only at the outset, to determine the terms of the contract, but all along the evolution of the contract. When the lenders' constraint is binding, the country is at the verge of failing its DSA. In this situation, the Fund expects no accumulation of private debt as the country's total indebtedness might become unsustainable otherwise.[2] It will therefore offer financial support to the country to ensure a certain level of consumption. However, it provides just enough resources for the country to pass its DSA, making sure that the Fund bears no expected loss and the present value of the country's future surpluses covers its private debt position. Furthermore, if the country decides to accumulate some additional amount of private debt, the Fund will withdraw. Given that, the private lenders fulfil the Fund's expectation and do not lend to the country as they fear that the country would default after the Fund's withdrawal. A negative spread arises in the security market since the Fund is restricting the provision of its insurance to the country, and concurrently it also appears to sustain the no-trade equilibrium in the private bond market.

As already mentioned our analysis is an extension of Ábrahám et al. (2019). The main difference is that we do not consider an exclusivity contract between the Fund and the contracting countries. We explicitly model the presence of private international lenders and that for two reasons. First, we know that if the entire position of the country is taken over by the Fund, then the debt becomes risk free. We also know that for some *small* but *strictly positive* debt level, any amount of debt less than this level is free from default risk. Such a debt level corresponds to the smallest default threshold in the pure incomplete market economy with default risk (Zhang, 1997). Now, for any given total debt position, as long as the Fund absorbs enough debt, the residual debt position in the private market becomes safe as well. By a continuity argument, there should always exist an effective intervention level that is less than 100% of the total debt position. However, to make this intuition precise, we need to specify the whole setup taking into account the coexistence of the private bond and the debt with the Fund. Second, assuming that the Fund absorbs part of the

[2] The lenders' constraint does not necessarily bind for all future states. Hence, the country can end up in a state tomorrow where lenders are capable of sustaining its indebtedness even if it borrowed more today.

countries' debt brings us closer to the existing lending institutions. As a matter of fact, when the ESM or the IMF intervene, they take over only part of the country's sovereign debt, the remainder being left on the market.

Having said that our Fund is not a one-to-one replication of existing multilateral lending facilities. Notably, we assume no seniority while the ESM and the IMF usually require seniority in their lending programs.[3] Moreover, while it is true that international lending institutions conduct DSA as a necessary condition to guarantee credits, it is not the case that their resulting debt contracts provide insurance against future DSAs, as the Fund does. In other words, international lending institutions base their lending policy on one of several scenarios – typically, the 'most likely one', the 'politically preferred' or the 'worst case' scenario. In contrast, the Fund contract risk-shares among these different scenarios or paths. That is it provides additional transfer in the worst scenario for higher payments in the best. Finally, the DSA conducted by some financial institutions such as the IMF does not require that debt is sustainable in order to lend. Most notably, exceptional lending can occur below some pre-set access limits regardless of the formulated DSA.

Our quantitative exercise follows the calibration of Ábrahám et al. (2019). More precisely, it relies on the period 1980 to 2015 for the Euro Area countries that were the most affected by the European sovereign debt crisis (i.e. Greece, Ireland, Italy, Portugal and Spain). In a later stage, however, one could focus on one of the aforementioned countries. Italy provides an interesting instance to study. Indeed, it has never required the help of the ESM but faces a public indebtedness above 100% of GDP and one of the largest spreads in the Euro Area. The specificity of Italy and its debt management has already been studied by Alesina et al. (1990) and, more recently, by Bocola and Dovis (2019).

The main results of our inquiry are twofold. First, on equilibrium path, the contracting country never defaults. Hence, its entire debt position becomes safe. This is due to the state-contigent transfers provided by the Fund. More precisely, the Fund offers a countercyclical policy meaning that drops in productivity are counteracted by increases in transfers to the country. The mechanism at work relies on the complementarity between bonds and Arrow securities. The country goes long in Arrow securities for relatively low productivity states and short for relatively high productivity states. Hence, if the country ends up in a low productivity state, the realized Arrow security will compensate for both the low productivity and the outstanding amount of debt, guaranteeing a certain level of consumption. Conversely, if the country ends up in a high productivity state, it will get more indebted as it went short on Arrow securities. However, the country's productivity and prospective borrowing are sufficiently large to enable the country to repay its outstanding debt without major impacts on its consumption. Thus, this policy not only prevents default, it also enables greater consumption smoothing and debt accumulation.

Second, without intervening on the private market, the Fund restraints the total indebtedness.

[3] IMF has a *de facto* seniority, but it is not a formal contractual feature (see Schlegl et al. (2019)). In opposition, the ESM has a *de jure* seniority with respect to the market. The only exception to this is Spain. The Spanish program was initially agreed with the EFSF with a standard *pari-passu* clause. The Spanish government managed to prolongate this feature into the ESM loan.

On the one hand, it prevents the country to accumulate today some level of debt it would like to repudiate tomorrow. On the other hand, it prevents the accumulation of debt lenders cannot credibly sustain. This second point brings us back to the DSA, which has to hold in every period and history. The Fund contract is therefore shaped by this DSA and, in view of this, the country cannot freely accumulate debt. When the lenders' constraint is binding, a negative spread appears. The country is then unable to borrow on the private bonds market. The result is similar to a default with two main differences. First, in the standard incomplete market economy, default is accompanied with positive spreads. However, with the Fund's intervention, the country is hedged in all circumstances. Thus the default set remains empty and no risk premium is attached on the bond price. Second, the default is a decision made by the borrower (i.e. in our case the country). In our environment, it is the lenders that decide to stop their lending activities.

The paper is organized as follows. We review the existing literature in Section 2. We lay down the economic environment in Section 3. Subsequently, we present the Planner's problem in Sections 4. In Section 5, we show how the Planner's problem can be decentralized as a competitive equilibrium with endogenous borrowing constraint. There we also discuss the equilibrium and the steady state properties of the model. Thereafter, we calibrate our model to stressed Euro Area countries in Section 6 and present the underlying results in Section 7. Finally, we conclude in Section 8.

2 Literature Review

This paper addresses four main strands of the literature. First, we contribute to the literature on financial stability and sovereign debt. Since the outbreak of the European debt crisis, many proposals have been developed to find alternatives to the existing debt overhang of European countries. One of them is the issuance of joint-liability eurobonds. The main idea is to generate a European safe asset to complement national government bonds. This additional source of safe assets should ease the debt management of member states as well as reduce the banks' balance sheet exposures towards national bonds (Giudice et al., 2019). However, such a scheme requires countries to be subject to symmetric shocks (Tirole, 2015). Besides the issuance of eurobonds, some studies propose the creation of a supranational institution. Furceri and Zdzienicka (2013) discuss the necessity of a fiscal risk sharing mechanism in the Euro Area. Beetsma et al. (2018) propose a clearing-house system of transfers to smooth out sector-specific shocks. Finally, Ábrahám et al. (2019) develop a Financial Stability Fund in the form of a long-term risk-sharing mechanism which is constrained-efficient. In a set of quantitative exercises, they show that such an institution would not only improve risk sharing among members, it would also ensure counter-cyclical fiscal policies. We contribute to this literature by further developing the structure of the aforementioned Fund. Most notably, we derive the optimal tripartite interaction between the Fund, the private international lenders and the contracting countries.

The second strand of literature one addresses is related to safe assets. Empirical studies seek to identify the observable determinants of a safe asset (Krishnamurthy and Vissing-Jorgensen, 2012;

He et al., 2016; Habib et al., 2020). Among other criteria, a key determinant is more often than not the strength of public institutions. Theoretical analysis are directed towards the fundamental determinants of a safe asset and their implications. Among many other contributions, Jiang et al. (2020) analyze the tradeoff between bondholders and taxpayers that enables the creation of risk-free debt, while Caballero et al. (2016) show that safe asset scarcity pushes economies into safety-trap recessions. Our analysis is closer to the former as it focuses on the mechanism enabling the Fund to produce safe assets in conjunction with the private lenders.

The third strand of literature we address is the one on optimal contracts. Our Fund is shaped by two-sided limited enforcement constraints and therefore relates to the seminal contributions of Kehoe and Levine (1993, 2001), Kocherlakota (1996) and Kehoe and Perri (2002) who were one of the first to analyze the dynamic of infinite-horizon general equilibrium models with limited enforcement constraints. Methodologically, our model relies on the Lagrangian approach developed by Marcet and Marimon (2019) which determines the relative Pareto weight as part of the state space. This weight is a sufficient statistics to keep track of the binding constraints as it represents the original Planner's weight plus the sum of all multipliers attached to the enforcement constraints. The Planner's problem is subsequently decentralized using the approach of Alvarez and Jermann (2000). Our analysis is close to Thomas and Worrall (1994) who study international lending contracts. The authors focus on the country's sovereignty constraint presenting the tradeoff between short-term gains of reneging the contract and long-term benefits of staying in the contract. Our contribution to this literature is to integrate the endogenous choice on private debt and to quantitatively assess its impact. Most notably, we extensively analyze the meaning and the consequences of the lenders' limited enforcement constraint.

The last contribution of this paper regards theoretical models of debt default. This literature starts with the seminal contribution of Eaton and Gersovitz (1981) and has then been subsequently extended by notably Aguiar and Gopinath (2006) and Arellano (2008). Most of this literature is now directed to explain the dynamic of emerging economies such as Argentina. More precisely, it seeks to mimic the observed pattern of interest rate spreads, debt, default rate and output in emerging markets. One of the difficulty is that calibrated models often underpredict the default rate of the sample countries (Arellano and Ramanarayanan, 2012). In view of this, our approach to this literature goes in the opposite direction. We seek the optimal design of a Fund to avoid any kind of debt repudiations. Our elaboration mostly relates to the work of Hatchondo et al. (2017), who consider the case of adding a non-defaultable bond into the otherwise standard defaultable bond economy. They find in a set of quantitative exercises that there are welfare gains by swapping defaultable bonds into non-defaultable bonds.

3 Environment

Following Ábrahám et al. (2019), we consider an infinite-horizon small open economy where the benevolent government acts as a representative agent. Its preferences are represented by the utility function of the form $U(c,n) := u(c) + h(1-n)$ where n is the labor, $1-n$ the leisure and c

the consumption. The utility function is differentiable and strictly concave with respect to both consumption and leisure. The government discounts the future at the rate $\beta < 1/(1+r)$, where r is the risk-free world interest rate.

The country has access to a labor technology $y = \theta f(n)$ subject to decreasing returns to scale, where $f'(n) > 0$, $f''(n) < 0$. Moreover, θ is a productivity shock assumed to be Markovian with $\theta \in (\theta_1, ..., \theta_N)$ and $\theta_i < \theta_{i+1}$. It represents the only source of uncertainty in the economy.

The country has two funding opportunities. First, it can borrow long-term defaultable bonds on the international market. Here, we follow the approach of Chatterjee and Eyigungor (2012) and assume that a fraction $1-\delta$ of the bond portfolio matures every period and the remaining fraction δ is rolled-over and pays a coupon κ. Second, the country can sign a state-contingent contract with the stability Fund. The Fund has also access to the international debt market and offers to the country a state-contigent contract with the following characteristics:

1. There is a strict risk-assessment of the borrower and, provided that the existing level of defaultable debt is sustainable if there is a Fund contract, then no other *ex-ante* conditionality is needed.

2. The Fund contract is a self-enforcing *long-term commitment* made at some initial date $t = 0$, between the Fund and the borrower.

3. There is two-sided limited enforcement since the contract accounts for the limited commitment on both contracting parties. On the one hand, the government is sovereign and, therefore, can default at will. On the other hand, the Fund has a free access to the international financial market and can withdraw whenever additional lending entails expected losses (i.e. when the liabilities of the country become, with positive probability, unsustainable).

4. The private bond market is *competitive*. The Fund accounts for the bond holdings of the country but takes its sovereign debt decisions as given. Only when its limited enforcement constraint is binding, it may restrict the country from issuing new debt.[4]

5. The country's liabilities (debts) with the Fund have no seniority with respect to the sovereign debt in the hands of other agents. This implies that whenever the country defaults, it does so on its entire debt position.

Note that those characteristics are interrelated. An alternative would be that the Fund only accounts for the liabilities of the country with the Fund and to assume that these liabilities are senior to the country's sovereign defaultable debt in the private bond market. However, this would require a strong *no bailout* commitment by the Fund: "in the case the country defaults on its sovereign debt (i.e. there is a debt crisis) the Fund does not intervene". We do not analyse this alternative here, since it does not seem very realistic.[5]

[4] We will also consider the alternative of not placing a restriction but simply announcing the probability of outright default, given that the private lenders will not lend in some future states.

[5] We plan to explore it, and compare it with our benchmark alternative, in future versions.

4 The Fund contract

In Ábrahám et al. (2019), the Fund has an *exclusive contract* with the borrowing country, in the sense that it absorbs the total amount of its sovereign debt – which, as it is common in the debt literature, also accounts for the private-sector debt of the country. In contrast, we assume that the Fund takes over only part of the debt. In other words, the contracting country has access to simultaneously the Fund and the private bond market, as long as it does not default.

We first consider the Fund contract as a solution to a Planner's problem with two types of agents: the risk-neutral lenders, whose discount rate is the risk-free rate in the bond market, and the risk-averse and more impatient sovereign country. Particularly, the Planner accounts for both the private international lenders and the Fund. It takes as given the country's borrowing and lending in the bond market, as well as the possibility that it can default on its private bonds and from the Fund. Similarly, it takes into account that the Fund having access to, and commitment with, the bond market can borrow and lend at the risk-free rate (i.e. will not lend to the country if the country's liabilities are not sustainable) and, being competitive, makes expected zero profits with the Fund contract. Now we explicitly define define the Planner problem. In the next section we decentralise the Fund contract and characterise the *recursive competitive equilibrium* of the economy.

4.1 The Sequential Form

The contracting problem between the country and the Fund takes into account the existence of a sequence of private bond positions $\{b(s^t)\}_{t=0}^{\infty}$, together with the underlying price sequence $\{q(s^t, \tau^f(s^{t+1}), b(s^{t+1}))\}_{t=0}^{\infty}$ in the private bond market. Note that we let the bond price depend on the Fund state-contingent transfers for next period.[6] The private bond sequence is determined by the borrower's choice in the private bond market. For simplicity, we assume that, in state $s^t = (s_0, \ldots, s_t) = (s^{t-1}, s_t)$, $b(s^t)$ is the amount of outstanding bonds. Moreover, the private bond sequence is assumed to satisfy the transversality condition:

$$\lim_{n\to\infty} \mathbb{E}\left\{ \left[\prod_{j=0}^{n} Q\Big(s^{t+j}, \tau^f\left(s^{t+j+1}\right), b\big(s^{t+j+1}\big)\Big) \right] b\big(s^{t+j+1}\big) \middle| s^t \right\} = 0, \quad \text{with}$$

$$Q\Big(s^{t+j}, \tau^f\left(s^{t+j+1}\right), b\big(s^{t+j+1}\big)\Big) = \frac{q\Big(s^{t+j}, \tau^f\left(s^{t+j+1}\right), b\big(s^{t+j+1}\big)\Big)}{1-\delta+\delta\kappa+\delta q\Big(s^{t+j+1}, \tau^f\left(s^{t+j+2}\right), b\big(s^{t+j+2}\big)\Big)}$$

Given $\{b(s^t)\}_{t=0}^{\infty}$, the **Fund's contacting problem** in *sequential form* reads

$$\max_{\{c(s^t), n(s^t)\}_{t=0}^{\infty}} \mathbb{E}\left[\mu_{b,0} \sum_{t=0}^{\infty} \beta^t U(c(s^t), n(s^t)) + \mu_{\ell,0} \sum_{t=0}^{\infty} \left(\frac{1}{1+r}\right) \tau(s^t) \middle| s_0 \right] \tag{1}$$

[6] With an abuse of notation here $\tau^f(s^{t+1})$ denotes the vector $\{\tau^f(s^{t+1})\}_{s^{t+1}|s^t}$, while, unless we say it otherwise, $\tau^f(s^t)$ denotes the s^t component of the vector $\{\tau^f(s^t)\}_{s^t|s^{t-1}}$. In the case of the uncontingent bond $b(s^{t+1})$ is just the bond amount purchased at s^t.

$$\text{s.t.} \quad \mathbb{E}\left[\sum_{j=t}^{\infty} \beta^{j-t} U(c(s^j), n(s^j)) \middle| s^t\right] \geq V^a(s_t), \tag{2}$$

$$\mathbb{E}\left[\sum_{j=t}^{\infty} \left(\frac{1}{1+r}\right)^{j-t} \tau(s^j) \middle| s^t\right] \geq Z - b(s^t), \tag{3}$$

$$\tau(s^t) = \theta(s^t) f(n(s^t)) - c(s^t), \quad \forall s^t, t \geq 0,$$

$$\text{with } \mu_{b,0}, \ \mu_{\ell,0}, \ \{b(s^t)\}_{t=0}^{\infty}, \text{ given.}$$

The contracting country consumes $c(s^t)$ and provides labor $n(s^t)$.[7] The variable $\tau(s^t)$ is the sum of the country's net savings in the private bond market and in the Fund. In other words, $\tau(s^t) \equiv \tau^f(s^t) + \tau^p(s^t)$ for all t and s^t, where

$$\tau^p(s^t) = q(s^t, \tau^f(s^{t+1}), b(s^{t+1})) \left[b(s^{t+1}) - \delta b(s^t)\right] - (1 - \delta + \delta\kappa) b(s^t), \quad \text{and}$$

$$\tau^f(s^t) = \theta(s^t) f(n(s^t)) - c(s^t) - \tau^p(s^t).$$

As there exists no capital, $\tau(s^t)$ is also the country's *current account*.[8] Whenever $\tau(s^t) < 0$ the country is a net borrower with respect to the rest of the world. The Fund's transfer, $\tau^f(s^t)$, is defined in a tautological way. Only when we come to the decentralization will we be able to properly define it in terms of asset positions and prices.

Equations (2) and (3) represent the *limited enforcement constraints* of the borrower and the lender, respectively. The borrower's outside option is to default and is given by $V^a(s_t)$, which only depends on the current state s_t. The underlying assumption is that if the country defaults from the Fund, it also defaults on its sovereign debt liabilities and then is never allowed to return to the Fund in the future. Alternatively, whenever the country defaults on its sovereign debt in the international capital market it also defaults from the Fund, since Fund's liabilities are not senior to privately held debt. In order to prevent that the Fund provides permanent transfers to a country – e.g. in order to prevent debt mutualization – we will assume that $Z = 0$, i.e. that in no state the Fund contract has expected losses. Then (3) shows the second aspect that makes the Fund contract different from an uncontingent defaultable debt contract: in states where lenders can nearly sustain the country's indebtedness – say, when (3) is binding at s^t – both lenders provide less resources to avoid losses that would go beyond the contract's terms. In other words, the Fund contract anticipates these states and limits the amount of lending, while with defaultable these states are anticipated by positive spreads.

One could consider an alternative formulation where Fund liabilities had seniority over privately held sovereign debt – for example, because the implicit cost of leaving the Fund are higher. We will consider this case, however it is a case that relies on a strong *non bailout* commitment from the Fund: the Fund does not act as a *crisis resolution* mechanism, when privately held sovereign debt is at risk of default. Our present formulation is closer to the current rules of public international lending institutions – such as, the IMF or the ESM. The Fund takes into account all the country's

[7]We do not distinguish between private and public consumption.

[8]That is, as in other models of sovereign debt without capital, c also imbeds the capital investment.

debt liabilities – within and outside the Fund – that satisfy the *Debt Sustainability Analysis* (DSA) in every possible state. The difference with current practices is that the DSA is usually only performed at the beginning of the contract, or a certain time intervals (e.g. Fund contracts are relatively short term contracts), while in our characterisation of the Fund contract, DSA – i.e. our (3) – is contingent in all the states that the contract specifies, including those states where limited enforcement constraints are binding. Moreover, while international lending institutions grant credit base on one of several scenarios, our Fund risk-share among the different plausible scenarios. [9]

With $\tau(s^t) \equiv \tau^f(s^t) + \tau^p(s^t)$, the Planner accounts for both the private international lenders and the Fund. In other words, the Fund contract account for the country's entire debt position. While the Fund contract *directly* specifies $\tau^f(s^t)$ taking as given $\tau^p(s^t)$, effectively the Planner is taking into account the total surplus $\tau^f(s^t) + \tau^p(s^t)$ when evaluating the participation constraint, since only in this way it is capable of consistently stabilizing the borrower's entire debt position. An equivalent interpretation is that the Fund stands ready to absorb the debt position of the borrower in the form of private bonds, and effectively there is complete credit (risk) transfer from the private bond investors to the Fund, up to certain limits implied by the participation constraints both from the Fund and the borrower.

To have a better idea of the link between the private lender's and the Fund's value, observe that, conditional on s^t,

$$\begin{aligned} V^l(s^t) &:= \mathbb{E}\left[\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j \tau(s^{t+j}) \middle| s^t\right] \qquad (4)\\ &= \mathbb{E}\left[\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j \left(\tau^f(s^{t+j}) + \tau^p(s^{t+j})\right) \middle| s^t\right] \\ &= \mathbb{E}\left[\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j \left(\tau^f(s^{t+j}) + \left[q(s^t, \tau^f(s^{t+1}), b(s^{t+1}))\left[b(s^{t+1}) - \delta b(s^t)\right] - (1-\delta+\delta\kappa) b(s^t)\right]\right) \middle| s^t\right]. \end{aligned}$$

Using the transversality condition on the bond choice, (4) simplifies into

$$V^l(s^t) = \mathbb{E}\left[\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j \tau^f(s^{t+j}) \middle| s^t\right] - b(s^t).$$

Imposing the present value constraint on Fund's lending $\{\tau^f(s^t)\}_{t=0}^{\infty}$ as

$$\mathbb{E}\left[\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j \tau^f(s^{t+j}) \middle| s^t\right] \geq Z,$$

then the overall participation constraint of the Fund is given by

$$V^l(s^t) = \mathbb{E}\left[\sum_{j=0}^{\infty}\left(\frac{1}{1+r}\right)^j \tau(s^{t+j}) \middle| s^t\right] \geq Z - b(s^t).$$

[9] Another difference is that in our framework, the Fund has no seniority over privately owned debt. This is in general not the case multilateral lending institutions intervene. While the IMF and the World bank are granted *de facto* seniority, other lending facilities such as the ESM enjoy *de jure* seniority with respect to the market. The only exception is Spain which managed to prolongate the *pari-passu* clause of the EFSF program to the ESM.

Solutions to the Fund's contacting problem are homogenous of degree one in $\mu = (\mu_b, \mu_\ell)$ and the initial relative Pareto weight $x_0 \equiv \frac{\mu_{b,0}}{\mu_{\ell,0}}$ is given by the initial break-even condition for the Fund

$$\mathbb{E}\left[\sum_{t=0}^{\infty}\left(\frac{1}{1+r}\right)^t \tau(s^t)\Big| s_0\right] = Z - b_0,$$

given the initial debt position in the private international market b_0. As said, we will assume that $Z = 0$, which is a (expected) zero profit condition for the Fund; given this, x_0 depends on (s_0, b_0) and 'the present value of all the expected primary surpluses' – more precisely, in our formulation, 'the present value of all the expected current account surpluses'. If without private debt there is an interior solution to the Fund's contracting problem, then an optimal solution exists and there are feasible paths of private debt, starting at b_0, but there is also an upper bound on how large the initial debt $-b_0$ can be. To sum up, at the outset of the contract the above break-even condition determines whether the contract is feasible. Once a contract is signed between the country and the Fund, the country's indebtedness is properly monitored and the Fund is ready to withdraw if the terms of the contract are not respected.

4.2 The Recursive Form

Using the approach of Marcet and Marimon (2019), we can state the **recursive Fund contract** as follows:

$$\begin{aligned} FV(s,x,b) = \mathcal{SP} \min_{\{\nu_b,\nu_l\}} \max_{\{c,n\}} \; & x\Big[(1+\nu_b)U(c,n) - \nu_b V^a(s)\Big] && (5)\\ & + \big[(1+\nu_l)\tau - \nu_l(Z-b)\big] \\ & + \frac{1+\nu_l}{1+r}\mathbb{E}\big[FV(s',x',b')\big|s\big] \\ \text{s.t.} \quad & \tau = \theta(s)f(n) - c, \\ & x' = \frac{1+\nu_b}{1+\nu_l}\eta x, && (6) \end{aligned}$$

where $\eta \equiv \beta(1+r) < 1$ and ν_b and ν_l are the normalized multipliers attached to the country's and the lender's limited enforcement constraints, respectively.[10] The private bond policy of the country, $b' = B(s, \tau^f, b)$ is specified below and is taken as given.[11] The value function takes the form of

$$\begin{aligned} FV(s,x,b) &= xV^b(s,x,b) + V^l(s,x,b), \text{ with} \\ V^b(s,x,b) &= U(c,n) + \beta\mathbb{E}\left[V^b(s',x',b')\,|s\right], \text{ and} \\ V^l(s,x,b) &= \tau + \frac{1}{1+r}\mathbb{E}\left[V^l(s',x',b')\,|s\right]. \end{aligned}$$

We obtain the optimal consumption and leisure policies, $c(s,x,b)$ and $n(s,x,b)$ by taking the first-order conditions of problem (5)

$$u'(c) = \frac{1+\nu_l}{1+\nu_b}\frac{1}{x} \quad \text{and} \quad \theta(s)f'(n) = \frac{h'(1-n)}{u'(c)},$$

[10] The normalization of the Pareto weights is the same as the one in Ábrahám et al. (2019).

[11] In this (Nash) specification of the Fund contract the effect of τ^f on $B(s,\tau^f,b)$ is not taken into account.

which results in a transfer policy $\tau(s,x,b) = \tau^f(s,x,b) + [q(s,\tau^f(s'),b(s'))(b(s') - \delta b) - (1 - \delta + \delta\kappa)b]$. Note that we implicitly assume that, $\tau^f(s,x,b)$ is consistent with $\tau^{f\prime}(s') \equiv \{\tau^f(s')\}_{s'|s}$, which will be in equilibrium.

4.3 The Country's Outside Option

We specify $V^a(s)$, which represents the outside option of the contracting country in (2). More precisely, the outside option is given by the autarky value of the standard incomplete market model with default (Eaton and Gersovitz, 1981; Aguiar and Gopinath, 2006; Arellano, 2008).

We assume that if the contracting country decides to default, it is excluded from the private bond market and the Fund. Once it has defaulted, it can reintegrate the private bond market with probability λ but cannot obtain the assistance of the Fund anymore. The Bellman equation for the outside option reads

$$V^a(s) = \max_n \ \left\{U(\theta^p(s)f(n),n)\right\} + \beta\mathbb{E}\left[(1-\lambda)V^a(s') + \lambda J_n^{bi}(s',0)\middle|s\right], \tag{7}$$

where $\theta^p \le \theta$ contains the penalty for defaulting. We assume an asymmetric default cost as in Arellano (2008), that is

$$\theta^p = \begin{cases} \bar{\theta}, & \text{if } \theta \ge \bar{\theta} \\ \theta, & \text{if } \theta < \bar{\theta} \end{cases} \quad \text{with } \bar{\theta} = \psi\mathbb{E}\theta$$

$V^a(s)$ corresponds to the value under financial autarky and J^{bi} to the value of reintegrating the private bond market without the Fund. More precisely,

$$\begin{aligned} J^{bi}(s,b) &= \max\{J_n^{bi}(s,b), V^a(s)\}, \text{ where} \\ J_n^{bi}(s,b) &= \max_{c,n,b'} U(c,n) + \beta\mathbb{E}\left[J^{bi}(s',b')\middle|s\right] \\ &\text{s.t.} \quad c + q^p(s,b')(b' - \delta b) \le \theta(s)f(n) + (1 - \delta + \delta\kappa)b. \end{aligned} \tag{8}$$

Lenders in the private bond market are competitive financial intermediaries. In expected terms, they make zero profits. The price schedule is given by

$$q^p(s,b') = \frac{\mathbb{E}\left[(1 - D(b'))[1 - \delta + \delta\kappa + \delta q(s', b''(s',b'))]|s\right]}{1+r},$$

where $D(b') = \{s' : J_n^{bi}(s',b') < V^a(s')\}$ is the default set. Hence, there is a default premium embedded in the private bond price. Having one-period debt (i.e. $\delta = 0$), the price schedule simplifies to $q^p(s,b') = \frac{\mathbb{E}[1-D(b')|s]}{1+r}$.

5 Decentralization of the Fund Contract

We decentralize the Fund contract to obtain the price schedule and asset positions related to the Fund contract. The decentralization presents many features that are absent in the original model. The main difference is that Ábrahám et al. (2019) design an exclusivity contract between the country and the Fund. In opposition, we explicitly model the joint existence of private international lenders and the Fund. This cohabitation offers a more general exposition of the problem and introduces a richer dynamic in equilibrium. The decentralization is based on the works of Alvarez and Jermann (2000) and Krueger et al. (2008).

5.1 The Decentralized Problem

The aim of the decentralisation is to obtain the current and future asset positions (a and a', respectively) and the underlying asset price, q^f, that corresponds to the Fund's transfer. Formally, one seeks the following relationship

$$\tau^f(s,x,b) = \sum_{s'|s} q^f(s',\omega'|s)(a'(s') - \delta a) - (1-\delta+\delta\kappa)a,$$

where $\omega = a+b$ is the state variable recording the entire debt position. We need to account for both a and b jointly because of the assumption of no seniority. The Fund's transfer is decentralized as a competitive equilibrium with endogenous borrowing constraints following Alvarez and Jermann (2000).[12]

At the start of a period, the country holds a portfolio a securities with respect to the Fund and a portfolio b with respect to the private bond market. A fraction $1-\delta$ of each portfolio matures today and the remaining fraction δ is rolled-over and pays a coupon κ. The country can trade in S state contingent securities $a'(s')$ with a unit price of $q^f(s',\omega'|s)$. Alternatively, it can also trade private bond b' with a unit price of $q^p(s,\omega')$. The budget constraint therefore reads

$$c + \sum_{s'|s} q^f(s',\omega'|s)(a'(s') - \delta a) + q^p(s,\omega')(b'-\delta b) \leq \theta(s)f(n) + (1-\delta+\delta\kappa)(a+b).$$

Thus, the country faces two funding opportunities. On the one hand, it can borrow from the Fund. On the other hand, it can borrow from the private bond market. The price of private debt depends on the entire debt position ω' and not only on b'. This is due to the fact that the country decides to default on its entire debt position. Hence, the risk premium on the private bond has to account for the debt position in the private bond market as well as in the Fund. The same holds true for price of Fund-provided assets.

The state contingent portfolio $a'(s')$ can be decomposed into a common bond $\bar{a}'$ that is independent of the next period state, traded at the implicit bond price $q^f(s,\omega') \equiv \sum_{s'|s} q^f(s',\omega'|s)$, and an insurance portfolio of S assets (Arrow securities) $\hat{a}'(s')$. Thus we have that $a'(s') = \bar{a}' + \hat{a}(s')$, $\bar{a}' = \left[\sum_{s'|s} q^f(s',\omega'|s)a'(s')\right]/q^f(s,\omega')$ and $\sum_{s'|s} q^f(s',\omega'|s)\hat{a}'(s') = 0$. The budget constraint can then be rewritten as

$$c + q^f(s,\omega')(\bar{a}' - \delta a) + \sum_{s'|s} q^f(s',\omega'|s)\hat{a}'(s') + q^p(s,\omega')(b'-\delta b) \leq \theta(s)f(n) + (1-\delta+\delta\kappa)(a+b).$$

[12] Other decentralizations are possible. For example, Dovis (2019) obtains state-contingent contracts by means of an active debt structure management and partial defaults.

The dynamic programming problem of the country (i.e. the borrower) with the aforedefined financial assets is

$$W^b(s,\omega) = \max_{\{c,n,b',\{a'(s')\}_{s'\in S}\}} U(c,n) + \beta\mathbb{E}\big[W^b(s',\omega')\big|s\big] \tag{9}$$

$$\text{s.t. } c + \sum_{s'|s} q^f(s',\omega'|s)(a'(s') - \delta a) + q^p(s,\omega')(b' - \delta b)$$

$$\le \theta(s)f(n) + (1-\delta+\delta\kappa)\omega$$

$$\omega'(s') = a'(s') + b' \ge \mathcal{A}_b(s'), \tag{10}$$

where $\mathcal{A}_b(s')$ is the endogenous borrowing limit of the country which is defined as

$$W^b(s',\mathcal{A}_b(s')) = V^a(s'), \tag{11}$$

and is *not too tight* in the sense of Alvarez and Jermann (2000). The contracting country faces two financing opportunities implying two different lenders. On the one hand, there are competitive international private lenders. On the other hand, there is the Fund whose problem is given by

$$W^f(s,a_l,b_l) = \max_{\{c_f,\{a_l'(s')\}_{s'\in S}\}} c_f + \frac{1}{1+r}\mathbb{E}\big[W^f(s',a_l'(s'),b_l')\big|s\big] \tag{12}$$

$$\text{s.t. } c_f + \sum_{s'|s} q^f(s',\omega'|s)(a_l'(s') - \delta a_l) = (1-\delta+\delta\kappa)a_l,$$

$$a_l'(s') + b_l' \ge \mathcal{A}_f(s',b_l'), \tag{13}$$

where b_l is the amount of assets provided by the private lenders and a_l the amount of assets provided by the Fund. Unlike the borrower, $\mathcal{A}_f(s',b_l)$ should not be interpreted as an endogenous borrowing limit. In fact, as we will see, it represents an endogenous net present value (NPV) limit, which we can define as

$$W^f(s',\mathcal{A}_f(s',b_l') - b_l', b_l') = Z.$$

The problem of the private competitive lenders is given by

$$W^P(s,a_l,b_l) = \max_{\{c_p,b_l'\}} c_p + \frac{1}{1+r}\mathbb{E}\big[W^p(s',a_l'(s'),b_l')\big|s\big] \tag{14}$$

$$\text{s.t. } c_p + q^p(s,\omega')(b_l' - \delta b_l) = (1-\delta+\delta\kappa)b_l,$$

$$a_l'(s') + b_l' \ge \mathcal{A}_p(s',a_l'). \tag{15}$$

Similar to the Fund, one has that

$$W^p(s',a_l',\mathcal{A}_p(s',a_l') - a_l') = b_l'.$$

Together the two NPV constraints give us

$$W^f(s',\mathcal{A}_f(s',b_l') - b_l', b_l') + W^p(s',a_l',\mathcal{A}_p(s',a_l') - a_l') = Z + b_l'. \tag{16}$$

The initial asset holdings of the country in the Fund, $a(s_0) = -a_l(s_0) = 0$, and in the private bond market, $b_0 = -b_{l,0} \leq 0$, are given.

All constraints defined above restrict the amount of debt the country can accumulate. On the one hand, $\mathcal{A}_b(s')$ prevents the country to accumulate today some amount of debt it would like to repudiate tomorrow. Hence, it prevents the country to default on the equilibrium path. On the other hand, $\mathcal{A}_f(s', b_l')$ and $\mathcal{A}_p(s', a_l')$ guarantee that the Fund and the private lender can credibly sustain the country's indebtedness. Specifically, when $Z = 0$, it ensures that the total level of the country's liabilities can be absorbed by the Fund without incurring permanent losses.

The combination of the first-order conditions of the country's problem with respect to c and $a'(s')$ gives the country's Euler equation for the Fund's bonds

$$q^f(s', \omega'|s)u'(c) - \gamma_b(s') = \beta\pi(s'|s)u'(c')\left[(1 - \delta + \delta\kappa) + \delta \sum_{s''|s'} q^f(s'', \omega''|s')\right], \tag{17}$$

where γ_b is the multipliers attached to the country's endogenous borrowing limit. Conversely, the first-order conditions with respect to c and b' gives the country's Euler equation for the private bonds

$$q^p(s, \omega')u'(c) - \sum_{s'|s} \gamma_b(s') = \beta \sum_{s'|s} \pi(s'|s)u'(c')\left[(1 - \delta + \delta\kappa) + \delta q^p(s', a'', b'')\right]. \tag{18}$$

Similarly, the first-order conditions of the decentralized Fund's problem with respect to c and $a_l'(s')$ gives the Fund's Euler equation

$$q^f(s', \omega'|s) - \gamma_f(s') = \frac{1}{1+r}\pi(s'|s)\left[(1 - \delta + \delta\kappa) + \delta \sum_{s''|s'} q^f(s'', \omega''|s')\right], \tag{19}$$

where γ_f is the multipliers attached to the Fund's endogenous NPV limit. Finally, the first-order conditions of the private lenders' problem with respect to c and b_l' gives the private lenders' Euler equation

$$q^p(s, \omega') - \sum_{s'|s} \gamma_p(s') = \frac{1}{1+r} \sum_{s'|s} \pi(s'|s)\left[(1 - \delta + \delta\kappa) + \delta q^p(s', a'', b'')\right], \tag{20}$$

where γ_p is the multipliers attached to the private lender's endogenous NPV limit. We will see in fact that both lender's multipliers coincide in equilibrium. Consequently, there will be no asset-return dominance as $\sum_{s'|s} q^f(s', \omega'|s) = q^p(s, \omega')$ for all (s, ω, ω').

5.2 Recursive Competitive Equilibrium and Steady State

Definition 5.1 (Decentralized Recursive Competitive Equilibrium)**.** *A* recursive competitive equilibrium (RCE) *is a sequence of prices $q^f(s', \omega'|s)$ and $q^p(s, \omega')$, value functions for the country, $W^b(s, \omega)$, the Fund, $W^f(s, a_l, b_l)$, and the private lenders, $W^p(s, a_l, b_l)$, borrowing limits, $\mathcal{A}_b(s')$, for the country, NPV limits, $\mathcal{A}_f(s', b_l')$, for the Fund and, $\mathcal{A}_p(s', a_l')$ for private lenders, as well as policy functions for (i) consumption, $c(s, \omega)$, $c_f(s, a_l, b_l)$ and $c_p(s, a_l, b_l)$, (ii) labor, $n(s, \omega)$, (iii) asset holdings in the Fund, $A(s, \omega)$ and $A_l(s, a_l, b_l)$, and (iv) asset holdings in the private bond market, $B(s, \omega)$ and $B_l(s, a_l, b_l)$, such that*

1. *Given value functions for the outside value options of the country,* $V^a(s')$*, and of the lenders,* $Z-b$*, as well as asset prices* $q^f(s',\omega'|s)$ *and* $q^p(s,\omega')$*,*

   (a) *the policy functions* $c(s,\omega)$*,* $n(s,\omega)$*,* $B(s,\omega)$ *and* $A(s,\omega)$*, together with the value function* $W^b(s,\omega)$*, solve the country's problem* (9) *with the endogenous borrowing limits,* $\mathcal{A}_b(s')$*,*

   (b) *the policy functions* $c_f(s,a_l,b_l)$ *and* $A_l(s,a_l,b_l)$*, together with the value function* $W^f(s,a_l,b_l)$*, solve the Fund's problem* (12) *with the endogenous NPV limit,* $\mathcal{A}_f(s',b_l')$*, and*

   (c) *the policy functions* $c_p(s,a_l,b_l)$ *and* $B_l(s,a_l,b_l)$*, together with the value function* $W^p(s,a_l,b_l)$*, solve the private lenders' problem* (14) *with the endogenous NPV limit,* $\mathcal{A}_p(s',a_l')$*.*

2. *The Fund-provided asset market clears,* $a'(s')+a_l'(s')=0$*.*

3. *The private asset market clears,* $b'+b_l'=0$*.*

4. *The product and labour markets clear,* $c(s,\omega)+c_f(s,a_l,b_l)+c_p(s,a_l,b_l)=\theta(s)f(n(s,\omega))$*.*

Definition 5.2 (High Implied Interest Rates)**.** *An allocation has high implied interest rates if for all* t *and* s^t

$$\mathbb{E}_0\sum_{t=0}^{\infty}Q^f\left(s^t,\omega(s^t)|s_0\right)\left[c\left(s^t,\omega(s^t)\right)+c_l\left(s^t,\omega(s^t)\right)\right]<\infty,$$

The intertemporal discount factor, $Q^f\left(s^t,\omega(s^t)|s_0\right)$, is defined below. This condition ensures that the present value of the total transfer is finite. Having decentralized the Planner's allocation, one needs to determine whether the Second Welfare Theorem holds.

Proposition 5.1 (Decentralized Equilibrium)**.** *Any Fund's transfer with high implied interest rates can be decentralized as a competitive equilibrium with endogenous borrowing limits.*

Proof. See Appendix A □

This proposition states that there is a direct correspondence between, on the one hand, ω and, on the other hand, x and b given by

$$u'(c(s,\omega))=\frac{1+\nu_l(s,x,b)}{1+\nu_b(s,x,b)}\frac{1}{x}.$$

In words, for a given s, if ω, x and b satisfy the above correspondence, then $c(s,\omega)=c(s,x,b)$, $c_p(s,a_l,b_l)=\tau^p(s,x,b)$, $c_f(s,a_l,b_l)=\tau^f(s,x,b)$, $c_p(s,a_l,b_l)+c_f(s,a_l,b_l)=\tau(s,x,b)$ and $n(s,\omega)=n(s,x,b)$. In that same logic, we have that $W^b(s,\omega)=V^b(s,x,b)$ and $W^p(s,a_l,b_l)+W^f(s,a_l,b_l)=V^l(s,x,b)$. Thus, the endogenous borrowing limits (11) and (16) are exactly and uniquely binding when they are binding in the Fund contract. This implies for the Fund-provided asset price that

$$q^f(s',\omega'|s)=\frac{1}{1+r}\pi(s'|s)\frac{u'(c')}{u'(c)}\eta\left[(1-\delta+\delta\kappa)+\delta\sum_{s''|s'}q^f(s'',\omega''|s')\right]$$

$$\text{if } \nu_b(x',s',b')=0 \text{ and } \nu_l(x',s',b')>0,$$

$$q^f(s', \omega'|s) = \frac{1}{1+r}\pi(s'|s)\left[(1-\delta+\delta\kappa) + \delta \sum_{s''|s'} q^f(s'', \omega''|s')\right]$$
$$\text{if } \nu_l(x', s', b') = 0 \text{ and } \nu_b(x', s', b') > 0.$$

Having properly defined the equilibrium equivalence between the Planner's and the decentralized problem, one can now characterize the steady state of the model. The definition of the steady state is related to the definition of an ergodic set of relative Pareto weights. The term ergodic refers to the fact that the relative Pareto weights in this set are aperiodic and recurrent with non-zero probability. In other words, the economy will move around the same set of relative Pareto weights over time and over histories.

Definition 5.3 (Steady State Equilibrium)**.** *A* steady state equilibrium *is related to a lower bound* $\underline{x} = \min_{s \in S}\{x : V^b(s, x, b) = V^a(s)\}$ *and an upper bound* $\overline{x} = \max_{s \in S}\{x : V^b(s, x, b) = V^a(s)\}$ *for the relative Pareto weights together with the shock space* S*. One distinguishes two kinds of steady states*

1. A perfect risk sharing steady state *is an equilibrium path in which* $\underline{x} = \overline{x}$ *implying that the relative Pareto weight* x_t *remains constant for all* $t > h$*, for some* $h \geq 0$*.*

2. An imperfect risk sharing steady state *is an equilibrium path in which* $\underline{x} < \overline{x}$ *implying that the relative Pareto weight* x_t *takes values from the ergodic set* $\{\underline{x}, \ldots, \overline{x}\}$ *for all* $t > h$*, for some* $h \geq 0$*.*

The lower bound of the ergodic set is determined by the lowest achievable relative Pareto weight in the contract. It represents the lowest value that the country accepts in the contract. In the case of equally patient agents (i.e. $\eta = 1$), the lower bound would be determined by $\min_{s \in S}\{x : V^l(s, x, b) = Z - b\}$. Any relative Pareto weight below this threshold will always be admitted by the lender. The upper bound represents the highest relative Pareto weight that makes the country's constraint bind. Any value above this will always be admitted by the country. Note that unlike the lower bound, the upper bound of the ergodic set is the same whether both agents are equally patient or not.

In our economy, $\eta < 1$ implying that the country is more impatient than the Fund. Moreover, as the law of motion of the relative Pareto weight is given by

$$x' = \frac{1+\nu_b}{1+\nu_l}\eta x,$$

the relative weight decreases over time, when none of the constraints are binding (i.e. $\nu_b = \nu_l = 0$). This reduction continues until the lower bound of the ergodic set is hit.[13] Note that if both agents would be equally patient (i.e. $\eta = 1$), the relative Pareto weight would remain stable when none of the constraints are binding. Finally, note that a contract is not feasible when $\min_{s \in S}\{x :$

[13]This is why we cannot determine the lower bound by $\min_{s \in S}\{x : V^l(s, x, b) = Z - b\}$. Being at the lender's lower bound x continues to diminish, until it reaches the borrower's lower bound.

$V^l(s,x,b) = Z - b\} < \min_{s \in S}\{x : V^b(s,x,b) = V^a(s)\}$ simply because none of the contracting parties would like to participate to the contract.

Lemma 5.1 (Bounds of the Ergodic set). *The bounds of the ergodic set solely depend on the current productivity state, $s \in S$.*

Proof. See Appendix A □

This lemma states that the bounds of the ergodic set are independent of b. In other words, the country's participation constraint is solely determined by the realized productivity state. This is because, in the Fund, the country's participation constraint is always satisfied meaning that the country is guaranteed to receive a minimal level of utility irrespective of its indebtedness. From the Fund contract FOCs in Section 4, its optimal labor is determined by the value of the future relative Pareto weight and the productivity state, while its optimal consumption is pinned down by the former. Moreover, the value of the relative Pareto weight does not dependent on the private bond choice since the country's outside option is solely dependent on the productivity state. Hence, the bounds of the ergodic set are independent of the level of private indebtedness.

Proposition 5.2 (No Default Equilibrium). *In a RCE, the contracting country does not default.*

Proof. See Appendix A □

This proposition directly follows from (11) or, equivalently, from (2). The Fund always provides state-contingent transfers to the country through a portfolio of Arrow securities and bonds. This sustains the chosen sequence of private bond, $\{b(s^t)\}_{t=0}^{\infty}$ and ensures that the country obtains at least the value of its outside option in any sate. Hence, given the transfer, the country is at most indifferent between reneging the contract or not and finds it optimal not to do so.

Having said that, the Fund does not accept to sustain all possible sequence of private bonds $\{b(s^t)\}_{t=0}^{\infty}$. We will see that, when (16) binds, the Fund is very strict and does not accept that the country accumulates debt in the private bond market.

5.3 The Fund Contract, Assets and Prices

The country faces two alternatives to purchase debt: the Fund and the private bond market. Besides bonds and unlike private lenders, the Fund also trades Arrow securities. This section therefore establishes the price dynamic and the optimal holdings of assets in the decentralized environment. Using the fact that the borrowing constraints of the borrower and the lender do not bind at the same time,[14]

$$q^f(s',\omega'|s) = \frac{1}{1+r}\pi(s'|s)\frac{u'(c(s',\omega'))}{u'(c(s,\omega))}\eta\left[(1-\delta+\delta\kappa) + \delta\sum_{s''|s'} q^f(s'',\omega''|s')\right]$$

[14] If both constraints would bind at the same time, no agreement could be reached between the country and the lenders. In other words, no contract would exist.

$$\text{if } \gamma_b(s') = 0 \text{ and } \gamma_l(s') > 0,$$

$$q^f(s', \omega'|s) = \frac{1}{1+r}\pi(s'|s)\left[(1-\delta+\delta\kappa) + \delta\sum_{s''|s'} q^f(s'', \omega''|s')\right]$$

$$\text{if } \gamma_l(s') = 0 \text{ and } \gamma_b(s') > 0.$$

Thus, the price is determined by the agent whose constraint is not binding (Krueger et al., 2008). It then follows that

$$q^f(s', \omega'|s) = \frac{\pi(s'|s)}{1+r}\left[(1-\delta+\delta\kappa) + \delta\sum_{s''|s'} q^f(s'', \omega''|s')\right]\max\left\{\frac{u'(c(s', \omega'))}{u'(c(s, \omega))}\eta, 1\right\}. \tag{21}$$

Given the above price schedule, the intertemporal discount factor is defined by

$$Q^f\left(s', \omega'|s\right) = \frac{q^f\left(s', \omega'|s\right)}{1-\delta+\delta\kappa+\delta\sum_{s''|s'} q^f\left(s'', \omega''|s'\right)}.$$

The implicit interest rate in the Fund is then defined by

$$r^f(s, \omega') := \frac{1}{Q^f(s, \omega')} - 1,$$

where $Q^f(s, \omega') = \sum_{s'|s} Q^f\left(s', \omega'|s\right)$. Observe that it suffices that the lenders' constraint is binding in one state $s' \in S$ to have that $\sum_{s'|s} Q^f(s', \omega'|s) > \frac{1}{1+r}$. Hence, in this case, $r^f < r$. Similarly, the implicit interest rate in the private bond market is given by

$$r^p(s, \omega') := \frac{1}{Q^p(s, \omega')} - 1,$$

where $Q^p(s, \omega')$ is determined in the same manner as $Q^f(s, \omega')$.

Having defined the price in the Fund, one should now specify its relationship with respect to the price in the private bond market. To this end, one should first determine the relationship between the private lender's and the Fund's NPV constraints. In a certain state, the Fund always plays the best response with respect to the constraint $\mathcal{A}_f(s', b_l')$. If the private lenders over-lend (from the perspective of the Fund), the Fund withdraws from the market. In this case, the private lenders face potential debt repudiations in the near future. Conversely, if the private lenders under-lend (from the perspective of the Fund), the Fund fills the gap by providing additional credit lines. In this case, the lender unnecessarily restricts its lending to the country. Hence, the private lenders best response is to always follow the DSA recommendation of the Fund.

Proposition 5.3 (Effective Lenders Constraint)**.** *In a RCE, in every states $s' \in S$, the private lender's lending, b_l', and the Fund's lending, a_l', are such that*

$$\mathcal{A}_f(s', b_l') = \mathcal{A}_p(s', a_l')$$

Proposition 5.3 implies that (13) and (15) refer to the same object in equilibrium. As a result, the price of one unit of debt in the Fund and in the private bond market is the same in all state and history. Indeed, looking at the Euler equations (19) and (20), one observes that the two prices coincide when the NPV constraints of both lenders bind at the same time.

Corollary 5.1 (Bond Price). *In a RCE, in every states,*

$$\sum_{s'|s} q^f \left(s', \omega(s')|s\right) = q^p(s, \omega'),$$

Proof. See Appendix A □

The analysis of the lenders' binding constraint is more delicate in the presence of long-term bonds. With one-period bonds, private lenders could immediately withdraw if they felt the country were likely to fail its DSA in the next period. Assuming maturities that go beyond one period forces the private lenders to be even more careful in the credit they grant to the country as they have to sustain the offered credit during the entire maturity length. Furthermore, if private lenders provide bonds with a given maturity, they cannot force the country to repay in advance what it owes. In other words, the country should always be able to maintain the maturity structure of its debt. Therefore, in all circumstances – even when the lenders' participation constraint binds, it must be that the country is offered $b' \geq \delta b$. That is, private lenders cannot completely withdraw as it was the case with short term bonds. They can nonetheless refuse to roll over the the maturing portion of the debt (i.e. $(1-\delta)b$).

This is exactly what happens when the lenders' constraint given in (16) is binding. The country would like to borrow more today with the promise that it would pay back tomorrow, but this is a non credible promise, from the perspective of the Fund. If the country borrows more today, there is a chance that its debt becomes unsustainable tomorrow.[15] Knowing that the country is at risk of failing its DSA, the Fund expects that $b' \geq \delta b$ and therefore provides some credit and Arrow securities to ensure that the country continues to obtain the appropriate amount of insurance. However, it provides just enough resources such that the country's indebtedness remains sustainable in all future states. Furthermore, if the country decides to roll over part of its maturing debt, $b' < \delta b$, the Fund will withdraw. Knowing this, private lenders fulfil the Fund's expectation and do not roll over the country's maturing debt as they fear that the country would default after the Fund's withdrawal. The negative spread therefore appears to sustain this no-trade equilibrium in the private bond market. Concurrently, it also restricts the provision of the Fund's insurance to the country.[16]

Proposition 5.4 (Effective Private Lending). *In a RCE, in the states in which (16) binds $b' \geq \delta b$.*

Proof. See Appendix A □

Proposition 5.5 (Debt Indetermination). *In all states s' in which (16) does not bind, the division of ω' between b' and $\bar{a}'$ is indeterminate.*

[15] We say that the country's debt might become unsustainable as the lenders' constraint does not necessarily bind for all $s' \in S$. Hence, if the country is lucky enough, it could end up in a state tomorrow where lenders could sustain its indebtedness even if it borrowed more today.

[16] Recall that with a positive spread, we have the standard debt dilution argument – i.e. the long-term debt becomes more expensive as the spread rises.

Proof. See Appendix A ☐

This proposition follows directly from Corollary 5.1 and the fact that when (16) does not bind, the country can equally access the private bond market and the Fund. As a matter of fact, when the lenders' participation constraint is not binding, debt is as expensive in the Fund as in the private bond market and the country can accumulate debt in both locations. Therefore, the country is indifferent between holding debt in the private bond market and in the Fund. This results to an indetermination that one resolve with the following corollary.

Corollary 5.2 (Effective Intervention of the Fund)**.** *The Fund's intervention is minimal if, given* (s, b)*, such that in all continuation states* s' *the constraint (16) is not binding,* $0 \leq \bar{a}'(s) \leq \delta b$*. If there is a RCE then there is a RCE where the Fund intervention is minimal.*

Proof. See Appendix A ☐

Corollary 5.2 resolves the previous indetermination. The Fund's credit line is set to its minimal level when (16) does not bind. This does not necessarily imply that the Fund solely provides Arrow securities in this situation.[17] As the level of private debt appears on the right-hand side of (16), one cannot always set $\bar{a}' = 0$. If the private lenders would absorb today the entire debt position of the country, they face the danger of violating the constraint $W^p(s', a'_l, b'_l) \geq b'_l$. To avoid that the debt burden becomes too large for private lenders, the Fund needs to take over part of the country's debt.

Using the intertemporal budget constraints, one can construct the *asset holdings* that make the consumption allocations in the Fund contract satisfy the present value of the budget. First, define the transversality condition of the borrower as

$$\lim_{t\to\infty} \mathbb{E}_t Q\left(s^{t+1}, \omega(s^{t+1})|s^t\right)\left[a\left(s^{t+1}, \omega(s^{t+1})\right) + b(s^{t+1})\right] = 0,$$

for all t and s^t, where

$$Q^f\left(s^{t+j}, \omega(s^{t+j})|s^t\right) = Q^f\left(s^{t+j}, \omega(s^{t+j})|s^{t+n-1}\right) Q^f\left(s^{t+n-1}, \omega(s^{t+n-1})|s^{t+n-2}\right) \cdot \ldots \cdot Q^f\left(s^{t+1}, \omega(s^{t+1})|s^t\right).$$

Recall that, under Corollary 5.1,

$$\sum_{s^{t+1}|s^t} q^f\left(s^{t+1}, \omega(s^{t+1})|s^t\right) = q^p\left(s^t, a(s^{t+1}), b(s^{t+1})\right) =: q\left(s^t, \omega(s^{t+1})\right) \text{ and}$$
$$\sum_{s^{t+1}|st} Q^f\left(s^{t+1}, \omega(s^{t+1})|s^t\right) = Q^p\left(s^t, a(s^{t+1}), b(s^{t+1})\right) =: Q\left(s^t, \omega(s^{t+1})\right),$$

for all t and s^t. Using the borrower's budget constraint, one gets

$$(a(s^t, \omega(s^t)) + b\left(s^t\right))(1 - \delta + \delta\kappa + \delta q\left(s^t, \omega(s^{t+1})\right)) = c\left(s^t, \omega(s^t)\right) + q\left(s^t, \omega(s^{t+1})\right) a\left(s^{t+1}, \omega(s^{t+1})\right) +$$

[17]This is a major difference compared to the case of one-period bonds. With short-term debt, when the lenders' participation constraint does not bind, the entire debt position of the country is located in the private market. The Fund solely trades Arrow securities.

$$q\left(s^t,\omega(s^{t+1})\right)b\left(s^{t+1}\right)-Y\left(s^t,\omega(s^t)\right),$$

where, $Y\left(s^t,\omega(s^t)\right)=\theta(s_t)f\left(n\left(s^t,\omega(s^t)\right)\right)$ for all t and s^t. Iterating forward the budget constraint and using the transversality condition as well as the equilibrium price relationship, one obtains

$$a\left(s^t,\omega(s^t)\right)+b\left(s^t\right)=\mathbb{E}_t\sum_{j=0}^{\infty}Q\left(s^{t+j},\omega(s^{t+j})|s^t\right)\left[c\left(s^{t+j},\omega(s^{t+j})\right)-Y\left(s^{t+j},\omega(s^{t+j})\right)\right].$$

Similarly, define the transversality condition of the lender as

$$\lim_{t\to\infty}\mathbb{E}_tQ\left(s^{t+1},\omega(s^{t+1})|s^t\right)\left[a_l\left(s^{t+1},\omega(s^{t+1})\right)+b_l(s^{t+1})\right]=0.$$

Using the consolidated budget constraint of both lenders, one gets

$$\begin{aligned}(a_l\left(s^t,\omega(s^t)\right)+b_l\left(s^t\right))(1-\delta+\delta\kappa+\delta q\left(s^t,\omega(s^{t+1})\right))=&c_l\left(s^t,\omega(s^t)\right)+q\left(s^t,\omega(s^{t+1})\right)a_l\left(s^{t+1},\omega(s^t)\right)+\\&q\left(s^t,\omega(s^{t+1})\right)b_l\left(s^{t+1}\right).\end{aligned}$$

Iterating forward the budget constraint and using the transversality condition as well as the equilibrium price relationship, one obtains

$$\begin{aligned}a_l\left(s^t,\omega(s^t)\right)+b_l(s^t)=&\mathbb{E}_t\sum_{j=0}^{\infty}Q\left(s^{t+j},\omega(s^{t+j})|s^t\right)c_l\left(s^{t+j},\omega(s^{t+j})\right)\\=&\mathbb{E}_t\sum_{j=0}^{\infty}Q\left(s^{t+j},\omega(s^{t+j})|s^t\right)\left[Y\left(s^{t+j},\omega(s^{t+j})\right)-c\left(s^{t+j},\omega(s^{t+j})\right)\right]\\=&-a\left(s^t,\omega(s^t)\right)-b\left(s^t\right)\end{aligned}$$

The market clearing condition implies that $a_l\left(s^t,\omega(s^t)\right)+a\left(s^t,\omega(s^t)\right)=0$ and $b(s^t)+b_l(s^t)=0$ for all t and s^t.

If the participation constraint of one of the contracting parties is binding, the borrowing limit for of the constrained agent in the decentralized economy is determined by

$$\mathcal{A}_b(s^t)=\mathbb{E}_t\sum_{j=0}^{\infty}Q\left(s^{t+j},\omega(s^{t+j})|s^t\right)\left[c\left(s^{t+j},\omega(s^{t+j})\right)-Y\left(s^{t+j},\omega(s^{t+j})\right)\right],\tag{22}$$

$$\mathcal{A}_l(s^n)=\mathbb{E}_t\sum_{j=0}^{\infty}Q\left(s^{n+j},\omega(s^{n+j})|s^n\right)c_l\left(s^{n+j},\omega(s^{n+j})\right),\tag{23}$$

where $\mathcal{A}_l(s^n)\equiv\mathcal{A}_f(s^n)=\mathcal{A}_p(s^n)$ under Proposition 5.3. Further note that one distinguishes between t and n with $t\neq n$ as the country's and the lenders' constraints cannot bind at the same time if the contract is feasible.

Two conclusions can be drawn from this definition. First and foremost, it is now clear that (13) represents a net present value (NPV) constraint. In any state, the decentralized asset portfolio between the country and the Fund is a whole plan of contingent asset position to the indefinite future, in opposition to the private bond contract which is a one period deal. The whole contingent plan of asset holdings corresponds to the whole plan of transfers $\{\tau(s^t)\}_{t=0}^{\infty}$, which is clearly not a one period decision. The fact that the whole plan can be determined recursively does not mean that the asset positions in s^{t+1} – that is $\omega(s^{t+1})$ – refer only to a set of contingent payoffs at $t+1$. Rather, $\omega(s^{t+1})$ represents the NPV of all future Fund's transfers starting from s^{t+1}. Therefore when (13) binds with strictly positive probability, the Fund refuses to grant an alternative plan embedded in some other $\tilde{\omega}(s^{t+1})$, which would render the NPV negative. Equivalently, this means that the Fund should not lend too much at too low a price or it would end up loosing money. Hence, the lender's constraint is a present value – or more lively, a no bailout – constraint, which is conceptually distinct from the borrower's borrowing constraint, (i.e. a sovereignty constraint).

The second point we want to highlight is that (22) and (23) do not depend on the private bond holdings. Thus, irrespective of the sequence of private bonds $\{b(s^t)\}_{t=0}^{\infty}$, as long as, the different sequences have the same initial starting point, $b(s_0)$ and can be sustained in the Fund contract, they will lead to the same present discounted value for the lenders and the borrower. This is a *Ricardian equivalence* result applied to equations (22) and (23).

Proposition 5.6 (Welfare Equivalence)**.** *Any sequence of private bond positions $\{b(s^t)\}_{t=0}^{\infty}$ being sustained in a RCE with the same initial $b(s_0)$ leads to the same welfare.*

Proof. See Appendix A □

6 Calibration

The calibration follows the one of Ábrahám et al. (2019) for stressed countries during the European debt crisis. The utility function is additively separable with respect to labor and leisure with constant relative risk aversion adapted from King et al. (1988).

$$U(c, n) = \frac{c^{1-\sigma_c} - 1}{1 - \sigma_c} + \gamma \frac{(1-n)^{1-\sigma_l} - 1}{1 - \sigma_l}.$$

The parameters σ_c is the coefficient of relative risk aversion and $-\sigma_l$ denotes the inverse of the Frisch elasticity of labor supply. Furthermore, γ is a coefficient measuring the disutility from work. The production function is Cobb-Douglas with parameter α

$$f(n) = n^{\alpha}.$$

Table 1 summarizes the value of each parameters. For the labor productivity shock θ, we use the Markov switching regime constructed by Ábrahám et al. (2019). The Markov chain is composed of 3 regimes of 9 shocks each – that is 27 labor productivity states in total. The first (last) regime has the lowest (largest) average productivity. Even though each regime is characterized by a different average productivity, they overlap between each other.

7 Results

The results of our quantitative exercise are presented in this section. First, we present and discuss the shape and the pattern of the policy functions and the financial variables. The focus is here to

Table 1: Parameter Values

α	β	σ_c	σ_l	γ	r	λ	ψ	Z
0.566	0.945	1	0.6887	1.4	0.0248	0.15	0.89	0

see whether the contracting country's entire debt position is safe with zero default risk and how this is achieved. Second, one analyzes the model outcome in steady state. Most notably, we simulate the dynamic of the economy in its ergodic set and compare the outcome with and without the Fund's intervention. Finally, one conducts a welfare analysis.

7.1 Policy Functions and Financial Variables

We first present the different policy functions and financial variables. Figure 1 depicts the former for zero private debt as a function of (s, z), while Figures 2, 4 and 3 depict the latter also as a function of (s, z) and for zero private debt. In appendix C, Figures C1 and C4 present the main policy functions and financial variables as a function of (s, ω) and (s, b), respectively. The labor productivity shocks are labeled by θ_i with $i = 1, ..., 27$ and where $\theta_i < \theta_{i+1}$. Moreover, we normalize the relative Pareto weights as $z \equiv \frac{x}{\eta}$. In the figures below, we focus on three main productivity shocks, namely the highest, θ_{max}, the median, θ_{med}, and the lowest, θ_{min}, ones. Furthermore, for some variables depending on the level of private debt, we present them as a function of $b = 0$ and $b = -0.75$, where the latter represents the highest level of debt private lenders can sustain in our calibration.

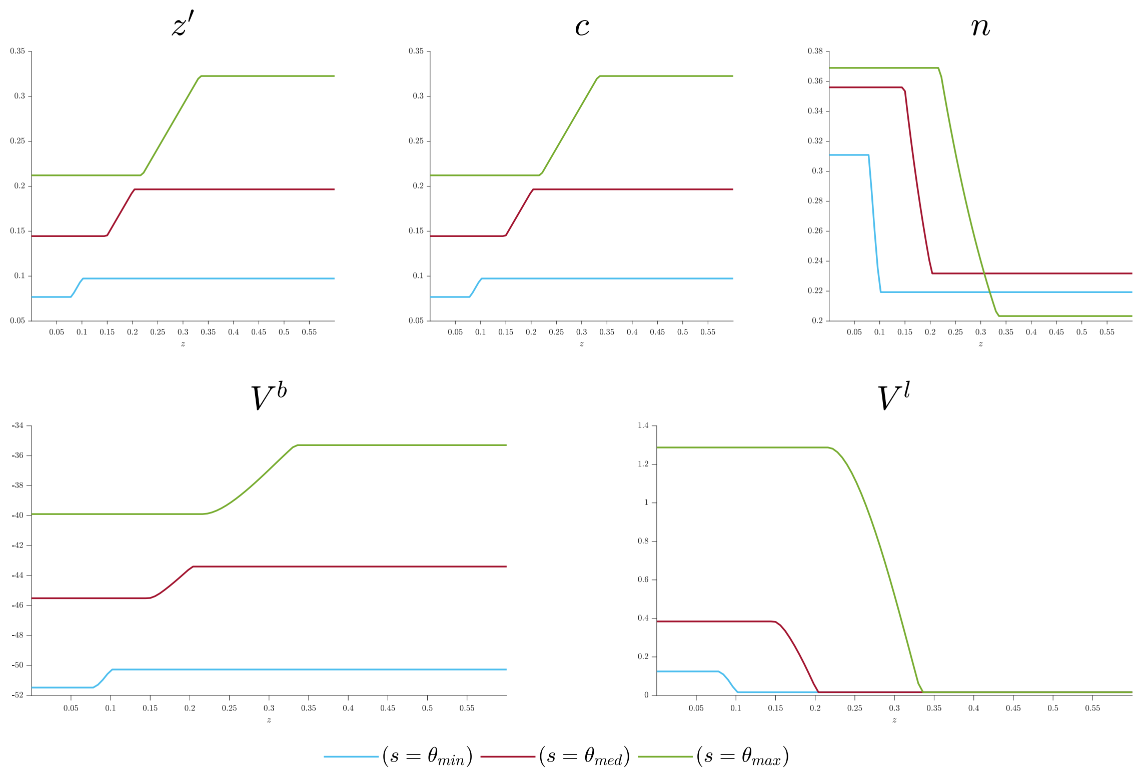
The top panel of Figure 1 presents the optimal policies with respect to the future relative Pareto weights, consumption and labor as function of (s, z). With a logarithmic utility, the consumption of the borrower is equal to the relative future Pareto weight, $c = z'$. Both c and z' are increasing, while n is decreasing in the current relative Pareto weight z. In each panel, the horizontal line on the left hand side is determined by the country's binding participation constraint, while the horizontal line on the right hand side is determined by the lenders' binding participation constraint. The line rejoining both horizontal lines is determined by the first best allocation and has a slope of $\eta < 1$. Consistent with Lemmas 5.1 the borrower's binding constraint does not depend on the level of private debt. However, the lenders' binding constraint does.

The value functions of the country, V^b, and the lenders, V^l, are presented next to the aforementioned policy functions. The shape of the value function is a by-product of the optimal policy functions. The horizontal lines on the left hand side for the country's continuation value and on the right hand side of the lenders' continuation value represent the respective autarky values. In line with the optimal policy functions, for a given labor productivity shock, the value functions of both the lenders and the country when the borrower's participation constraint binds are independent of the level of private debt.

Now we turn to the financial variables depicted in Figure 2.[18] The first row of the figure

[18] In appendix C, Figure C3 presents the same financial variables but for different levels of private debt.

Figure 1: Optimal Policies with Zero Private Debt as Function of (s, z)

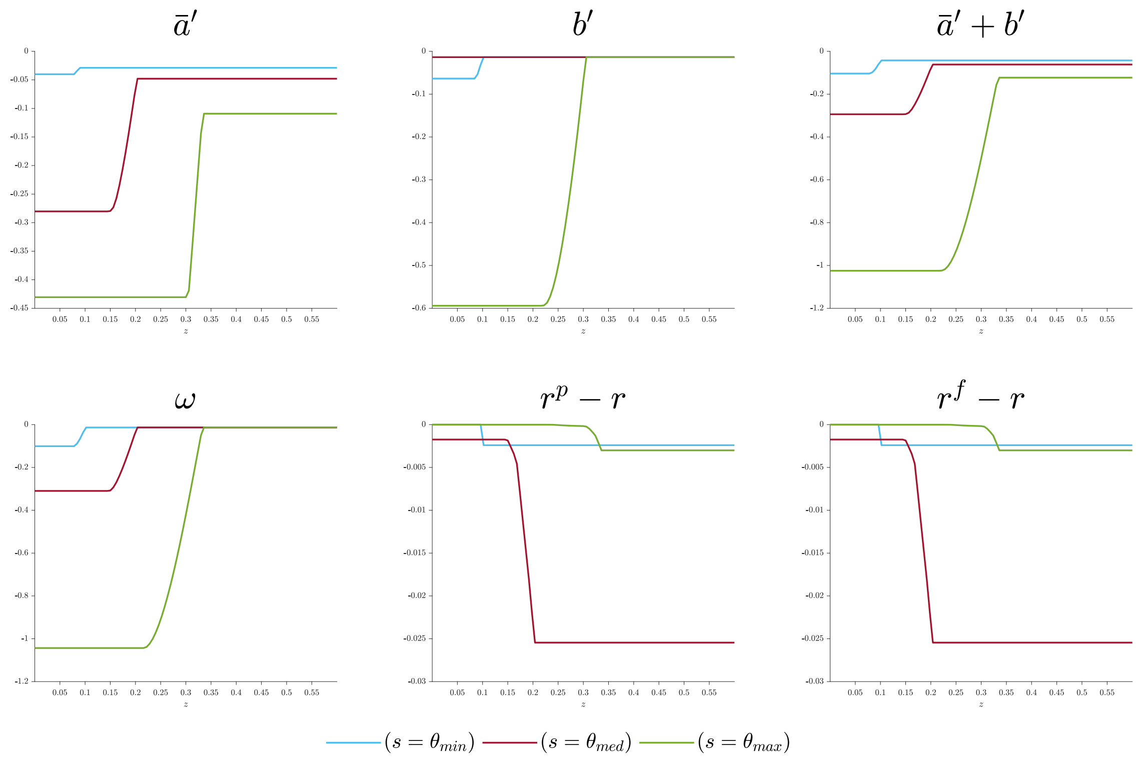


represents the prospective debt holdings of the country. Consistent with the definition of effective intervention in Corollary 5.2, when the lenders' constraint does not bind, the credit line of the Fund is minimal. This does not mean that the majority of the debt is held in the private bond market, though. Conversely, when the lenders' participation constraint binds, private lenders do not roll-over the country's debt. With zero initial private debt this translates into a complete stop of private lending activities. In general, the debt accumulation is largely reduced. As we will see this is because the country has a limited access to Arrow securities when the lenders' participation constraint binds.

The second row of Figure 2 depicts the current asset holdings and the interest spreads. One sees that when the lenders' participation constraint is binding, ω is very close to zero because of Proposition 5.4 and the fact that $Z = 0$ and $b = 0$. It might not exactly be equal to zero depending on the value of the total surplus and how large the negative spread is. This nonetheless tells us that if the lenders' participation constraint is binding today then the value of the country's debt is in great part offset by the value of the realized Arrow security. Hence, when the lenders' participation constraint binds, the country is limited in the trade of Arrow securities and bonds. This limitation ensures that the country does not violate the NPV constraint of the lenders.

Regarding interest rates, the Fund's and private market's spreads are nil when the lenders'

Figure 2: Financial Variables with Zero Private Debt as Function of (s, z)

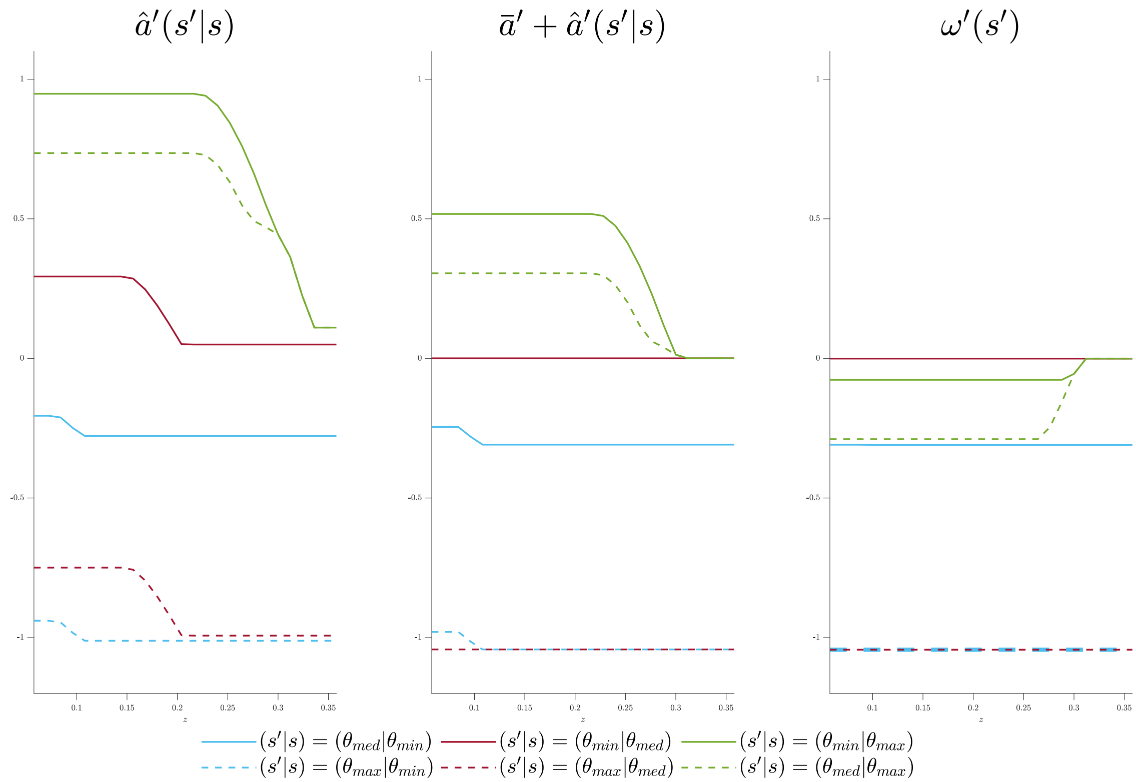


constraint is not binding consistent with Proposition 5.2.[19] In opposition, spreads are negative when the lenders' constraint is binding. As one can see, the negative spread remains relatively modest. Furthermore, it is larger the more constraints are binding in $s' \in S$. Hence, it relates to the extend of insurance required in each future state. A negative spread reduces the trade of Arrow securities in the binding states $s' \in S$. To see why, recall that the holdings of Arrow securities are defined such that $\sum_{s'|s} q^f(s', \omega'|s)\hat{a}'(s') = 0$. Hence, when $Q^f(s', \omega'|s) > \frac{\pi(s'|s)}{1+r}$ for some s' with $\pi(s'|s) > 0$, the country has to reduce its holdings of Arrow securities in the binding states to satisfy the aforementioned clearing condition. As a result, if the lenders' constraint binds in many future states, little hedge is offered by the Fund limiting the accumulation of debt.

Figure 3 presents the holdings of Arrow securities. This figure is key in explaining the insurance mechanism provided by the Fund. First, one clearly sees that the country goes long in the transition between a relatively high productivity state to a relatively low productivity state. The opposite is true for short positions. Hence, Arrow securities prevent large drops in consumption when the labor productivity suddenly decreases. That is, the holding of Arrow securities is procyclical. In other words, the insurance is large when the productivity state is high as well. Second, one observes that the gross holding of Arrow securities decreases in the relative Pareto weight. In other words, the insurance provided by Arrow securities reduces as z increases. As explained before, this is due

[19]The default set of the economy with and without the Fund is presented in Figure C5 in Appendix C.

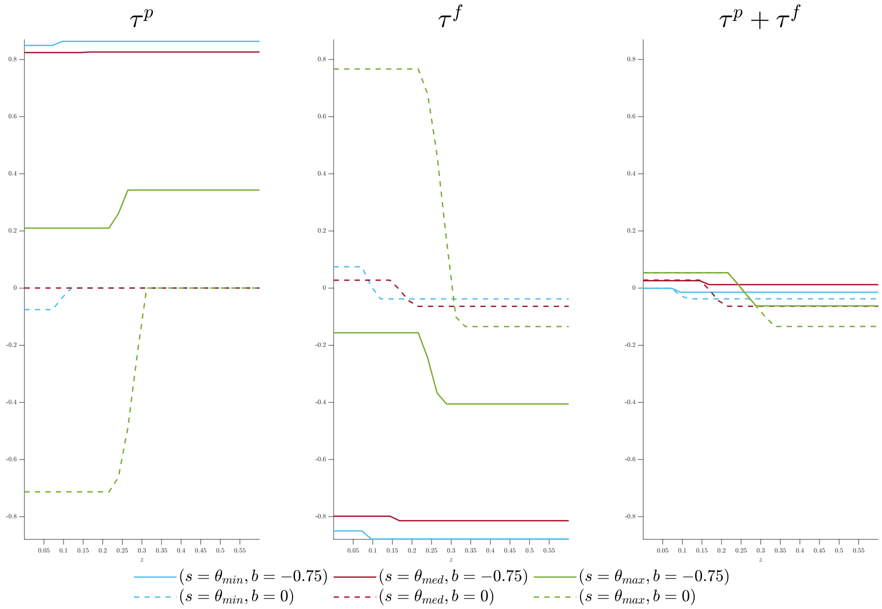
Figure 3: Arrow Securities with Zero Private Debt as Function of (s, z)



to the negative spread. The last aspect of Arrow securities one wants to highlight is that they are always traded in complement to bonds. The relationship is pretty clear on the two panels on the right-hand side of the figure. When the lenders' constraint is binding, $\bar{a}' + \hat{a}'(s')$ is close to zero for the same reasons mentioned in the comment of ω. Correspondingly, notice the flat profile of $\omega'(s')$. Those two panels deliver the same message, namely that Arrow securities hedge the credit position of the country. It then logically follows that any limit to the trade of Arrow securities translates into a limitation of debt accumulation.

The financial variables presented in Figure 2 and 3 do not depend on the private bond holdings. The ones presented in Figure 4 do. The Fund's primary surplus, τ^f, represents the net savings of the country in the Fund. As the relative Pareto weight increases towards the value at which the lenders' participation constraint binds, the surplus becomes negative. The opposite is true when the relative Pareto weight is decreasing. Thus, the surplus is procyclical or if one prefers the deficit is countercyclical. As already mentioned, this procyclicality is the key mechanism preventing default. Next to the net savings in the Fund, one has the net savings in the private bond economy, τ^p. The pattern here is the opposite of the one observed before, reflecting the hedging property of the Fund. The last panel of Figure 4 depicts the total net savings, $\tau^f + \tau^p$, which correspond to the current account of the contracting country. It follows the same pattern as τ^f but is independent of b. The total surplus is therefore procyclical (or countercyclical if one refers to primary deficits)

Figure 4: Transfers as Function of (s, b, z)



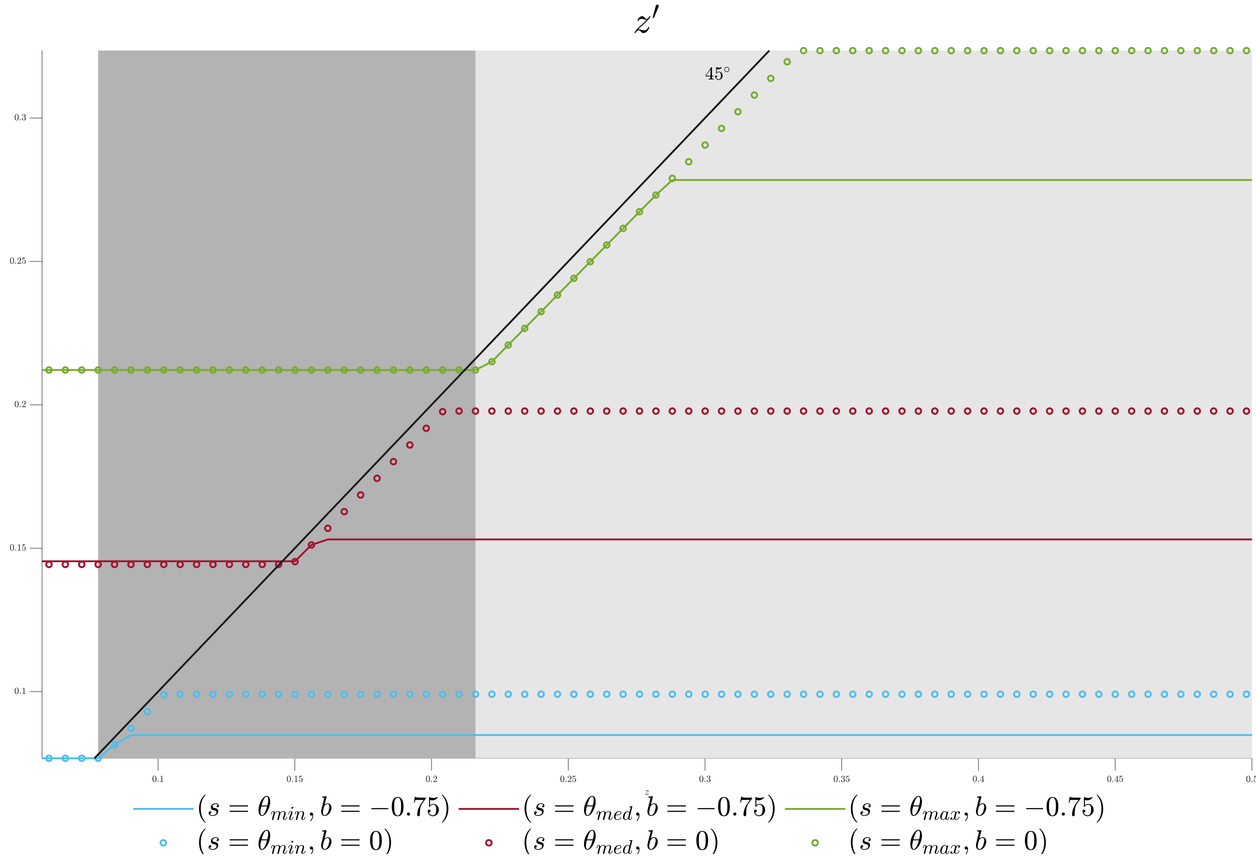
as well. Furthermore, it remains modest compared to τ^f or τ^p, reflecting the fact that positions in the private bond market are counterbalanced by positions in the Fund.

7.2 Steady State Analysis

Figure 5 displays the evolution of the relative Pareto weight in steady state. This variable is key as it determines the dynamic of the whole economy. The dark grey region represents the ergodic set given in Definition 5.3. It is delimited by a lower bound of $\underline{z} = 0.07$ and an upper bound of $\overline{z} = 0.21$. The light grey region represents the basin of attraction of the ergodic set. As one can clearly see the upper and lower bounds of the set do not coincide. Thus, we are in the case of an imperfect risk sharing steady state. As noted earlier, the line characterizing the first best in our economy is below the 45° line as the country is relatively more impatient than the lenders. This means that whenever none of the constraints is binding, the relative Pareto weight decreases. It continues to do so until it hits the value at which the country's participation constraint is binding. This is different than the case of equally patient agents where the relative Pareto weight remains constant when none of the constraints is binding.

One can illustrate the movement of the relative Pareto weights in the ergodic set with the following example. For simplicity one assumes that the level of private debt remains constant. Suppose one starts in the ergodic set on the first best line of the median shock (red non-horizontal line) with a relative Pareto weight of say $z = 0.18$. There, neither of the two participation constraints binds. If the economy remains in this state, the relative Pareto weight decreases until it reaches the country's binding constraint at around $z = 0.15$. At this point, consider that the economy moves

Figure 5: Evolution of the Relative Pareto Weight in Steady State as a Function of (s, b, z)



to the highest productivity state. There, the value $z = 0.15$ is now too low for the country – its participation constraint therefore binds. The Planner will then increase the relative weight and set it to the minimum level to make the country indifferent between reneging the contract or not – that is $z = 0.21$. As long as the productivity state does not change, the economy remains there. Now assume that the economy pass from the highest to the lowest productivity state. There, the lenders' participation constraint binds and the Planner will decrease the weight to $z = 0.1$.

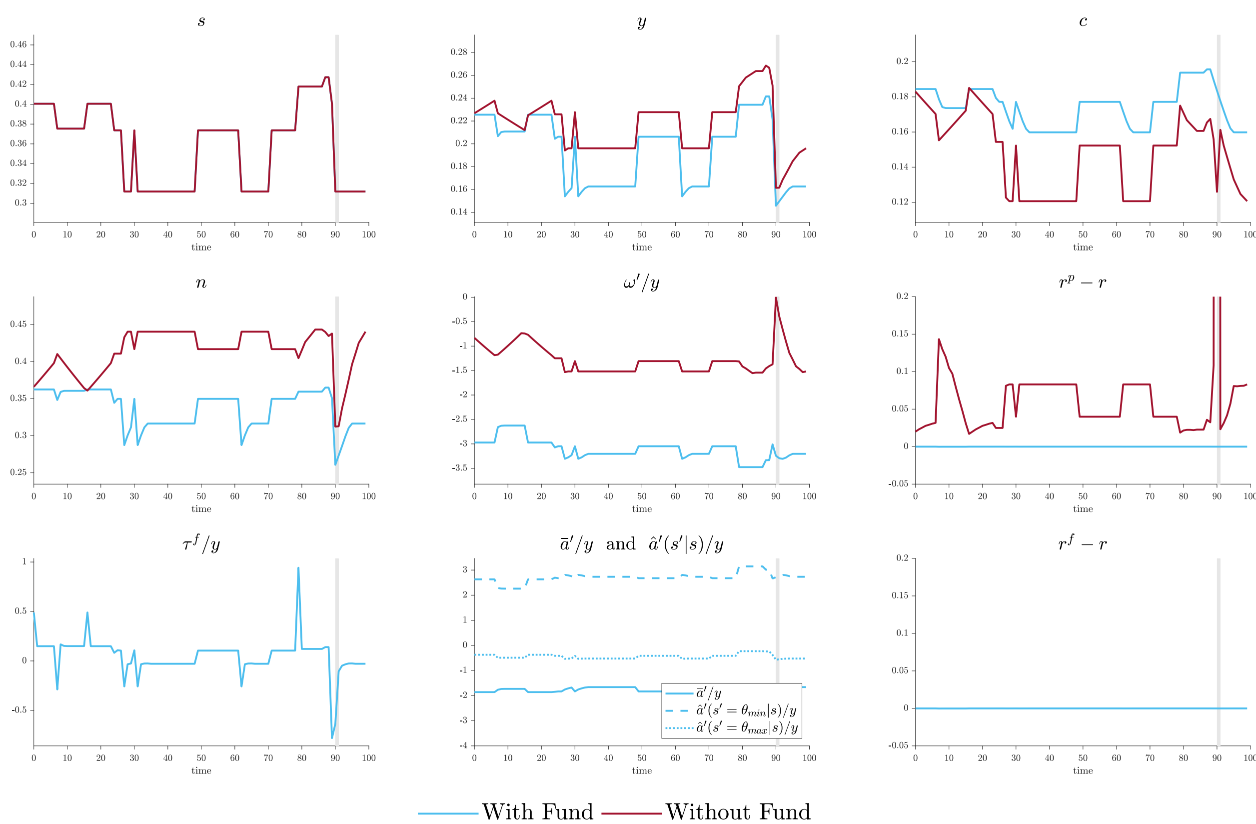
To have a better idea of the model's dynamic in steady state, we simulate the economy within the ergodic set of relative Pareto weights. We conduct two exercises. First, one simulates the economy in normal time. For this purpose, one generates one history of shocks for 400 periods in steady state. To avoid that the initial conditions blur the results, the first 300 periods are discarded. Second, one simulates the economy in crisis time starting with the lowest Pareto weight in the ergodic set, the lowest labor productivity and an initial debt level drawn from the ergodic set. To gauge the impact of the Fund's intervention in each exercise, we simulate both the economy with and without the Fund in parallel.[20]

Figure 6 depicts the simulation in normal time. The red line represents the economy without

[20]The economy without the Fund is a standard incomplete market economy in the spirit of Arellano (2008).

the Fund and the blue line the economy with the Fund. The grey region depicts the periods in which the country devoid of the Fund's intervention decides to default. Defaults of the incomplete market economy are the consequence of a drop in labor productivity combined with a relatively large private indebtedness. The possibility of default creates positive spreads on the private bond market. In opposition, one sees that the Fund's intervention prevents defaults and the underlying risk premium attached to the bond price. When the labor productivity largely drops, the Fund transfers resources to the country. This sustains the current private indebtedness and let the country accumulate debt at the risk-free rate.

Figure 6: Simulation – Normal Time



With the Fund's intervention, the economy has a more stable consumption path over time. Hence, the country avoids the major fluctuations of consumption that characterizes the standard incomplete market economy with defaults. Moreover, as it trades bonds at the risk-free rate without restrictions, it can afford a greater consumption. Most notably, the contracting country is able to accumulate private debt at the risk-free rate in regions where it would normally default without the Fund. This is entirely due to the fact that debt positions are hedged by Arrow securities. To get a sense of the insurance component, we display the Arrow securities purchased today for the highest and the lowest states tomorrow. Three points deserve to be noted. First, the portfolio of Arrow securities is procyclical as it follows the exact same pattern as the shock process. Second, it evolves in the opposite direction as private debt. Thus, debt positions are hedged by Arrow securities. Third, the positions taken in Arrow securities are substantial. If one focuses on $\hat{a}'(s'|s)$ for $s' = \theta_{min}$, we see that it amounts on average 300% of GDP. Instead of looking at the Arrow securities one can observe the Fund's primary surplus, τ^f, which also moves procyclically and largely oscillates around zero since $Z = 0$.

Figure 7: Simulation – Crisis Time

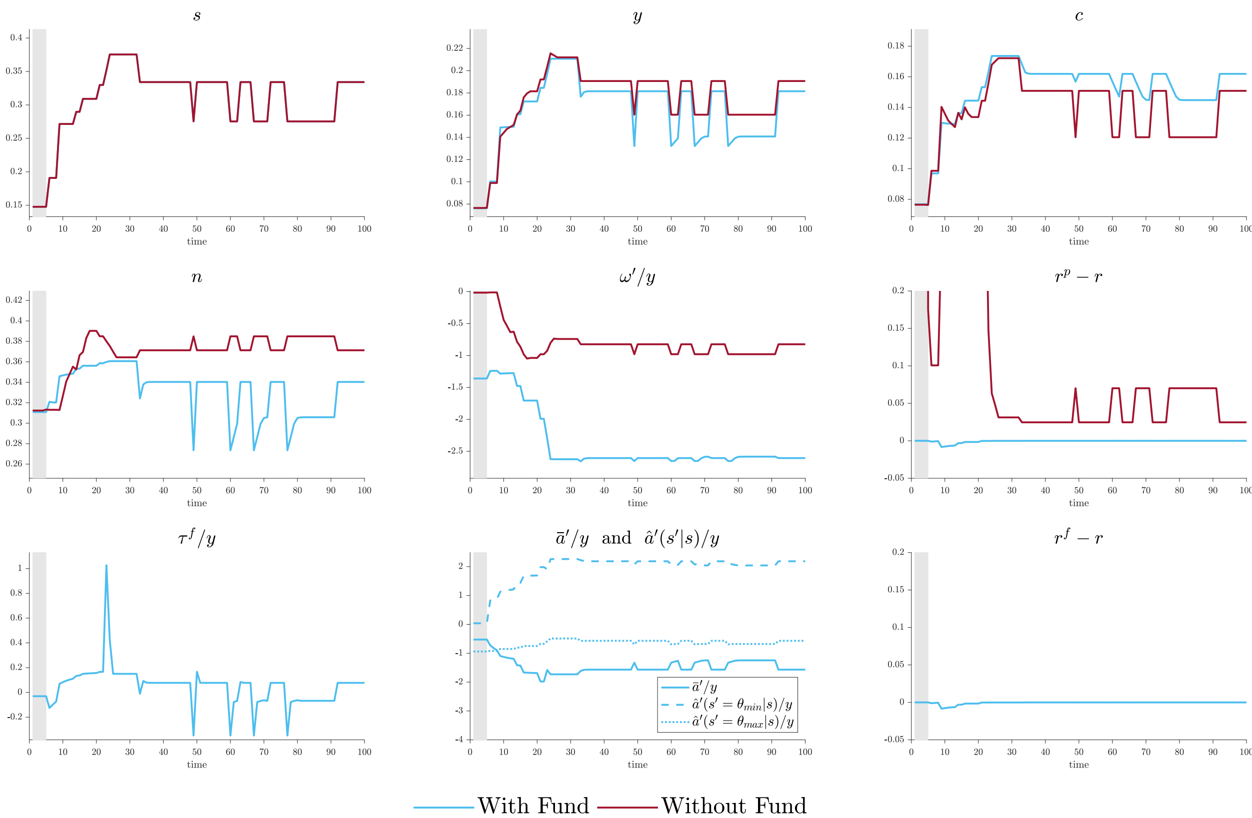


Figure 7 depicts the simulation in crisis time. One sees that in the immediate outbreak of the crisis, the country without the Fund defaults and is excluded from the market. In opposition, the Fund's intervention prevents default and enables the country to maintain its large indebtedness. As the country starts in the lowest possible productivity state, its prospective holdings of Arrow securities is mostly negative. Indeed, the country cannot end up in a worst state and therefore needs little insurance. Owing to the Fund's intervention, the interest rate spread remains nil. In the counterfactual economy without the Fund, the spread explodes and records numerous aftershocks. Finally, the Fund's intervention enables the country to maintain a larger consumption while working less than in the economy without the Fund.

Comparing the two simulations, one concludes three main points. First, default in the incomplete market economy is frequent and can arise in normal times as well. This generates discontinuities in both bond holdings and consumption. Second, as time goes by, one observes a wedge between the two economies. Without the Fund, the country cannot accumulate large amounts of private debt without incurring a large risk premium. Hence, the country without the Fund is unable to increase (or even maintain) its consumption level over time. In other words, the Fund's intervention enhances consumption smoothing. Third, with the Fund, the country has access to insurance and therefore can accumulate large amounts of debt in the private bond market. Only very rarely can the economy without the Fund accumulate as much debt as the other economy. Thus, the Fund's intervention enlarges the country's capacity to accumulate debt.

Figure 8: Impulse Response Functions

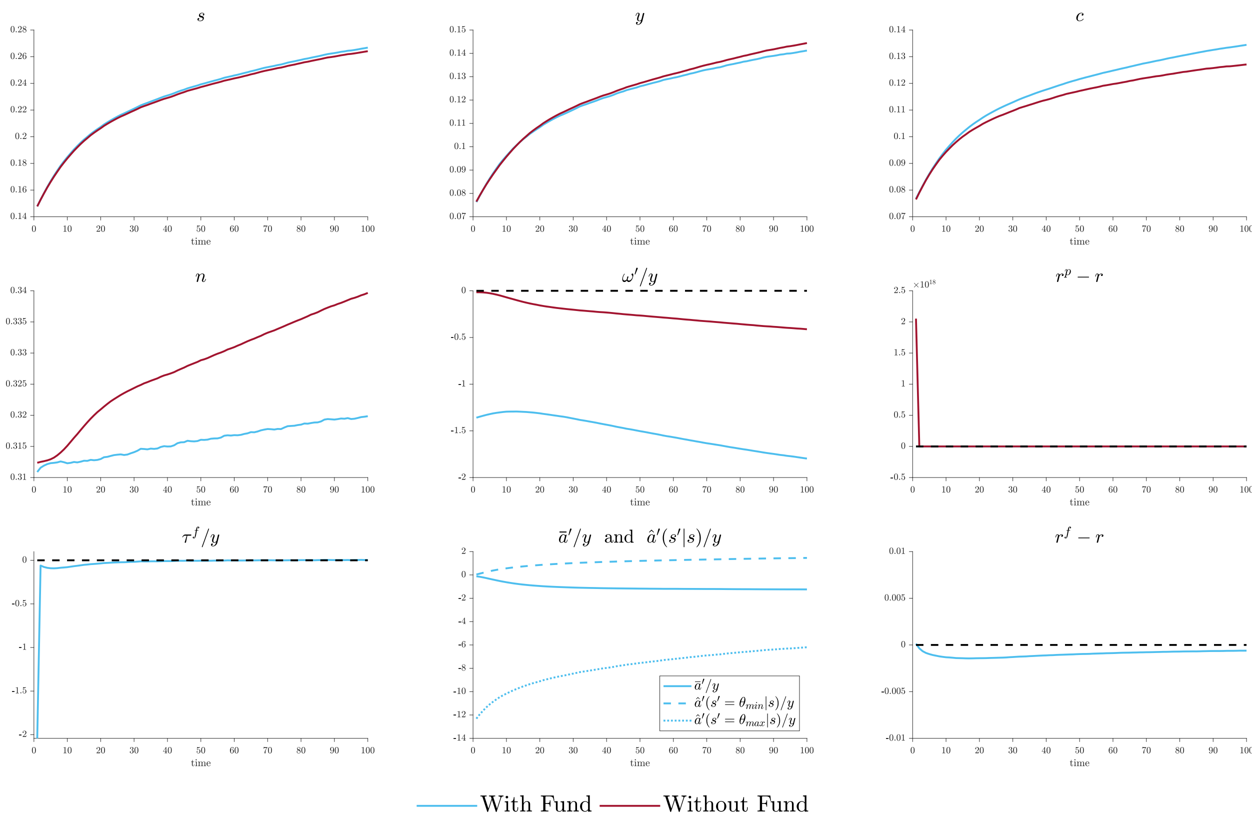


Figure 8 depicts the impulse response functions resulting from a stark negative shock. The responses are computed as the mean of 60'000 independent shock histories starting with the lowest productivity shock θ_{min}, the lowest Pareto weight in the ergodic set and and initial debt hodlings drawn from the ergodic set. The depicted patterns confirm our earlier arguments.

In the very first periods following the shock's realization, the Fund transfers resources to the contracting country. This prevents a large decrease in consumption and a large increase in labor supply. Still, the consumption is the same in the two economies as one starts with the lowest relative Pareto weight in the ergodic set. However, without the Fund's intervention, the country repudiate its debt and is obliged to provide more labor to avoid a massive reduction in consumption. Thus, the immediate impact of a sudden low productivity shock is more severe in the absence of the

Fund. In the long run, the country without the Fund is likely to repudiate debt again and therefore reaches more rapidly its steady state level of consumption. Notice that the shock value might diverge between the economy with and without the Fund in the default region as the defaulting economy is subject to a penalty.

Regarding the financial variables, one observes that the Fund's intervention enables a larger debt holdings in the private bond market in the short and the long run. Without the Fund, many countries in the simulation default implying $b' = 0$ and $r^p - r \to \infty$. The last row of Figure 8 explains how the Fund provides insurance. In the immediate outbreak of the shock, the country receives resources from the Fund. In terms of prospective Arrow securities, the country has a short gross position as it starts in the lowest productivity state and therefore cannot get worst in the future. Subsequently, the country gets less short and more long on Arrow securities as the labor productivity improves.

7.3 Welfare Analysis

Table 2 depicts the welfare comparison between the economy with and without the Fund. The first column represents the welfare gains of the Fund's intervention in consumption equivalent terms at zero initial debt holdings.[21] Recall that the country which has access to the Fund can hold debt in the Fund or in the private bond market. Thus, to adequately compare the two economies, we compare them for the same *total* debt holdings. That is, the welfare comparisons are computed at the points where $\omega = 0$ for the economy in the Fund and at $b = 0$ for the economy outside the Fund. The welfare computation is explained in Appendix B.

Table 2: Welfare Comparison at Zero Initial Debt

Productivity State	Welfare Gains (%)	Maximal Debt absorption (% of GDP)	
		Without Fund $\max \frac{-b'}{y}$	With Fund $\max \frac{-\omega'}{y}$
$s = \theta_{min}$	7.53	2	135
$s = \theta_{med}$	5.97	159	173
$s = \theta_{max}$	5.06	274	395
average	6.46		

Welfare gains are important with the Fund's intervention. With zero initial debt, the consumption-equivalent welfare gains span between 5.06% and 7.53%. Moreover, the largest welfare gains are recorded in low productivity states. Thus, the Fund's intervention is mostly valued when the country is in a difficult economic situation. As mentioned above, welfare gains are the consequence of two main features of the Fund's intervention. First, the Fund provides state-contingent transfers

[21]We evaluate welfare gains at zero initial debt for the same reasons highlighted in Ábrahám et al. (2019).

and therefore enhances consumption smoothing. Second, it enables a greater accumulation of debt in general.

The latter argument can be better illustrated when looking at the two last columns of Table 2. The difference between the debt capacities of the economy with and without the Fund is noticeable. The Fund's intervention enables the country to accumulate a greater amount of debt in all cases. Notably, in the absence of the Fund's intervention, the country's absorbing capacity is mostly limited in low productivity states. Two main reasons lie behind this. First, there is no default cost in low productivity states owing to the asymmetric structure of the default penalty. Second, due to the relatively high persistence of shocks, a low productivity state today implies a high spread on almost all prospective private debt levels as the country has little chance to repay what it owes unless a better shock realizes. Those two factors increase the risk premium embedded in the private bond price and therefore greatly reduce the country's borrowing capacity in the absence of the Fund's intervention.

8 Conclusion

The purpose of this paper was to design the optimal interaction between a Financial Stability Fund, private competitive international lenders and a country. It therefore extends the work of Ábrahám et al. (2019) by not having the Fund absorbing all the sovereign debt of a country. The Fund's long-term contracts is shaped by two-sided limited enforcement constraints. On the one hand, the government is sovereign and, therefore, can default at will. On the other hand, the Fund has a free access to the international financial market and can withdraw whenever additional lending entails expected losses (i.e. when the liabilities of the country become, with positive probability, unsustainable). The country's constraint therefore represents a sovereignty constraint, whereas the lenders' limited enforcement constraint can be interpreted as a DSA. The country can borrow long-term defaultable bonds on the private international market, while receiving state-contingent transfers from the Fund. The Fund contract has no seniority with respect to the privately held sovereign debt and, therefore, takes the private debt into account.

In equilibrium, the Fund prevents the country from defaulting on its entire debt position without intervening on the private market. As a result, the debt in the private international market becomes risk-free, although it is constrained when the lenders' limited enforcement constraint binds. A special attention is drawn on this occasion. The lenders' constraint internalizes a pecuniary externality that competitive private lenders usually do not: the fact that marginal lending can result in debt becoming unsustainable. The Fund contract involves DSA not only at the outset, to determine the terms of the contract, but all along the evolution of the contract. When the lenders' constraint is binding, the country is at the verge of failing its DSA. In this situation, the Fund expects no accumulation of private debt and therefore offers financial support. However, it provides just enough resources for the country to pass its DSA. Most importantly, it grants this financial assistance solely under the condition of no additional private debt. Given that, the private lenders fulfil the Fund's will and a negative spread appears to sustain this no-trade equilibrium in the private bond market.

In following versions, we will provide a more accurate calibration of a Euro Area stressed country. Most notably, one will concentrate on the case of Italy. Having not received the help of the ESM and currently recording one of the largest spreads in the Euro Area, this country offers the opportunity of a very interesting quantitative counterfactual analysis. Additionally, we will study the case in which the Fund has seniority over private lenders, and compare both Funds' designs.

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Appendix

A Proofs

PROOF OF PROPOSITION 5.2. We conduct a proof by contradiction. The present proof only considers the economy in equilibrium. It might be that default occurs off equilibrium path. This situation is, however, outside the scope of the proposition. As a preliminary note, observe that

$$\frac{\partial W^b(s,\omega)}{\partial \omega} = u'(c) > 0,$$

for any $s \in S$. As defined above, the borrowing limit is not too tight implying that

$$W^b(s', \mathcal{A}_b(s')) = V^a(s').$$

Now assume by contradiction that there exists a total amount of debt $\tilde{\omega}'(s') > \mathcal{A}_b(s')$ such that

$$W^b(s', \tilde{\omega}'(s')) < V^a(s'),$$

for a given state $s' \in S$. However, this directly contradicts the fact that the value of the borrower is increasing in ω. Therefore, the constraint stating that $\omega'(s') \geq \mathcal{A}_b(s')$ ensures that the country does not default on equilibrium path. □

PROOF OF PROPOSITION 5.1. Following Alvarez and Jermann (2000) we prove the proposition by construction. First notice that our problem is formed by a concave objective function and a convex feasible set. Thus, the Euler equations and the transversality conditions are sufficient for a maximum. The Fund's asset price is determined by the agents whose constraint is not binding. That is

$$q^f(s', \omega'|s) = \frac{\pi(s'|s)}{1+r}\left[(1-\delta+\delta\kappa) + \delta \sum_{s''|s'} q^f(s'', \omega''|s')\right] \max\left\{\frac{u'(c(s', \omega'))}{u'(c(s,\omega))}\eta, 1\right\}.$$

From the first-order condition of the Fund's contracting problem we have

$$u'(c(s,x,b)) = \frac{1+\nu_l(s,x,b)}{1+\nu_b(s,x,b)}\frac{1}{x}.$$

Moreover, from the law of motion of Pareto weights, we get that

$$x' = \frac{1+\nu_l(s,x,b)}{1+\nu_b(s,x,b)}\eta x.$$

This implies that

$$\frac{x'}{x} = \frac{1+\nu_l(s,x,b)}{1+\nu_b(s,x,b)}\eta = \frac{1}{u'(s,x,b)}\eta.$$

Recall that the lenders' utility is linear because of risk neutrality and $\eta \equiv \beta(1+r)$. In state (s,x,b), if the country is unconstrained, $\nu_b(s,x,b) = 0$, whereas if the lenders are unconstrained, $\nu_l(s,x,b) = 0$.

As a result, the maximum marginal rate of substitution is attained by the unconstrained contracting party in state (s, x, b). This, together with the above definition of asset price, ensures that the Euler equation of the country in (17) as well as the Fund in (19) are satisfied in all periods and states.

However, the country has two Euler equations: one for the Fund's bonds in (17) and the other for the private bonds in (18). Proposition 5.3 together with Corollary 5.1 ensure that the former is consistent with the latter. The last point one needs to verify is whether the transversality condition holds true. For the borrower one has that

$$
\begin{aligned}
&\lim_{t\to\infty} \mathbb{E}_t \beta^t Q\left(s^{t+1}, \omega(s^{t+1})|s^t\right) u'\left(c\left(s^t, \omega(s^t)\right)\right)\left[a(s^{t+1}, \omega(s^{t+1})) + b(s^{t+1})\right] \\
&= \lim_{t\to\infty} \mathbb{E}_t \Bigg[\beta^t u'\left(c\left(s^t, \omega(s^t)\right)\right) q^f\left(s^{t+1}, \omega(s^{t+1})|s^t\right) \\
&\times \mathbb{E}_{t+1} \sum_{j=0}^{\infty} Q\left(s^{t+j+1}, \omega(s^{t+j+1})|s^t\right)\left[c\left(s^{t+j}, \omega(s^{t+j})\right) - Y\left(s^{t+j}, \omega(s^{t+j})\right)\right]\Bigg] \\
&= \lim_{t\to\infty} \beta^t u'\left(c\left(s^t, \omega(s^t)\right)\right) \mathbb{E}_{t+1} \sum_{j=0}^{\infty} Q\left(s^{t+j+1}, \omega(s^{t+j+1})|s^t\right)\left[c\left(s^{t+j}, \omega(s^{t+j})\right) - Y\left(s^{t+j}, \omega(s^{t+j})\right)\right]\Bigg] \\
&\le u'(c(s_0, x_0, \omega_0))\mathbb{E}_{t+1} \sum_{j=0}^{\infty} Q\left(s^{t+j+1}, \omega(s^{t+j+1})|s^t\right)\left[c\left(s^{t+j}, \omega(s^{t+j})\right) - Y\left(s^{t+j}, \omega(s^{t+j})\right)\right]\Bigg] \\
&\le u'(c(s_0, x_0, \omega_0))\mathbb{E}_{t+1} \sum_{j=0}^{\infty} Q\left(s^{t+j+1}, \omega(s^{t+j+1})|s_0\right)\left[c\left(s^{t+j+1}, \omega(s^{t+j+1})\right) + c_l\left(s^{t+j+1}, \omega(s^{t+j+1})\right)\right]\Bigg] \\
&= 0
\end{aligned}
$$

Where the last inequality follows from the Definition 5.2. It is straightforward to extend the argument to the lender's transversality condition. □

Proof of Lemma 5.1. We conduct a proof by construction. The law of motion of relative Pareto weights is given by

$$x' = \frac{1+\nu_b}{1+\nu_l}\eta x.$$

The consumption is determined by

$$c = u'^{-1}\left(\frac{1+\nu_l(s,x,b)}{1+\nu_b(s,x,b)}\frac{1}{x}\right) = u'^{-1}\left(\frac{x'}{\eta}\right) =: u'^{-1}\left(z'\right),$$

where $z' \equiv \frac{x'}{\eta}$ is the normalized relative Pareto weight and ν_b and ν_l are the normalized multipliers attached to the country's and the lenders' participation constraints, respectively. The labor is determined by

$$\theta(s)f'(n) = \frac{h'(1-n)}{u'(c)},$$

$$n = g(s, c)$$

where $g(\cdot,\cdot)$ is the function returning the optimal labor from the FOC. As one can see, the optimal consumption is solely determined by the relative Pareto weight, while the optimal labor is pinned down by the value of consumption – and therefore by the relative Pareto weight – and the productivity state. The country's instantaneous utility is then given by

$$U(c,n) = U\left(u'^{-1}\left(z'\right), g\left(s, u'^{-1}\left(z'\right)\right)\right)$$

which solely depends on the relative Pareto weight and the productivity state. As this holds for every periods and states, the continuation value will be independent on b as well. Given this and the fact that the country's outside option solely depends on the productivity state, so does the relative Pareto weight when the country's constraint binds. □

Proof of Proposition 5.3. We conduct a proof by construction. Assume that the private lenders' NPV constraint is binding but the Fund's one does not for a given $s' \in S$. We have that for a given $\omega(s') = a_l'(s') + b_l'$,

$$a_l'(s') + b_l' > \mathcal{A}_f(s', b_l'),$$
$$a_l'(s') + b_l' = \mathcal{A}_p(s', a_l').$$

Which implies by definition that

$$W^f(s', a_l', b_l') > Z$$
$$W^p(s', a_l', b_l') = b_l'$$

If instead the private lenders would accept to lend more such that $\tilde{b}_l' > b_l'$ and $\omega(s') = \tilde{a}_l'(s') + \tilde{b}_l'$ and

$$\tilde{a}_l'(s') + \tilde{b}_l' > \mathcal{A}_f(s', b_l'),$$
$$\tilde{a}_l'(s') + \tilde{b}_l' > \mathcal{A}_p(s', a_l'),$$

they would in fact be better off as

$$W^p(s', \tilde{a}_l', \tilde{b}_l') > \tilde{b}_l' > b_l' = W^p(s', a_l', b_l').$$

Hence, when the Fund's NPV does not bind, the private lenders' NPV does not as well.

Now assume the opposite situation. The Fund's NPV constraint is binding, while the private lenders' NPV constraint does not. Similar to the previous case, We have that for a given $\omega(s') = a_l'(s') + b_l'$,

$$a_l'(s') + b_l' = \mathcal{A}_f(s', b_l'),$$
$$a_l'(s') + b_l' > \mathcal{A}_p(s', a_l').$$

Using the argument we made above, the private lenders will always prefer to lend more as b_l'. The Fund will then reduce its credit line as the private lenders offer more debt. The private lenders eventually find a level of debt $\tilde{b}_l' > b_l'$ such that $\omega(s') = \tilde{a}_l'(s') + \tilde{b}_l'$ and

$$\tilde{a}_l'(s') + \tilde{b}_l' = \mathcal{A}_f(s', b_l'),$$

$$\tilde{a}_l'(s') + \tilde{b}_l' = \mathcal{A}_p(s', a_l').$$

The private lenders will always find the above level $\tilde{b}_l'$. Assume on the contrary that they do not. The private lenders will continue to lend up to the point where $\bar{a}' = 0$. Above that point any additional level of private lending, say $\ddot{b}_l'$, will cause $\mathcal{A}_f(s', b_l') < \ddot{a}_l'(s') + \ddot{b}_l'$. When this arises, the Fund withdraws and lets the private lenders with a substantial level of debt the country is likely to repudiate in the near future. The private lenders will therefore never want to risk this situation and will never reach this point. In other words, the private lenders will always follow the DSA recommendation of the Fund. □

Proof of Corollary 5.1. We conduct a proof by construction. Following Proposition 5.3, we do not distinguish between the Fund and the private lenders. We refer to two lending entities as the lenders. One distinguishes three cases:

1. The country's and lenders' participation constraints are not binding

   The lenders' Euler equation reads

$$q^f(s', \omega'|s) = \frac{\pi(s'|s)}{1+r}\left[(1-\delta+\delta\kappa) + \delta \sum_{s''|s'} q^f(s'', \omega''|s')\right],$$

$$q^p(s, \omega') = \sum_{s'|s} \frac{\pi(s'|s)}{1+r}\left[(1-\delta+\delta\kappa) + \delta q^p(s', \omega'')\right],$$

   and the country's Euler equations are

$$q^f(s', \omega'|s) = \beta\pi(s'|s)\frac{u'(c(s', \omega'))}{u'(c(s, \omega))}\left[(1-\delta+\delta\kappa) + \delta \sum_{s''|s'} q^f(s'', \omega''|s')\right]$$

$$q^p(s, \omega') = \beta \sum_{s'|s} \pi(s'|s)\frac{u'(c(s', \omega'))}{u'(c(s, \omega))}\left[(1-\delta+\delta\kappa) + \delta q^p(s', \omega'')\right]$$

   If none of the two constraints is ever binding,

$$\begin{aligned}
\sum_{s'|s} q^f(s', \omega'|s) &= \beta \sum_{s'|s} \pi(s'|s)\frac{u'(c(s', \omega'))}{u'(c(s, \omega))}\left[(1-\delta+\delta\kappa) + \delta \sum_{s''|s'} q^f(s'', \omega''|s')\right] \\
&= \sum_{s'|s} \pi(s'|s)\frac{1}{1+r}\left[(1-\delta+\delta\kappa) + \delta \sum_{s''|s'} q^f(s'', \omega''|s')\right], \\
q^p(s, \omega') &= \beta \sum_{s'|s} \pi(s'|s)\frac{u'(c(s', \omega'))}{u'(c(s, \omega))}\left[(1-\delta+\delta\kappa) + \delta q^f(s', \omega'')\right] \\
&= \sum_{s'|s} \pi(s'|s)\frac{1}{1+r}\left[(1-\delta+\delta\kappa) + \delta q^p(s', \omega'')\right],
\end{aligned}$$

   It then follows that

$$Q^p(s, \omega') = \sum_{s'|s} Q^f(s', \omega'|s).$$

2. The country's participation constraint is not binding and the lenders' participation constraint binds

   The lenders' Euler equation reads

$$q^f(s',\omega'|s)-\gamma_f(s')=\frac{\pi(s'|s)}{1+r}\left[(1-\delta+\delta\kappa)+\delta\sum_{s''|s'}q^f(s'',\omega''|s')\right],$$

$$q^p(s,\omega')-\sum_{s'|s}\gamma_p(s')=\sum_{s'|s}\frac{\pi(s'|s)}{1+r}\left[(1-\delta+\delta\kappa)+\delta q^p(s',\omega'')\right],$$

   where $\sum_{s'|s}\gamma_p(s')=\sum_{s'|s}\gamma_f(s')$ under Proposition 5.3. The country's Euler equations are

$$q^f(s',\omega'|s)=\beta\pi(s'|s)\frac{u'(c(s',\omega'))}{u'(c(s,\omega))}\left[(1-\delta+\delta\kappa)+\delta\sum_{s''|s'}q^f(s'',\omega''|s')\right],$$

$$q^p(s,\omega')=\beta\sum_{s'|s}\pi(s'|s)\frac{u'(c(s',\omega'))}{u'(c(s,\omega))}\left[(1-\delta+\delta\kappa)+\delta q^p(s',\omega'')\right].$$

   If the country's participation constraint never binds,

$$\sum_{s'|s}q^f(s',\omega'|s)=\beta\sum_{s'|s}\pi(s'|s)\frac{u'(c(s',\omega'))}{u'(c(s,\omega))}\left[(1-\delta+\delta\kappa)+\delta\sum_{s''|s'}q^f(s'',\omega''|s')\right]\quad\text{and}$$

$$\sum_{s'|s}q^f(s',\omega'|s)>\sum_{s''|s'}\frac{\pi(s'|s)}{1+r}\left[(1-\delta+\delta\kappa)+\delta\sum_{s''|s'}q^f(s'',\omega''|s')\right],$$

   Moreover,

$$Q^p(s,\omega')=\sum_{s'|s}Q^f(s',\omega'|s).$$

3. The country's participation constraint binds and the lenders' participation constraint is not binding

   The lenders' Euler equation is

$$q^f(s',\omega'|s)=\frac{\pi(s'|s)}{1+r}\left[(1-\delta+\delta\kappa)+\delta\sum_{s''|s'}q^f(s'',\omega''|s')\right],$$

$$q^p(s,\omega')=\sum_{s'|s}\frac{\pi(s'|s)}{1+r}\left[(1-\delta+\delta\kappa)+\delta q^p(s',\omega'')\right],$$

   and the country's Euler equations are

$$q^f(s',\omega'|s)u'(c(s,\omega))-\gamma_b(s')=\beta\pi(s'|s)\frac{u'(c(s',\omega'))}{u'(c(s,\omega))}\left[(1-\delta+\delta\kappa)+\delta\sum_{s''|s'}q^f(s'',\omega''|s')\right],$$

$$q^p(s,\omega')u'(c(s,\omega)) - \sum_{s'|s}\gamma_b(s') = \beta\sum_{s'|s}\pi(s'|s)\frac{u'(c(s',\omega'))}{u'(c(s,\omega))}\left[(1-\delta+\delta\kappa)+\delta q^p(s',\omega'')\right].$$

If the lenders' participation constraint never binds,

$$\sum_{s'|s} q^f(s',\omega'|s) > \beta\sum_{s'|s}\pi(s'|s)\frac{u'(c(s',\omega'))}{u'(c(s,\omega))}\left[(1-\delta+\delta\kappa)+\delta\sum_{s''|s'} q^f(s'',\omega''|s')\right] \quad \text{and}$$

$$\sum_{s'|s} q^f(s',\omega'|s) = \sum_{s''|s'}\frac{\pi(s'|s)}{1+r}\left[(1-\delta+\delta\kappa)+\delta\sum_{s''|s'} q^f(s'',\omega''|s')\right],$$

Moreover,

$$Q^p(s,\omega') = \sum_{s'|s} Q^f(s',\omega'|s).$$

From those three cases, one can conclude that the bond price in the private market, $q^p(s,\omega')$, is always equal to the price in the Fund, $\sum_{s'|s} q^f(s'|s,\omega)$. As a result, the division of debt between b' and $\bar{a}'$ will be indeterminate if the country can freely access the Fund and the private bond market as we show in Proposition 5.5. □

Proof of Proposition 5.6. We conduct a proof by construction. Consider two sequences of private bonds $\{b(s^t)\}_{t=0}^{\infty}$ and $\{\tilde{b}(s^t)\}_{t=0}^{\infty}$ satisfying the definition of a RCE with $\tilde{b}(s_0) = b(s_0)$ and $\tilde{b}(s^t) \neq b(s^t)$ for all $t > 0$ and $s^t \neq s_0$. Hence, at $t = 0$, the budget constraint reads

$$\begin{aligned}
a(s_0,\omega(s_0))(1-\delta+\delta\kappa) = &\sum_{s^1|s_0} q^f\left(s^1,\omega(s^1)|s_0\right)\left(a\left(s^1,\omega(s^1)\right)-\delta a(s_0,\omega(s_0))\right) + q^p\left(s_0,\omega(s^1)\right)\left(b\left(s^1\right)-\delta b\left(s_0\right)\right)+\\
&c\left(s_0,\omega(s_0)\right) - b\left(s_0\right)(1-\delta+\delta\kappa) - Y\left(s_0,\omega(s_0)\right),\\
a(s_0,\tilde{\omega}(s_0))(1-\delta+\delta\kappa) = &\sum_{s^1|s_0} q^f\left(s^1,\tilde{\omega}(s^1)|s_0\right)\left(a\left(s^1,\tilde{\omega}(s^1)\right)-\delta a(s_0,\tilde{\omega}(s_0))\right) + q^p\left(s_0,\tilde{\omega}(s^1)\right)\left(\tilde{b}\left(s^1\right)-\delta \tilde{b}\left(s_0\right)\right)+\\
&c\left(s_0,\tilde{\omega}(s_0)\right) - \tilde{b}\left(s_0\right)(1-\delta+\delta\kappa) - Y\left(s_0,\tilde{\omega}(s_0)\right).
\end{aligned}$$

Given that $\tilde{b}(s_0) = b(s_0)$ and the initial asset holdings in the Fund being $a\left(s_0,\omega(s_0)\right) = a\left(s_0,\tilde{\omega}(s_0)\right) = 0$, it holds that $\omega(s_0) = \tilde{\omega}(s_0)$. The two budget constraints can therefore be combined resulting to the fact that

$$a(s^1,\omega(s^1)) + b\left(s^1\right) = a(s^1,\tilde{\omega}(s^1)) + \tilde{b}\left(s^1\right),$$

where we used Corollary 5.1. Iterating forward the same argument for $t > 0$, we obtain that

$$a(s^t,\omega(s^t)) + b\left(s^t\right) = a(s^t,\tilde{\omega}(s^t)) + \tilde{b}\left(s^t\right),$$

or equivalently,

$$\mathbb{E}_t \sum_{j=0}^{\infty} Q\left(s^{t+j}, \omega(s^{t+j})|s^t\right)\left[c\left(s^{t+j}, \omega(s^{t+j})\right) - Y\left(s^{t+j}, \omega(s^{t+j}\right)\right]$$
$$= \mathbb{E}_t \sum_{j=0}^{\infty} Q\left(s^{t+j}, \tilde{\omega}(s^{t+j})|s^t\right)\left[c\left(s^{t+j}, \tilde{\omega}(s^{t+j})\right) - Y\left(s^{t+j}, \tilde{\omega}(s^{t+j}\right)\right],$$

for all t and s^t. The generalisation of the argument for any t extensively relies on the fact that the alternative private bond sequence $\tilde{b}(s^t) \neq b(s^t)$ is consistent with (13) for all t and s^t.

Thus, a given sequence of private bonds $\{b(s^t)\}_{t=0}^{\infty}$ for which the country's problem with borrowing limits $\mathcal{A}_b(s^t)$ and the lender's problem with NPV limits $\mathcal{A}_f(s^t, b(s^t))$ and $\mathcal{A}_p(s^t, a(s^t))$ have a solution, the alternative private bond sequence $\{\tilde{b}(s^t)\}_{t=0}^{\infty}$ that can be sustained as a RCE with $\tilde{b}(s_0) = b(s_0)$ and $\tilde{b}(s^t) \neq b(s^t)$ for all $t > 0$ and $s^t \neq s_0$ is equivalent to $\{b(s^t)\}_{t=0}^{\infty}$. □

Proof of Proposition 5.4. We conduct a proof by construction. We have established that whenever the lenders' borrowing limit binds, private lending activities with the country stops. Moreover, in this case $Q^f(s^{t+j}, \omega(s^{t+j})|s^t) > \frac{\pi(s^{t+j}|s^{t+1})}{(1+r)^j}$, while when the lender's constraint does not bind $Q^f(s^{t+j}, \omega(s^{t+j})|s^t) = \frac{\pi(s^{t+j}|s^{t+1})}{(1+r)^j}$. As the country is borrowing from the Fund when the lender's constraint binds, $Y(s^{t+j}, \omega(s^{t+j})) - c(s^{t+j}, \omega(s^{t+j})) \leq 0$.[22] It then follows that

$$\sum_{j=1}^{\infty} Q^f(s^{t+j}, \omega(s^{t+j})|s^t)\left[Y(s^{t+j}, \omega(s^{t+j})) - c(s^{t+j}, \omega(s^{t+j}))\right]$$
$$\leq \sum_{j=1}^{\infty} \frac{\pi(s^{t+j}|s^{t+1})}{(1+r)^j}\left[Y(s^{t+j}, \omega(s^{t+j})) - c(s^{t+j}, \omega(s^{t+j}))\right].$$

When the lender's constraint is binding in s^{t+1}, $b(s^{t+1}) = 0$, such that

$$\mathcal{A}_f(s^{t+1}) = \sum_{j=1}^{\infty} Q^f(s^{t+j}, \omega(s^{t+j})|s^t)\left[Y(s^{t+j}, \omega(s^{t+j})) - c(s^{t+j}, \omega(s^{t+j}))\right],$$

Under Proposition 5.3 with $W^l = W^f + W^p$, we have

$$W^f(s^{t+1}, \mathcal{A}_f(s^{t+1})) = Z - b(s^{t+1})$$
$$= \sum_{j=1}^{\infty} \frac{\pi(s^{t+j}|s^{t+1})}{(1+r)^j}\left[Y(s^{t+j}, \omega(s^{t+j})) - c(s^{t+j}, \omega(s^{t+j}))\right],$$

which implies that

$$\mathcal{A}_f(s^{t+1}) \leq Z - b(s^{t+1}).$$

□

[22] It is not possible that the lenders' constraint binds when $Y(s^{t+j}, \omega(s^{t+j})) - c(s^{t+j}, \omega(s^{t+j})) > 0$ as the country is saving meaning that lenders receive resources from the country and not the opposite.

Proof of Proposition 5.5. We conduct a proof by construction. One distinguishes two cases:

Wen (16) does not bind, the budget constraint reads

$$c + q^p(s,\omega')(b' - \delta b) + \sum_{s'|s} q^f(s',\omega'|s)(a'(s') - \delta a) = \theta(s)f(n) + (1 - \delta + \delta\kappa)(b + a).$$

Given that $\sum_{s'|s} q^f(s',\omega'|s)\hat{a}(s') = 0$ and Corollary 5.1, it can be rewritten as

$$c + q^f(s,\omega')(b' - \delta b) + q^f(s,\omega')(\bar{a}' - \delta\bar{a}) = \theta(s)f(n) + (1 - \delta + \delta\kappa)(b + a),$$
$$c + q(s,\omega')(\bar{\omega}' - \delta(b + a)) = \theta(s)f(n) + (1 - \delta + \delta\kappa)(b + a).$$

Having the same price and being equally accessible, private and Fund-provided bonds are prefect substitute. It is then clear that the decomposition of $\bar{\omega}'$ between b' and $\bar{a}'$ is indeterminate.

Proof of Corollary 5.2. We conduct a proof by construction. Setting $\bar{a}' = 0$ implies that the country exclusively accumulates debt in the private bond market, resolving the indetermination. None of the debt is located in the Fund which solely provides insurance. Hence, the Fund intervention is set to its minimum. To see this consider the budget constraint without effective intervention

$$c+q^p(s,\omega')(b'_{ne}-\delta b_{ne})+q^f(s,\omega')(\bar{a}'_{ne}-\delta a_{ne})+\sum_{s'|s} q^f(s',\omega'|s)\hat{a}_{ne}(s') = \theta(s)f(n)+(1-\delta+\delta\kappa)(b_{ne}+a_{ne}),$$

and with effective intervention

$$c + q^p(s,\omega')(b'_e - \delta b_e) + \sum_{s'|s} q^f(s',\omega'|s)\hat{a}_e(s') = \theta(s)f(n) + (1 - \delta + \delta\kappa)(b_e + a_e).$$

Consumption and labor are the same in those two cases. Only the asset positions are different. Under Corollary 5.1, we have that $b'_e = b'_{ne} + \bar{a}'_{ne}$. However, $b_{ne} \neq b_e$ and $a_{ne} \neq a_e$. Since $\bar{a}'_e = 0$ and $b'_e = b'_{ne} + \bar{a}'_{ne}$, it logically follows that $|a_{ne}| > |a_e|$. To see this, assume that we are in a given state $s \in S$. As a result, $a_{ne} = \bar{a}_{ne} + \hat{a}_{ne}(s)$, while $a_e = \hat{a}_e(s)$. Since we assume the same consumption and labor and $b'_e = b'_{ne} + \bar{a}'_{ne}$, we have that $\hat{a}_e(s) = \hat{a}_{ne}(s)$, implying that $|a_{ne}| > |a_e|$. Thus, the Fund's share of debt under the effective intervention is set to its minimum as it is only composed of the realised Arrow securities and does not contain any proper credit component.

However, it is not always possible to set $\bar{a}' = 0$ if one does not want the constraint $W^p(s', a'_l, b'_l) \geq b'_l$ to be violated. More precisely, the maximal level of debt the private lender can absorb is give by

$$b' \geq Z - \min_{s' \in \tilde{S}} \{W^l(s', \mathcal{A}_l(s', b'))\},$$

where $W^l = W^p + W^f$ and $\mathcal{A}_l(s', b') = \mathcal{A}_f(s', b') = \mathcal{A}_p(s', a')$ under Proposition 5.3. Moreover, $\tilde{S}$ designate the set of all $s' \in S$ such that $\pi(s'|s) > 0$. □

B Welfare Calculations

This section describes how the welfare gains depicted in Table 2 are computed. Similar to Ábrahám et al. (2019), define value of the country for a sequence $\{c(s^t), n(s^t)\}$ starting from an initial state at $t = 0$ as

$$V^b(\{c(s^t), n(s^t)\}) = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t U(c(s^t), n(s^t)) = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \left[\log(c(s^t)) + \gamma \frac{(1 - n(s^t))^{\sigma_n} - 1}{1 - \sigma_n} \right],$$

where the last equality is obtained from the functional form considered in Section 6. One denotes the country's allocations with the Fund by $\{c^f(s^t), n^f(s^t)\}$ and the allocations without the Fund by $\{c^i(s^t), n^i(s^t)\}$. The value for the borrower with and without the Fund is given by

$$\begin{aligned} W^{bf}(s, \omega) &= W^{bf}(\{c^f(s^t), n^f(s^t)\}), \quad \text{and} \\ V^{bi}(s, b) &= V^{bi}(\{c(s^t), n(s^t)\}), \end{aligned}$$

respectively. To properly compare the two economies, we consider the point where $\omega = b =: o$. Thus (s, o) represents the initial state for both economies. Now define

$$V^{bi}(s, o; \chi) = V^{bi}(\{(1 + \chi)c(s^t), n(s^t)\}),$$

where $\chi(s, o)$ represents the consumption-equivalent welfare gain of the Fund's intervention. It then directly follows that the welfare gain is computed in the following way

$$V^{bi}(s, o; \chi) = W^{bf}(s, o).$$

Given the above functional form, we have that

$$\frac{\log(1 + \chi)}{1 - \beta} + V^{bi}(s, o) = V^{bf}(s, o).$$

The welfare gain therefore boils down to

$$\chi(s, o) = \exp\left[(V^{bf}(s, o) - V^{bi}(s, o))(1 - \beta)\right] - 1.$$

We concentrate our analysis to the case in which $o = 0$.

C Additional Tables and Figures

Figure C1 depicts the main policy functions and financial variables as a function of (s, ω). More precisely, it presents the aforementioned statistics for the largest, the median and the lowest labor productivity shocks. The dynamic is fairly similar to what we have highlighted in Section 7. This is because there is a close correspondence between ω and x and b as discussed in Section 5.

Figure C4 depicts the main policy functions and financial variables as a function of (s, b). Most notably, it present the aforementioned statistics for the largest and lowest labor productivity shocks

θ_{max} and θ_{min}, as well as, the largest and lowest relative Pareto weights z_{max} and z_{min}, respectively. Regarding the consumption policy function, one clearly sees the intervention of the Fund. As a matter of fact, the consumption remains constant and only diverges in the labor productivity states and the relative weights. The same is true for labor. This is because positions are fully insured by the Fund given (s, z) irrespective of the level of private indebtedness. This is a direct consequence of Lemma 5.1 and Proposition 5.4.

The country's and lenders' value functions deliver the same message as the above policy functions. Neither of them depend on the value of private debt.

Regarding the prospective asset positions. The level of b' and $\bar{a}'$ are flat with respect to b. Again this is because the country's and the lender's participation constraints are specific to the value of the relative Pareto weight and the labor productivity but do not depend on any particular level of private debt.

Regarding the net savings, we note three main elements. First, the net savings in the Fund, τ^f, is increasing in the level of b. This is because the Fund hedges the positions taken in the private bond market. Second, the net savings in the private bond economy, τ^p, display the opposite pattern. Private net savings are decreasing in b. Finally, the total net savings (i.e. $\tau^f + \tau^p$) are flat in the level of private debt.

On the right panel of the last row of Figure C4, one can observe the interest rate spread in the private bond economy and in the Fund. Again, in line with Corollary 5.1, both spreads are aligned. The spreads are independent of b and negative for high values of the relative Pareto weights (i.e. when the lenders' participation constraint binds).

Figure C5 presents the default set of the economy with and without the Fund's intervention. The former is depicted on the right hand side and the latter on the left hand side of the figure. Without the Fund's intervention, the country defaults at different levels of labor productivity and different levels of debt depending on the labor productivity regime. In regimes of greater average labor productivity, the country defaults on relatively higher debt levels or even decides not to default. The hilly pattern of the default set is explained by the fact that the different regimes overlap between each others. With the Fund's intervention, the country never defaults consistent with Proposition 5.2.

Figure C1: Optimal Policies with Zero Private Debt as Function of (s, ω)

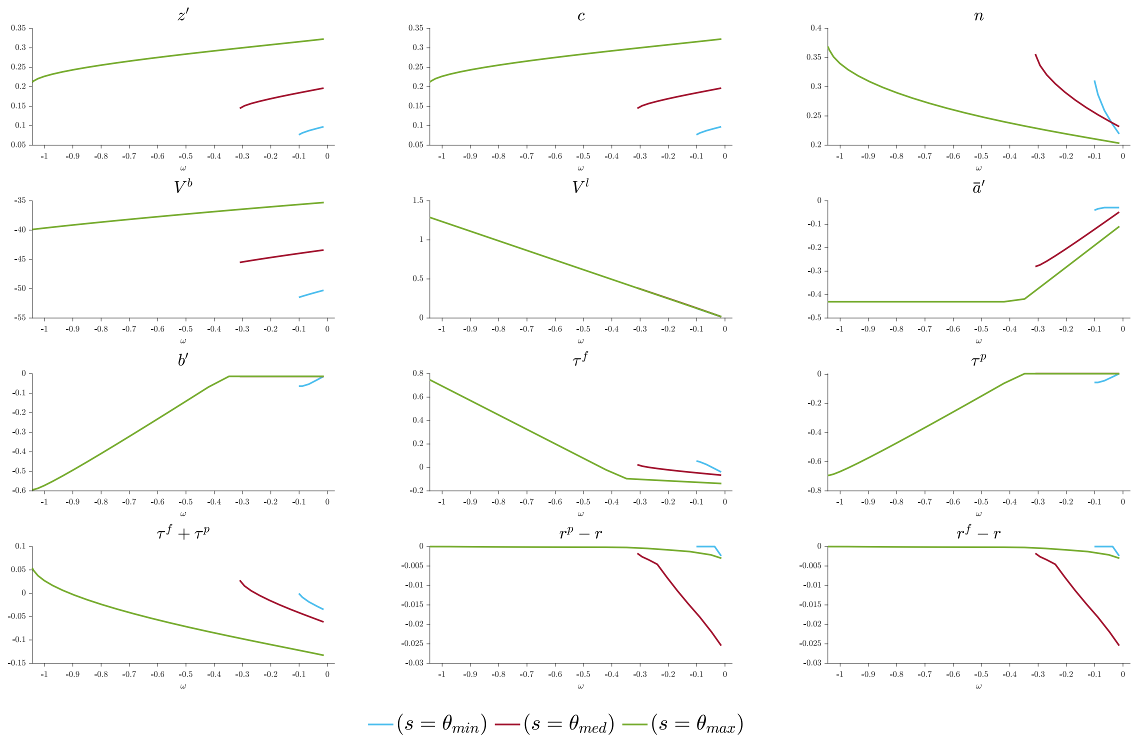


Figure C2: Optimal Policies for Different Levels of Private Debt as Function of (s, ω)

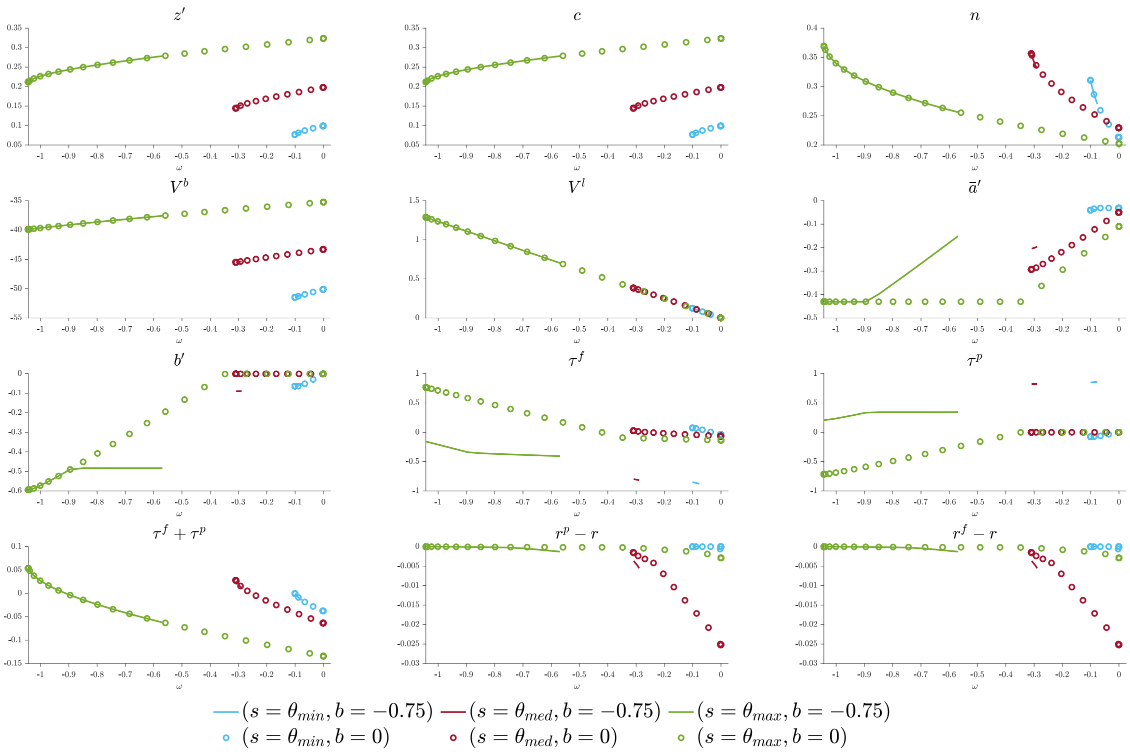
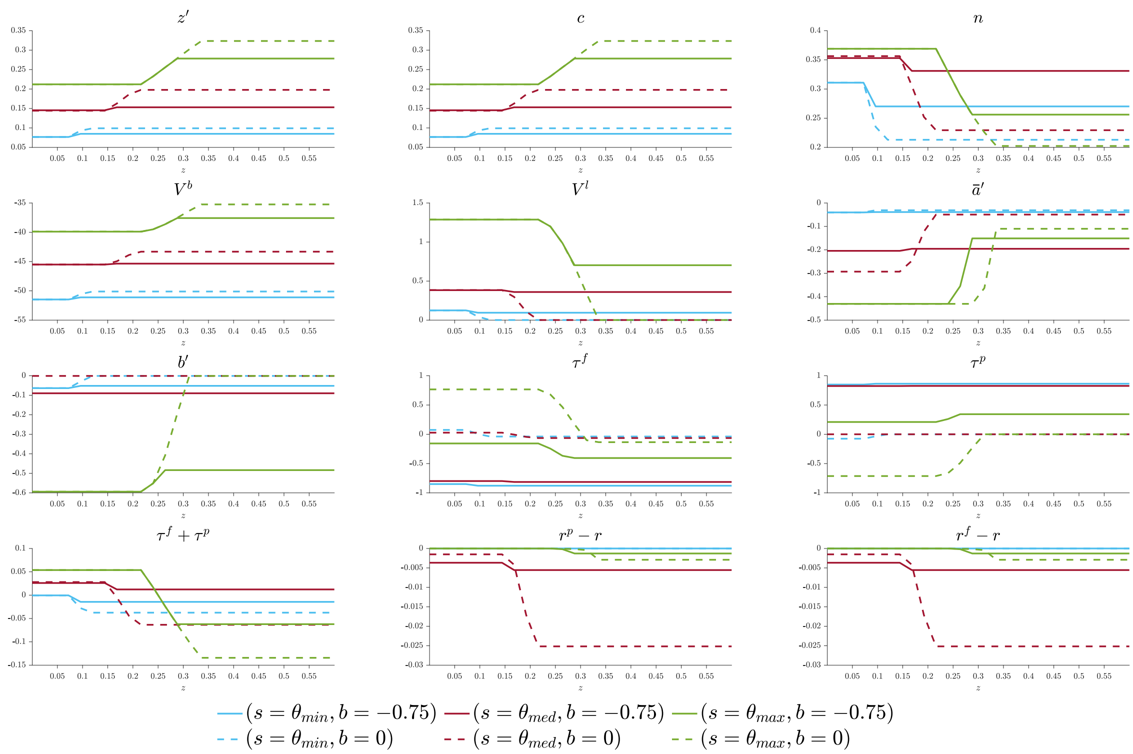


Figure C3: Optimal Policies for Different Levels of Private Debt as Function of (s, z)



Note: write here.

Figure C4: Optimal Policies as Function of (s, b)

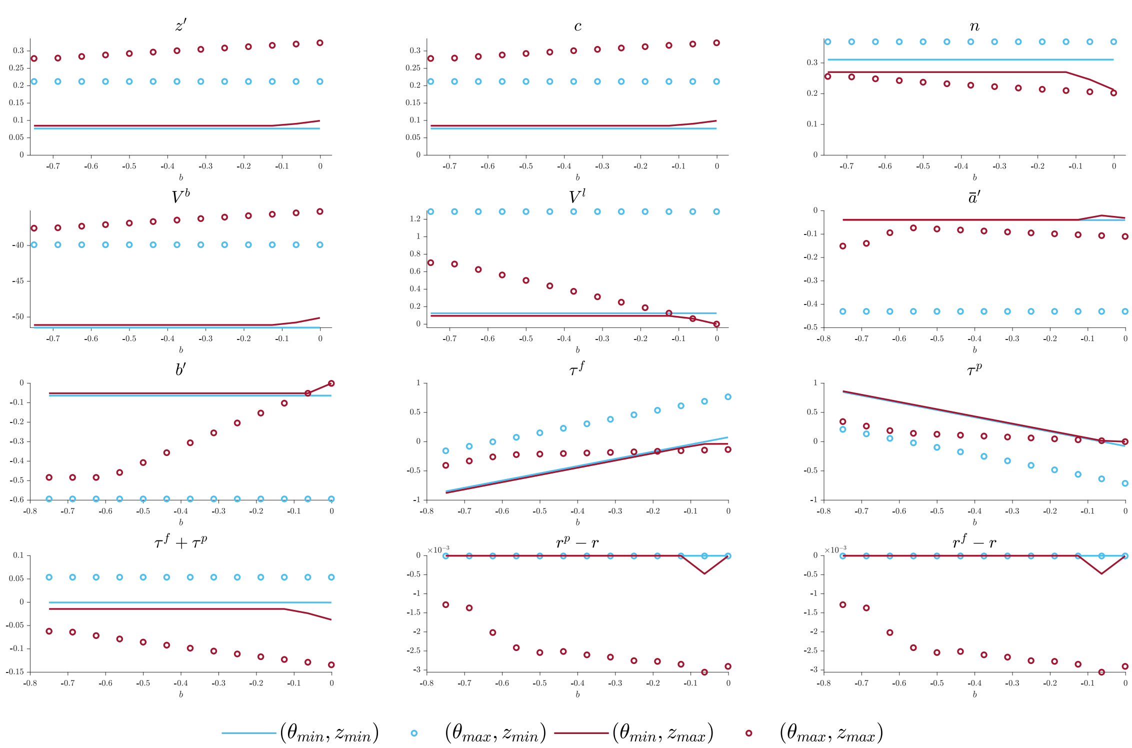


Figure C5: Default Set as a Function of (s, b)