Rational Expectations and General Equilibrium: A Theoretical Review*

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First version: November 27, 2011
This version: September 19, 2017

1 Introduction

This year is the fiftieth anniversary of the publication of Muth's (1961) seminal paper on rational expectations. Being almost neglected in its first decade, the profound perception and spectacular vision of this paper revolutionized the whole discipline of economics from the beginning of the second decade of its birth, and ever since. Given the central role played by expectations in all branches of economic dealing with dynamic problems, one can hardly imagine nowadays that how could a theory being built up, a hypothesis being tested and a meaningful insight being drawn, without explicit or implicit resorting to rational expectations—or, to the opposite, deviating from rational expectations while putting it as the benchmark.

However, to properly understand above assertions upon the influence of rational expectations, one has to get rid of the naive impression of thinking about rational expectations as simply positing the conditional expectations operator at several places of some equations in a model. Quite to the contrary, the profound implications of rational expectations can only be fully appreciated within a complete dynamic structure, though the structure itself could be as simple as the cobweb model initially employed by Muth. This structure, of course, is part of a more broader theory upon which the investigation of dynamic economic phenomena is based.

The core principle established by Muth, summarized in the term rational expectations, is drastically straightforward: Whatever are expectations hold by those forward-looking agents, must they

*This paper was originally submitted to the Department of Economics, SUNY at Stony Brook as the author's second year paper in the PhD program.

coincide with the endogenous outcome of the underlying theory; or in other words, nonetheless in a reverse direction, we economists who are contemplating a economic theory have to insist on rendering no more information to agents than what they would actually possess were they to conduct actions in a situation we are modeling. Essentially, this calls for an equilibrium approach, out which the prediction of the relevant theory can be endogenous.

Carrying out any analysis under this principle would effectively require a framework describing uncertainty and information. Muth's (1961) example illustrated how this could be down by appealing a stochastic framework, and further showed what was the implication of the principle he set forth in the same example. More generally however, this grand proposal can hardly have any impact in economics had it not been incorporated into the theory of general equilibrium. The combination of the brilliant idea and the powerful workhorse has prevailed in economics ever since its first appear in the beginning of 1970s.

Two pathbreak works, Radner (1972) and Lucas (1972), kicked off a new era of economics. Casting in a standard Arrow-Debreu type general equilibrium framework, Radner (1972) demonstrates how, by resuming perfect foresight approach, an essentially equivalence of rational expectations, can be utilized to suspend with time zero complete markets framework with an genuine sequential markets framework. Designed in a remarkably ingenious way, Lucas (1972) shows how, by resuming an extension of rational expectations to the case of differential information, can be used to construct a Phillips curve with neutrality results, which was then the central debate of macro policies. Notably, in the subsequent development, the roles played by Radner and Lucas just switched: Radner continued his early works (1966, 1972) to establish more general existence results of revealing equilibrium (1979), the same equilibrium concept used in Lucas (1972); whereas Lucas went on to advocate the equilibrium business cycle approach, which explicitly based on the perfect foresight/rational expectations approach.

We believe it is now appropriate to track the path of the evolution of rational expectations in economics, and to clarify different equilibrium concepts built on rational expectations. Ultimately we hope this survey serves the objective of better understanding and appreciating the intertwining connection between rational expectations and general equilibrium theory, which in our opinion, is definitely one of few nexus central to most economic analyses nowadays. In doing this, we will not confine to a particular field in which rational expectations plays a role, which is the typical form of surveys on rational expectations, but try to give an integrated review of related works, including both micro- and macro-economics. However, we do posit our discussion upon a unified platform, namely general equilibrium theory, as we should argue in much more detail below that the notion of rational expectations is deeply connected with, or to say actually grounded upon, the of equilibrium in economics.

In adopting a unified perspective, it is necessary to have a unified framework depicting uncertainty and information structure underlying our discussion. We present this framework in section 2, probably being detailed more than we need. Nonetheless, the basic idea is to encompass as many cases as possible which have been used in economics, especially in macroeconomics, in a unified framework, in order to demonstrate that our discussion in what follows does not restrain to unnecessarily special cases.

Section 3 is devoted to a detailed discussion about the early history of rational expectations, illustrating the historic background around which rational expectations was proposed. Next we discuss an alternative theory of expectations, implicity expectations, proposed by Mills about the same time as Muth's rational expectations theory, to highlight the tension then confronting empirical analysts and the reactions from them. It also serves the purpose of better appreciating the breakthrough content of rational expectations. Lastly, we discuss extensively on Muth's own expressions of rational expectations.

The main body of this survey is in section 4. We start by reviewing the equilibrium notion itself, based upon discussion of Hahn and Radner, to facilitate uncovering the logical connection between rational expectations and equilibrium theory. In next subsection, we review briefly the works in temporary equilibrium, which stands as a contrast of the rational expectations approach within the same general equilibrium theory. The following subsection presents a detailed discussion of what we classify as the perfect foresight equilibrium, with rational expectations equilibrium as one special example. A crucial difference between perfect foresight approach, with a more broadly used approach called fulfilled expectations, is emphasized at the end of the subsection. In the last subsection, we present even more extensive exposition of revealing equilibrium, which has been named as rational equilibrium in micro general equilibrium theory following Radner (1979). The informational efficiency of this type of equilibrium is discussed in some more detail, since it is the distinct feature of the revealing equilibrium and directly links to the efficient market hypothesis well-known in finance. A comparatively comprehensive survey of results in partially revealing equilibrium is presented at the end of the subsection, as few such surveys are available in the literature

2 Frameworks of Uncertainty and Information

To facilitate our discussion of the literature, a general framework of uncertainty and information in an *economy* is described in this section, and all notations and assumptions are fixed throughout this paper.[1]

[1] All the measure theoretic and probabilistic concepts and notations adopted in what follows are standard. See, for example, Doob (1994) and Durrett (2010, Ch.1 and Appendix).

The economy is populated by a set A of agents, e.g. consumers and/or producers etc., with a denoting a generic agent. The number of agents in the economy could either be finite or continuum. In the former case, for simplicity we shall write $A = \{1, \ldots, a, \ldots, A\}$, where A denotes the number of agents as well (there should be no confusion of the meaning of A according to the context); while in the later case, A is supposed to be an atomless measure space specified by the triple $(A, \mathcal{A}, \lambda)$, for example $A = [0, 1]$ and $\mathcal{A}$ and λ are the properly chosen σ-field and the associated measure respectively.

In this economy, time is discrete, specified by the index set $\mathcal{T} = \{1, \ldots, \mathcal{T}\}$ where $\mathcal{T}$ denotes the both the horizon, which could either be finite or infinite, and the set (again, there should be no confusion according to the context).

Suppose all (underlying) states of the world at time $t \in \mathcal{T}$ are completely contained in a set S_t, called the *state space* at time t. Denote $s_t \in S_t$ a generic element of S_t, which could also be called the *shock* to the economy. Define $S = \prod_{t\in\mathcal{T}} S_t$, called the *overall state space*, with a generic element $s = \{s_t\}_{t\in\mathcal{T}}$, which is called a *state* of the economy, prescribing a complete *history* or *path* of the economy from the very beginning to the very ending. Note S summarizes the *uncertainty* in the economy fully, and since S is the product of S_t, no particular path s is excluded a priori. Also, call $s^t = \{s_1, \ldots, s_t\}$ a *partial history* up through time t, or in Radner's (1982) words, an *elementary event* at time t, and denote $S^t = \prod_{\tau\leq t} S_t$ the set of elementary events at time t. In addition, define $s_t^{t+T} \equiv \{s_{t+1}, \ldots, s_{t+T}\} \in S_t^{t+T} \equiv \prod_{1\leq\tau\leq T} S_{t+\tau}$ for $T \geq 0$ with $s_t^t \equiv s^t$ and $S_t^t \equiv S^t$. For mathematical preciseness, we assume S_t to be a *complete separable metric space* (Polish space henceforth), thus S is a Polish space as well. Let $\mathcal{S}_t$ denote the σ-field of Borel sets on S_t, and correspondingly let $\mathcal{S} = \otimes_{t\in\mathcal{T}}\mathcal{S}_t$ denote the usual product Borel σ-field on S. Moreover, let $\mathcal{S}^t = \otimes_{\tau\leq t}\mathcal{S}_\tau$ and any set $E \in \mathcal{S}^t$ is called an *event* at time t. As a result, we have a natural filtration $\{\mathcal{S}^t\}_{t\in\mathcal{T}}$ with $\mathcal{S}^t \subset \mathcal{S}^{t+1}$, i.e. $\mathcal{S}^{t+1}$ is a refinement of $\mathcal{S}^t$. Implicitly, each $\mathcal{S}^t$ is identified as a sub σ-field of $\mathcal{S}$ by the inclusion map $A \mapsto A \times \prod_{\tau\geq 1} S_{t+\tau}$ for all $A \in \mathcal{S}^t$. At numerous occasion, it would be desirable to have a measure m defined for sets in $\mathcal{S}$ (as in the discussion of section 4), then the uncertainty in the economy is fully summarized by the measure space $(S, \mathcal{S}, m)$.[2]

Note all these specifications are just a generalization of the (finite) time-event tree model,

[2]As a caveat, the measure m would have nothing to do with any probabilities (beliefs) on $(S, \mathcal{S})$ to be introduced below, except a mild consistency condition stating that if $m(B) = 0$ for some $B \in \mathcal{S}$, i.e. B is negligible in $\mathcal{S}$, then any probability measure to be defined on $(S, \mathcal{S})$ should put zero probability for B, in the case that $|S|$ is continuum. Here m serves merely as a measurement of the amount of states of the world. To fix idea, one example of $(S, \mathcal{S}, m)$ goes as follows: For all t, $S_t \equiv \mathbb{R}^k$ for some (large enough) integer k, $\mathcal{S}_t$ is the Borel field on $\mathbb{R}^k$ equipped with Lebesgue measure m_t; then $S = \mathbb{R}^{k\times\{1,2,\ldots\}}$, $\mathcal{S}$ is the product field, and m is product measure $\otimes_{t\geq 1} m_t$. In addition, according to the established convention above, the measure m^t on S^t defined as $\otimes_{1\leq\tau\leq t} m_t$.

a device presented first in Debreu (1959, Ch. 7).[3] Yet this framework is motivated by eventually accommodating general theory of probability with the analysis of dynamic stochastic general equilibrium models or any other stochastic modeling of economic behavior.

For each agent $a \in A$, let a's *information* at time t be defined as a sub σ-field of $\mathcal{I}_t^a \subset \mathcal{S}_t$. Correspondingly, we could define a's *information structure* $\mathcal{I}^a$ as $\otimes_{t\in\mathcal{T}}\mathcal{I}_t^a$ and its historic information $\mathcal{I}^{at}$ as $\otimes_{\tau\leq t}\mathcal{I}_\tau^a$. Again, $\{\mathcal{I}^{at}\}_{t\in\mathcal{T}}$ represents a's filtration. If $|S_t|$ (the cardinal number) is finite, then a's information at time t is necessarily a partition of S_t, and a is unable to distinguish between any two states belonging to the same set in the partition. Different agent may have different information structure, so that asymmetric information is possible in this general framework. Further denote $\mathcal{P}_t \equiv \bigvee_{a\in A}\mathcal{I}_t^a$ the coarsest common refinement of agents' information,i.e. the *pooled information*, at time t; and $\mathcal{C}_t \equiv \bigwedge_{a\in A}\mathcal{I}_t^a$ the finest common coarsening of agents' information, i.e. the *common information*, at time t. Likewise, the overall pooled information $\mathcal{P}$ of the economy is the product σ-field $\otimes_{t\in\mathcal{T}}\mathcal{P}_t$, and the overall common information $\mathcal{C}$ is $\otimes_{t\in\mathcal{T}}\mathcal{C}_t$, in together with filtrations $\{\mathcal{P}^t\}_{t\in\mathcal{T}}$ and $\{\mathcal{C}^t\}_{t\in\mathcal{T}}$ defined in the obvious way as well. We shall refer to the tuple $\langle\mathcal{S}, (\mathcal{I}^a)_{a\in A}\rangle$ the *information structure* in the economy. Also note that, under this formulation, since all *events* are defined with respect to a filtration, it follows that all agents have *perfect memory*, or *perfect recall*, of all past states.

We prefer to interpret the uncertainty and information structure presented above as *exogenous*, or *action irrelevant*, that is they are unaffected by the agents' actions and the resulting endogenous variables, like prices, etc. In addition, we assume the resolution of time t uncertainty takes place at the beginning of the period, so that every agent a knows to which sets in $\mathcal{I}^{at}$ the true state of the world belongs. However, in certain case, notably in temporary equilibrium theory to be discussed in Section 4, some endogenous variables will also enter the state space of the world.[4] However, we focus on the former case in the remaining of this section.

One advantage of assuming exogeneity of uncertainty is that a stochastic process $\{X_t\}_{t\in\mathcal{T}}$ can readily be used to take into account both uncertainty and information structure. Formally, for each $t \in \mathcal{T}$, X_t is defined on a probability space $(\Omega, \mathcal{F}, \mu)$ with $X_t(\Omega) = S_t$, i.e. S_t is the image of X_t. A mild consistency condition is that when S_t is continuum, then $\mu(X_t \in B) = 0$ for $B \in \mathcal{S}_t$

[3] One mild difference between this general formulation of the time-event tree and the typical finite version used in general equilibrium literature is that, in the later case, time $t+1$ successors of different time t nodes may well be different, i.e. certain elementary events are excluded in the model a priori. To encompass this situation in our formulation, it amounts to either redefine the set of elementary events as $F^{t+1} = \{s^{t+1} \in S^{t+1} \mid s_{t+1} \in \varphi_t(s_t),\ s_t \in F^t\}$ recursively with $\varphi_t : F_t \to \mathcal{S}_{t+1}$ for all $t \in \mathcal{T}$ and $F_1 \in \mathcal{S}_1$, or assume $\mathbb{P}$ and $\mathbb{Q}^a$ (defined below) put zero measure on those precluded states $s \in S$.

[4] Hirshleifer and Riley (1979) adopts the term "event uncertainty" for exogenous uncertainty, like technological uncertainty, and preserve "market uncertainty" for the endogenous one, including, among other things, the search behavior in the market process.

and $m_t(B) = 0$. Since a Polish space could well be infinitely dimensional, we assume each X_t can be written as $(X_t^0, (X_t^a)_{a\in A})$, in which X_t^0 represents the aggregate shock (or signal) while X_t^a represents the idiosyncratic shock (or signal) to agent a. In taking this form, we are actually assuming that $s_t = (s_t^0, (s_t^a)_{a\in A})$. We do not exclude the possibility that two agents receive the same shock, i.e. $X_t^a = X_t^b$ for $a \neq b$. In this case, one common component X_t^α where $\alpha = \{a, b\}$ replaces X_t^a and X_t^b in X_t. Consequently, the information structure in the current specification can be defined in a more meaningful way, relative to the explicitly given σ-field above, by letting $\mathcal{I}_t^a = \sigma(X_t^0, X_t^a) \subset \sigma(X_t)$, where $\sigma(X)$ denotes the sub σ-field generated by X. Note that all the probabilistic properties, for example, the stationarity of $\{X_t^0\}$ or the Markovian property over t of $\{X_t^a\}$ for each a and mutually independence of $\{X^a\}$ across a, to name just a few, are simultaneously determined by the measure μ over Ω; and it is assumed that this μ is unaffected by the agents, therefore one can interpret this measure to be the *objective* measure. Moreover, the special case of *homogeneous information* is represented by the condition that $\mathcal{I}_t^a = \mathcal{I}_t^b$ for all $a,\ b \in A$ and $t \in \mathcal{T}$. This is in turn equivalent to $\sigma(X_t^a) = \sigma(X_t^b)$. Since X_t^a is only a shortcut indicating for each agent a the (private) state, we could conveniently assume $X_t^a = X_t^b$ m-a.s. if $\sigma(X_t^a) = \sigma(X_t^b)$, or in plain words, identical information means identical shock. Therefore, under homogeneous information, only aggregate shocks matter, and $X_t = X_t^0$.

One remark is in order. On one hand, when $|A|$ is finite or countably infinite, a stochastic process $\{X_t\}$ with *any* given distribution can always be constructed, that is there exists a probability space $(\Omega, \mathcal{F}, \mu)$ such that μ gives the desired distribution. On the other hand, when A is continuum, for example $A = [0, 1]$, a typical index space used in the literature, then it is desirable to have the process $Y(a, \varpi) \equiv X^a(\varpi)$, where $X^a(\varpi) = (X_t^a(\varpi)_{t\in\mathcal{T}})$ is defined on some probability space $(\Upsilon, \mathcal{G}, \nu)$ for each a, to be *jointly measurable* on $A \times \Upsilon$ with respect to the product σ-field $\mathcal{A} \otimes \mathcal{G}$, which implies for ν-almost $\varpi \in \Upsilon$, $Y(a, \varpi)$ is $\mathcal{A}$-measurable. This property is particularly useful in transferring longitudinal distribution to cross-section distribution, especially in the case that each agent faces i.i.d. shocks, i.e. $Y(a, \cdot)$ is independent across a with identical distribution, where the induced distribution over A is exactly the same as the distribution for each a. Models studied in Bewley (1986), Huggett (1993), Aiyagari (1994), and Krusell and Smith (1998) all share this feature. However, as pointed out by Judd (1985) and Feldman and Gilles (1985), it's impossible to construct a nontrivial process $Y(\cdot, \cdot)$ measurable with respect to the product σ-field when A is the unit interval and $\mathcal{A}$ is the Lebesgue σ-field. However, recently this deficiency has been completely resolved by Sun (2006, 1998) and Sun and Zhang (2009) by enriching the Lebesgue unit interval based on some non-standard analysis techniques. Moreover, as pointed out in Hammond and Sun (2003, 2008), it's still possible to get the similar (but weaker) results via a "finite approximation" as in the usual law of large number, without using the enriched space. Based on these results, we simply assume that when A is continuum, an appropriate probability space

$(\Omega, \mathcal{F}, \mu)$ exists with $\Omega = A \times \Upsilon$, on which the process $\{X^a(\omega)\}$ is defined,[5] is $\mathcal{A}$-measurable and has the desired distribution.[6]

In the preceding framework, the very nature of uncertainty in the economy, especially the temporal unfolding of the future uncertainty, is actually absorbed into the abstract probability space $(\Omega, \mathcal{F}, \mu)$. Despite providing a powerful analytic framework, it brings in no less problems than it helps resolve. Among other things, one compelling question arises: Are we postulating too much uncertainty by rendering a possibly too "big" *sample space* Ω? As made obvious by measure theory, by enlarging the underlying probability space, all kinds of uncertainty — represented by arbitrarily distributed random variables — could be readily included into the probabilistic framework, yet having nothing to do with any perceivable uncertainty in the working economy. In other words, purely extrinsic uncertainty which could have nothing to do with the fundamentals of the economy, like "sunspot" introduced by Cass and Shell (1983), can be easily incorporated into the probabilistic foundation of a model either consciously or unconsciously. And perhaps it is the latter situation that has ever since caused both intriguing and controversial conceptual problems.

In this regard, one advantage of the basic time-event tree framework is its explicitness of perceived uncertainty in the economy. Thus it is desirable to consider a combination of the two frameworks above. Formally, we shall take use of a technique, made well-known by Doob (1953, Ch.1), by identifying the stochastic process $\{X_t\}$, involving all the uncertainty and no more, with its *sample path* $s = \{s_t \in S_t \mid s_t = X_t(\omega), \forall\, t \in \mathcal{T}\}$ for any given ω. Note this relation actually defines a transformation $T : \Omega \to S$. In this way, we can go one step back by adopting the state space S as the sample space of the new probability space $(S, \mathcal{S}, \mathbb{P})$, where $\mathbb{P}$ is induced by the probability measure μ. More precisely, the process $\{X_t\}$ is transformed into a new process $\{\tilde{X}_t\}$ via $\tilde{X}_t(s) \equiv s_t = X_t(T^{-1}s)$, and $\mathbb{P}$ is induced by $\mathbb{P}(\Lambda) = \mu(T^{-1}\Lambda)$ for $\Lambda \in \mathcal{S}$. As an implication of the consistency condition for μ, $\mathbb{P}$ is absolutely continuous with respect to m when $|S|$ is continuum.

In what follows, we suspend the tilde on head of the process $\{\tilde{X}_t\}$ defined on $(S, \mathcal{S}, \mathbb{P})$. It then follows that the probabilistic properties of this newly defined $\{X_t\}$ is completely determined by $\mathbb{P}$, and the relevant information structure is still given by $\mathcal{I}_t^a = \sigma(X_t^0, X_t^a)$, yet now $\mathcal{I}_t^a \subset \mathcal{S}_t$ as in the first specification. One note that defined in above way, $\mathbb{P}$ inherits the objective nature of measure μ. In the case of homogeneous information, all previous discussions still apply, i.e. $X_t = X_t^0$ and $\mathcal{I}_t = \mathcal{I}_t^a$ for all a. In addition, since no one in the economy knows any event beyond those

[5]This is a slight abuse of notation, as in fact we have $\omega = (a, \varpi)$. However, for coherence, we still write $X^a(\omega)$ instead of $X^a(\varpi)$, identifying that a plays no role in the parenthesis.

[6]Apart from the measurability issue, it is also noticed that when $|A|$ is continuum, a Polish space S_t fails to be a proper range space for defining $\omega \mapsto \{X^a(\omega)\}_{a \in A}$. A simple counter example is $\mathbb{R}^{[0,1]}$, which effectively equals to the space of all functions from $[0, 1]$ to $\mathbb{R}$ and fails to be a Polish space.

contained in $\mathcal{I}$, effectively we can assume $\mathcal{I} = \mathcal{S}$, redefine S if necessary. To incorporate the Bayesian point of view, one could assume explicitly that agent a holds a prior belief, or subjective probability, defined by a probability measure $\mathbb{Q}^a$ on $(S, \mathcal{S})$. In this way, the stochastic process $\{X_t\}$ remains the same, for the sample paths are unchanged when $\mathbb{P}$ is replaced by $\mathbb{Q}^a$, so does the information structure $\{\mathcal{I}^a\}$. However the entire probabilistic properties of $\{X_t\}$ are determined by the associated measure, hence completely different under $\mathbb{Q}^a$ versus under $\mathbb{P}$. Still, the consistency condition applies that Q^a is absolutely continuous with respect to m when $|S|$ is continuum.

In summary, we have three frameworks: Time-event tree model, fundamental probability model, and sample-path probability model.[7] The time-event tree model was first employed in studies of general equilibrium theory and game theory in 1950s and 1960s.[8] Notably, modeled in this way, uncertainty and information have no explicit connection with probability, neither objective nor subjective. However, as in contrast, subsequent development of economic theory greatly advocates and explicitly employs formal probabilistic specifications.

3 Modeling Expectations in the "Early Days"

As early as 1930s, economists have identified the central role played by *expectations*, especially in thinking about the dynamic aspects of the economy. This point is made evident in Hicks (1946 [1939], Ch. IX, pp. 116–117):

> A rise in the price of a commodity exercises, at once, only a small influence upon

[7]To clarify the distinction among these three approaches, consider the following example: Let $A = \{a, b\}$, $\mathcal{T} = 1$, and $S = \{s_1, s_2, s_3\}$. There is no aggregate shock (signal), and the three states are $s_1 = (\mathfrak{f}, \alpha)$, $s_2 = (\mathfrak{f}, \beta)$ and $s_3 = (\mathfrak{g}, \gamma)$, i.e. a receives shock $\mathfrak{f}$ in both state 1 and 2, and $\mathfrak{g}$ in state 3, whereas b receives shock α, β and γ in state 1, 2 and 3 respectively. Naturally $\mathcal{S} = 2^S$, $\mathcal{I}^a$ is generated by $\{s_1, s_2\}$ and $\{s_3\}$, and $\mathcal{I}^b = \mathcal{S}$. Suppose the underlying probability space $(\Omega, \mathcal{F}, \mu)$ consists of the unite interval $(0, 1]$ endowed with the Lebesgue σ-field and the Lebesgue measure. Define $X : \Omega \to S$ as $X(\omega) = s_1$ for $\omega \in (1, 1/3]$, $X(\omega) = s_2$ for $\omega \in (1/3, 2/3]$ and $X(\omega) = s_3$ for $\omega \in (2/3, 1]$. Then s_i occurs with probability $1/3$ for each i. Thus μ induces $\mathbb{P}$ over S, and we have exactly $\mathcal{I}^a = \sigma(\tilde{X}^a)$, where $\tilde{X}^a(s_i) = s_i^a$. By enlarging Ω to the unit square $(0, 1] \times (0, 1]$, one could either consciously or unconsciously introduce extrinsic uncertainty represented by an extra random variable Y depending on the second coordinate of the unit square while taking values in S in the same way as X, i.e. $Y(\cdot, \omega_2) = s_1$ for $\omega_2 \in (0, 1/3]$; in contrast X only depends on the first. In this way, Y is independent of X, and the probability of observing a state, to say s_1, changes from $1/3$ to $5/9$.

[8]The basic time-event tree model, the one without information heterogeneity, was first fully presented in Debreu (1959, Ch. 7), while the notion of "state contingency" was introduced by Arrow (1953), which in turn was rooted in Hicks (1946 [1939]), see Arrow (1978, pp. 2–3). The information heterogeneity was first proposed and studied in Radner (1968). However, a general characterization of information structure has already emerged in the studies of extensive game, see for example Kuhn (1953). All these early contributions assume finite states and use partition to represent information.

> the supply of that commodity; but it sets entrepreneurs guessing whether the higher price will continue. If they decide that it probably will continue, they may start upon the production of a considerably increased supply for a future date. This decision will affect their current demand for factors . . . Similarly, the current supply of a commodity depends not so much upon what the current price is as upon what entrepreneurs have expected it to be in the past . . . The actual current price has a relatively small influence. [And] this is the first main crux of dynamic theory.

However, in the "early days" — 1950s and beginning of 1960s — of modeling expectations in a fashion of modern economic theory, the prevailing perspective was still the one set forth by Keynes (1936, p. 148):

> It would be foolish in forming our expectations, to attach great weight to matters which are very uncertain. It is reasonable, therefore, to be guided to a considerable degree by the facts about which we feel somewhat confident, even though they may be less decisively relevant to the issue than other facts about which our knowledge is vague and scanty. For this reason the facts of the existing situation enter, in a sense disproportionately, into the formation of our long-term expectations; our usual practice being to take the existing situation and to project it into the future, modified only to the extent that we have more or less definite reasons for expecting a change.[9]

Just got out of one longest uncertain period — the Great Depression, in which past experiences of economic life seemed to be not applicable any more, in plus the subsequent turmoil of World War II — in the twentieth century, it was hardly conceivable of modeling expectations of individuals with any confidence put in anticipating the possible uncertain future.

The then prevailing framework of modeling expectations is *adaptive expectations* model exemplified by the works of Cagan (1956) and Nerlove (1958). This type of model is motivated directly by a need of formulating expectations of agents, including but not limited, either consumer's expectation of next year's price level, or entrepreneur's expectation of next quarter's demand, to complete the description of equations in a econometric model aiming to explain observed data. In other words, it was largely those researches doing empirical work who were in need of creating a model of expectation formation. In light of Keynes' point of view, the natural hypothesis of expectation formation is the adaptive one.

However, the ad hoc nature of adaptive expectation, in light of our framework of uncertainty in the preceding section, can be easily illustrated. Consider the specification used in Lovell (1961):

[9] Quoted from Magill and Quinzii (1996, p. 21).

Let X_t be the level of sales at time t and $\hat{X}_t$ be the expected level at time $t-1$. Then the adaptive expectation hypothesis is stated as follows:

$$\hat{X}_t = \rho X_{t-1} + (1-\rho)X_t, \qquad 0 \leq \rho \leq 1.$$

This is equation (3.5) in Lovell (1961). The justification of this hypothesis goes like this: "If the firm's adjustment of the simple, naive projection based on definite information is in the right direction, the level of sales actually expected would fall between the two extremes of static and perfect forecasting." (Lovell, 1961, p. 305.)[10] The essential problem for this specification is that, when an entrepreneur needs to form the expectation at time $t-1$, X_t is *unknown* unless in a completely deterministic environment, since X_t depends on time t information which is simply not available to the entrepreneur. Yet this poses no problem to the econometrician, who is looking at all data as historic events.

At the end of 1950s, at least two people came up with alternative models about expectation formation. One is Edwin S. Mills (1957a, 1957b), and the other one is John F. Muth (1960, 1961).[11] Lovell (1986) calls Mills' approach as *implicit expectations* in order to make a distinction from Muth's *rational expectations* approach. Both approaches are motivated by investigating inventory management problems confronting firms,[12] and both had an equation $X_t = \hat{X}_t + u_t$ describing the *expectation error* (Mills, 1957a Eq. 2.1, 1957b Eq. 2; Muth, 1961 Eq. 3.2), where u_t represents the random error with mean zero which will be known at time t only. Hence both introduce explicitly the stochastic nature into the expectation formation. However a simple but vital distinction dooms the implicit expectations approach to be less appealing from both theoretic and empirical aspects. This vital distinction can be illustrated as follows.

Consider first Mill's approach. For a given behavior equation

$$Z_t = \cdots + a\hat{X}_t$$

including $\hat{X}_t$ as one of the explanatory variable, where all other explanatory variables are collected in $\cdots$, Mills approach makes use of $\hat{X}_t = X_t - u_t$ to substitute out $\hat{X}_t$, which yields

$$Z_t = \cdots + aX_t + v_t$$

[10] Lovell also cited Keynes (1936, p. 51): "...it is sensible for producers to base their expectations on the assumption that the most recently realized results will continue except in so far as there are definite reasons for expecting a change."

[11] See Young and Darity (2001) for an excellent historic account of the early history of rational expectations and its impacts on other branches of economics well before the 1970's rational expectations revolution.

[12] This is made clear for Mills' approach directly from the titles of his paper. For the other one, as noted in Young and Darity (2001), Muth was then a junior faculty at GSIA, Carnegie Institute of Technology, participating the project "Planning and Control of Industrial Operations", in together with C. Holt, F. Modigliani and H. Simon, of which the results were summarized in Holt et al. (1960). It was during this project, Muth's idea of rational expectations was developed and formalized.

with $v_t = -au_t$. Then run the regression of Z_t using above equation, and it follows that the consistency estimator $\hat{a}$ effectively requires the underlying assumption of (conditional) *uncorrelatedness* of X_t and u_t. However, as our discussion in section 2 indicates, in general X_t hinges on the information at time t which is provided by u_t in this case,[13] thus generally it's impossible to have $\mathbb{E}(X_t u_t) = 0$.

On the contrary, in Muth's approach, the maintained assumption is the expectation $\hat{X}_t$ being always equal to the expectation conditional on time $t-1$ information, i.e. $\hat{X}_t = \mathbb{E}_{t-1}X_t$.[14] Therefore, $u_t = X_t - \mathbb{E}_{t-1}X_t$ and $\mathbb{E}(u_t \mathbb{E}_{t-1}X_t) = 0$ always hold. Indeed, this follows the standard property of conditional expectation, see Shiller (1978, Sec. 2). So in rational expectations approach, the expectation error is required to be uncorrelated with the expectation rather than with the *actual* outcome as in Mills' approach.

To better appreciate Muth's approach, it is valuable to recall two "conclusions from studies of expectations data" that motivated Muth's work (Muth, 1961, p. 316):

> 1. Averages of expectations in an industry are more accurate than *naive* models and as accurate as elaborate equation systems, although there are considerable cross-sectional differences of opinion.
>
> 2. Reported expectations generally underestimate the extent of changes that actually take place. [Our emphasis.]

The first conclusion suggests that on average, expectations held by agents (at time $t-1$) are more accurate than the *naive* models of expectations, i.e. adaptive expectations alike, would suggest. In other words, the actual expectations held by agents should have an somewhat "optimal" property. Thus one is led naturally to assume $\hat{X}_t = \mathbb{E}_{t-1}X_t$, since it is well known that $\mathbb{E}_{t-1}X_t$ has the minimum *mean square error* among all forecast of X_t based on time $t-1$ information. That is, $\mathbb{E}_{t-1}X_t$ solves the minimization problem $\min \mathbb{E}(X_t - Y_{t-1})^2$ subject to Y_{t-1} being $\mathcal{S}^{t-1}$-measurable. The second conclusion suggests that the variance of the actual observation should be greater than the expected value, that is $\mathit{var}\, X_t > \mathit{var}\, \hat{X}_t$. This is readily verifiable if $\hat{X}_t = \mathbb{E}_{t-1}X_t$, since it implies $\mathit{var}\, X_t = \mathit{var}\, \hat{X}_t + \mathit{var}\, u_t$ where u_t is the expectation error.

[13] Since $\hat{X}_t$ is assumed to be the expectation formed at time $t-1$, it necessarily is determined by information at that time. Mathematically, this means $\hat{X}_t$ is $\mathcal{S}^{t-1}$-measurable. Yet the equation $X_t = \hat{X}_t + u_t$ dictates that X_t also contains information at t, which is imparted by u_t.

[14] In the context of the probabilistic framework presented in section 2, $\mathbb{E}_{t-1}X_t$ is defined as $E(X_t|\mathcal{S}^{t-1})$, i.e. the expectation conditional on all information available to the agent (here a representative agent is implicitly assumed). Since the exogenous disturbance v_t is the only source of uncertainty (another implicit assumption), we have $\mathcal{S}^{t-1} = \sigma(v_\tau, \tau \le t-1)$, the conditional expectation equals to $\mathbb{E}(X_t|v_\tau, \tau \le t-1) \equiv \mathbb{E}(X_t|\sigma(v_\tau, \tau \le t-1))$, which is denoted by $\mathbb{E}_{t-1}X_t$ for simplicity.

Muth (1961, p. 316) goes on stating the characteristic principle of the rational expectations approach:

> I should like to suggest that expectations, since they are *informed predictions* of future events, are essentially the same as the *predictions of the relevant economic theory* ... We call such expectations "rational." ... The [rational expectations] hypothesis can be rephrased a little more precisely as follows: That expectations of firms (or, more generally, the subjective probability distribution of outcomes) tend to be distributed, for the same *information set*, about the prediction of the theory (or the "objective" probability distributions of outcomes). [Our emphasis.]

The implications of "predictions of the relevant economic theory" are twofold: First, a formal adoption of the stochastic model implies that any prediction of the economic theory should be the (mathematical) expectation conditional on the known information set, so that informed predictions of agents can be accounted meaningfully in the theory; second, whatever the (unobserved) expectations are, they should be compatible with the structure governing the stochastic environment associated to the economic theory. In Muth's own words (1961, p. 316): "The hypothesis asserts ...: (1) Information is scarce, and the economic system generally does not waste it. (2) The way expectations are formed depends specifically on the structure of the relevant system describing the economy."

When the process $\{X_t\}$ is exogenously given, $\mathbb{E}_{t-1}X_t$ can be computed directly. However, an ingenious and essential contribution in Muth (1961) is its treatment of the expectations of the time t endogenous variables held by agents at time $t-1$. By assuming $\hat{X}_t = \mathbb{E}_{t-1}X_t$, the paper shows, through a simple example, how the whole process of $\{X_t\}$ is *endogenously determined* in a way that is compatible with the equations about the stochastic environment, which essentially characterize some *equilibrium* conditions of the model.

Although the few impacts of rational expectations upon economics were observed in the entire decade of 1960s (Young and Darity, 2001), Muth's paper has made it clear that two ingredients are essential "to make dynamic economic models complete" (Muth, 1961, p. 315): A formal stochastic environment, inherited from the statistics tradition, and structural relations built on the stochastic environment, to be contemplated from the equilibrium theory.

4 Expectations in Equilibrium Theory

4.1 The Notion of Equilibrium

Before going into details of particular equilibrium theory with expectations, it is worthwhile to discuss first the notion of equilibrium in economic theory a little bit. After all, introspection helps to clarify several subtle aspects of this central notion in economic theory.

In the inaugural lecture at Cambridge University, Hahn (1973, p. 47) stated "a very weak casual proposition" as the motivation of equilibrium notion:

> This is that no plausible sequence of economic states will terminate, if it does so at all, in a state which is not an equilibrium. The argument is straightforward; agents will not continue in actions in states in which preferred or more profitable ones are available to them nor will mutually inconsistent actions allow given prices to persist.

And after giving a more explicit specification of the environment, dating, uncertainty and information, etc., Hahn (1973, p. 59) went on to state a rather general definition of equilibrium

> An economy is in equilibrium when it generates messages which do not cause agents to change the theories which they hold or the policies which they pursue.

Here, agents' theories refer to the information structure they have, possibly including their prior beliefs, as those introduced in section 2, and their policies refer to the optimal decisions, given their information structure and the messages generated by the economy.

In a comprehensive survey of intertemporal equilibrium, Radner (1991, p. 438) decomposed Hahn's main theme into three conditions for equilibrium over time by citing Hicks' works: (i). The Hicks-Nash condition; (ii). fulfilled expectations; and (iii). market clearing. For the first condition, Radner quoted from Hicks (*Causality and Economics*, p. 45): "All opportunities for advantageous change that are presented within the model must be taken." The second condition refers to Hicks (1946 [1939], pp. 132–133):

> Equilibrium over time ... suggests itself when we start to compare the price-situations at any two dates ... The [equilibrium] condition [is] that the prices realized on the second Monday are the same as those which were previously *expected* to rule at that date ... remember the expectations of entrepreneur are in fact not precise expectations of particular prices, but partake more of the character of probability distribution, then it becomes evident that the realized prices can depart to some extent from those prices expected as most probable ...

However, an even subtler point in Hahn's definition is not elicited in Radner's summary. As Hahn (1973, pp. 60–61) put it:

> What is required is a frequency distribution of prices conditional on exogenous events which in some precise send corresponds closely enough with the prior conditional distributions held by agents . . . The traditional notion of an equilibrium which I described at the outset requires the equilibrium actions of agents to be consistent, whereas I have the weaker requirement that they not be *systematically* and *persistently* inconsistent. [Our emphasis.]

Evidently, the first part of Hahn's paragraph corresponds exactly to Muth's rephrased hypothesis of rational expectations, whereas the second part is typically used to justify the coincidence of individuals' forecasts and the equilibrium outcome.[15]

Notably, both Hicks-Nash condition and market clearing condition are well accounted for in the paradigm established by Arrow and Debreu (1954) and Debreu (1959) for Walrasian economy. However, the fulfilled expectation condition — or more general, the role of expectations — in a dynamic equilibrium simply can not be investigated within this paradigm, for the Arrow-Debreu model assumes once-for-all complete contingent commodity market.[16] In order to have it be possible to account for any kind of expectation, one must introduce the sequential markets.

4.2 Temporary Equilibrium

In the inceptive chapter initializing the analysis of intertemporal equilibrium,[17] Hicks (1946 [1939], Ch.IX, pp. 124–126) outlined explicitly a framework of how the decisions of agents hinge on the expectations of next period market condition in his famous Weekly economy:

> The plans which are adopted in any given week depend not only upon current prices

[15] This was originally used by Muth (1961, p. 318): "If the prediction of the theory were *substantially* better than the expectations of the firms, then there would be opportunities for the 'insider' to profit from the knowledge." [Our emphasis.] In an extraordinary review of rational expectations, Kantor (1979, p. 1424) related Muth's words directly to the optimal utilization of information: "Profitable opportunities to exploit available information will be exercised in a competitive world. Rational expectations are profit maximizing expectations." This is in accordance with the assertion of rational expectations hypothesis set forth by Muth himself.

[16] We acknowledge the contribution of L. McKenzie (1954, 1959), yet still choose to conform the convention by calling it Arrow-Debreu model. See Weintraub (2011) for a comprehensive retrospection of McKenzie's contribution to general equilibrium theory.

[17] According to Milgate (1979), four names are attributed to the initial development of intertemporal equilibrium: Lindahl, Myrdal, Hicks and Hayek. However, it is recognized that Hicks (1946 [1939], first edition published in 1939) set forth the first integrated investigation in this realm.

> but also upon the planner's expectations of future prices. We shall ... [assume] that every individual has a *definite* idea of what he expects any price which concerns him to be in any future week ... there will be a certain figure, or range of figures, which they consider most probable, but deviations from this most probable value on either side are ... possible ... When we are concerned with the determination of plans, we must suppose the expectations of the planners to be adjusted for risk ... Further ... the willingness to bear any particular risk ... will be appreciably affected by the riskness involved in the rest of the plan. [Our emphasis.]

All these considerations of expectations and riskness arising from an ambiguous role of uncertainty, formed the foundation of extending Marshall's temporary equilibrium, which was indeed a static equilibrium model, to Hick's temporary equilibrium with certain dynamic favor.

In the post war era, Stigum (1969, 1972) was the first to construct general equilibrium model in the fashion as the Arrow-Debreu paradigm, and built explicitly on Hicks' idea of a *definite*, or a *fixed*, expectation, defined as a function mapping current period market conditions to a distribution of next period price. This basic setup in fact departs from our basic framework of uncertainty in that now prices, which are endogenous variables to be determined as the market outcome, are regarded as part of the *state* of the world affecting agents payoffs. See, e.g. Jordan (1976) for a model in which the only source of uncertainty is the price in the second period.

The idea of temporary equilibrium has intrigued a large body of research papers in 1970s, with application in fields like monetary economics (addressing the problem that money has no positive value in a standard Walrasian equilibrium) and macroeconomics, especially in attempting to build a new foundation for Keynsian economics. In 1980s, a branch of this line of work transferred into building an endogenous cycle theory for short run economic fluctuation, e.g. Grandmont (1985). For an comprehensive survey of works in this area in 1970s and 1980s, see Grandmont (1977, 1991).

4.3 Perfect Foresight Equilibrium

Following the seminal contribution of Radner (1972), *perfect foresight equilibrium* with sequential markets has became a new paradigm in dynamic equilibrium theory.[18] In Radner's (1991, p. 438)

[18] Some authors in the literature also use the term *correct expectations equilibrium*, *Walrasian expectation equilibrium*, or simply *expectations equilibrium*, to label the equilibrium notion in Radner's paper, e.g. Magill and Quinzii (1996) and Glycopantis and Yannelis (2005). Others use directly *Radner equilibrium*, e.g. Mas-Colell et al. (1995) and Glycopantis and Yannelis (2005). In Radner (1982), the approach adopted is termed as *perfect foresight*, where the equilibrium is called plainly *equilibrium of plans and expectations*. We follow Radner's terminology for the approach by calling it perfect foresight equilibrium. The same term is used in Duffie and Sonnenschein (1989). Regarding the

own words, this equilibrium concept "is perhaps the closest in spirit to the Arrow-Debreu theory ... [and] is the closest to the notion of 'equilibrium over time,' described in *Value and Capital*."

Formally, in this type of model information structure $\langle S, \mathcal{I} \rangle$ is homogeneous, i.e. $\mathcal{I}_t = \mathcal{I}_t^a = \mathcal{I}_t^b$ for all $a, b \in A$, while agents are allowed to retain different prior beliefs, i.e. $\mathbb{Q}^a \neq \mathbb{Q}^b$. A consumption plan, or more generally, a trade plan z_t^a for each $a \in A$ and a price system p_t at time t, are defined as functions from S to the corresponding commodity space and price space. More specifically, a trade plan $z_t^a = (z_{t\tau}^a)$ may include both current period trading z_{t0}^a in the spot markets and state contingent commitments $z_{t\tau}^a$ for time $t + \tau$ with $\tau \geq 1$ in the futures markets. Each component of the trading plan, $z_{t\tau}^a$ for $\tau \geq 0$, is a function defined on $S^{t+\tau}$ and is required to be at most $\mathcal{I}^{t+\tau}$-measurable.[19] In words, a trade plan at time t specifies trading volumes of both current deliveries and future deliveries for all states wherever relevant markets exist. Moreover, the associated price system $p_t = (p_{t\tau})$ is an array of functions with $p_{t\tau}$ defined on $S^{t+\tau}$ for $\tau \geq 0$.

In contrast to the temporary equilibrium approach, perfect foresight equilibrium requires, rather than merely allow, agents to hold expectations about the future market conditions, among which the most important ones are expectations regarding to future market prices.[20] Since each agent is allowed to trade only in markets for limited future contingent deliveries, i.e. markets are incomplete, each agent necessarily has one budget constraint for each date-event pair, i.e.

$$\sum_{\tau \geq 0} \sum_{s_t^{t+\tau}} p_{t\tau}(s^t, s_t^{t+\tau}) z_{t\tau}^a(s^t, s_t^{t+\tau}) = 0$$

defined by the price expectations for all $s^t \in S^t$ under notations of section 2.[21] Note also that the sets over which τ and $s^{t+\tau}$ are summed are specified by the relevant market structure. This formulation differs from the standard Arrow-Debreu model in which a consolidated budget constraint

status of this equilibrium notion, in the comment of Radner's survey (1991), Duffie (1991) put it as follows: "Roy Radner's (1972) notion of 'equilibrium of plans, prices, and price expectations,' ... is now the closest thing to a *standard paradigm* for intertemporal general equilibrium." [Our emphasis.]

[19]Depending on the specification of market structures, state contingent commitments $z_{t\tau}$ that actually exist may well be measurable only with respect to a sub σ-field of $\mathcal{I}^{t+\tau}$. In contrast, future spot markets are always assumed to be complete ex ante, i.e. z_{t0}^a is always $\mathcal{I}^t$-measurable. See Radner (1972) or Radner (1982) for a comprehensive exposition of market structures in terms of measurability requirements.

[20]Note *future market prices* differ from *futures market* prices. The latter ones are prices observable from currently open market, whereas the former ones can only be observed from markets in the future. Also, to be precise, the price expectations $\{p_t\}$ are formed prior the start of the history and fixed throughout.

[21]When S is continuum, the summation used in the above equation becomes an integral with respect to the measure m defined in section 2. Note in this case, to make sure the integral of the product of price function and trade function to be well-defined, i.e. finite valued, more regularity conditions are in order. For example, a typical requirement is that $z_{it\tau}^a \in \mathscr{L}^\infty(S^{t+\tau}, \mathcal{S}^{t+\tau}, m^{t+\tau})$ and $p_{it\tau} \in \mathscr{L}^1(S^{t+\tau}, \mathcal{S}^{t+\tau}, m^{t+\tau})$ for each commodity i. See Bewley (1972) for details.

is used via the postulation of complete markets for all state contingent deliveries prior the actual beginning of the economy. Further more, the assumption of incomplete markets is essential for imposing sequential (state contingent) budget constraints, for otherwise one could easily transform the state contingent budget constraints into a consolidated time-0 constraint using an absence-of-arbitrage argument (see, for example Geanakoplos 1990). Since every agent's preference is defined over the space of feasible trade plans along all possible paths of the economy, whenever it attempts to find the optimal trade plans, it must hold expectations of market prices in future periods, contingent on each event, in order to optimize its objective function subject to well-defined sequential budget constraints.

Under this setup, a perfect foresight equilibrium is defined as an expected price system $p = (p_t)_{t \in \mathcal{T}}$ and a trade plan $z^a = (z_t^a)_{t \in \mathcal{T}}$ for each $a \in A$ such that: (i) Given price expectation p, z^a is optimal for each a, and (ii) trade plans are mutually consistent, i.e. markets clear $\sum_a z_t^a = 0$ for each date-event pair.[22] In the equilibrium, the economy evolves over time with markets open sequentially where actually trades take place according to the trade plans derived from the equilibrium price expectations. Since all trades are optimal, no revision is needed at each coming date-event pair. As a result, the markets clear at the expected equilibrium prices, and it is in this sense that perfect foresight equilibrium is also self-fulfilling.

Radner (1972) demonstrates the existence of a perfect foresight equilibrium for an exchange economy under arbitrary market structures for state contingent commitments. However, the proof hinges on an additional restriction on short sales, which is showed later on to be critical for the existence of equilibrium with incomplete markets by a stimulating paper of Hart (1975). An attempt to relax this artificial restriction and to incorporate more general assets structures rather than merely futures contracts leads to a boom in research in 1980s, out of which the general equilibrium theory of incomplete markets is developed.[23]

One notable feature of perfect foresight equilibrium is that every agent in the economy should hold identical expectations of prices to be prevailing in all future markets. This requirement seems

[22]When A is continuum, then the summation should be replaced by an integral with respect to the measure λ introduced in section 2. See the previous footnote.

[23]In Radner's (1972) original formulation, there was no formal role for both nominal and (generic) real assets, but only (a special case of) numeraire assets, since only futures markets for single consumption good exist. Hart (1975) generalized Radner's formulation to include generic real assets, i.e. contingent deliveries of consumption good bundles, and provided counter examples under standard assumptions (convex preference, etc.) showing that (perfect foresight) equilibrium with incomplete markets may not exist in absence of the artificial limitation on short sales employed by Radner (1972). For general equilibrium theory with incomplete markets and *finite states*, see the special issue in *Journal of Mathematical Economics* and the extensive survey by Geanakoplos (1990); for the case with *infinite states*, see the special issue of the same journal with an introduction by Duffie (1996). Also, Magill and Quinzii (1996) provides an excellent textbook account of the theory.

to be particularly stringent in sight that no Walrasian auctioneer even exists in current period to announce the price system for the future, since the associated auctioneer appears only when the markets open in the future. However, one notes that such a criticism is just one specific aspect of the more general problem, namely common knowledge of rationality, which is inherent to all equilibrium concepts based on Nash equilibrium (see Aumann 1987, 1995, 1997 for detailed discussion). In addition, the equilibrium prices expectations, when viewed as functions defined on S, could have nothing to do with agents' prior beliefs, unless agents' preferences rely on prior beliefs, like in the case of expected utility hypothesis. Indeed, under expected utility hypothesis, if agents have heterogenous beliefs about S, then the equilibrium (subjective) distributions of prices are necessarily different across agents.

One last remark has to do with the relationship between rational expectations and perfect foresight equilibrium. As we have argued in section 3, the concept of rational expectations is in fact grounded on an equilibrium theory. More precisely, if, instead of assuming heterogeneous beliefs, assume all agents in the economy have the same knowledge of the objective probability $\mathbb{P}$ of the stochastic process summarizing the uncertainty, then the equilibrium outcomes, notably the distribution of equilibrium prices $\{p_t\}$, will coincide with the distribution of the price expectations hold by agents, under perfect foresight equilibrium, hence fully satisfy the original formulation of rational expectations by Muth as presented in section 3. Essentially, rational expectations equilibrium is one particular type of perfect foresight equilibrium.[24] Moreover, the prominent feature of the use of conditional expectation operator $\mathbb{E}_t$ in the rational expectations literature, mostly in macroeconomics ever since the so called rational expectations revolution in 1970s,[25] now can be viewed as an outcome of the underlying equilibrium, provided that the preferences are expressed in the form of expected utility. For instance, in a prototypical RBC model like Hansen (1985), the intertemporal Eular equation has the following form:

$$u'(c_t(s^t)) = \beta\mathbb{E}_t[u'(c_{t+1}(s^t, s_{t+1}))(1 - \delta + r_{t+1}(s^t, s_{t+1}))]$$

where $\{s_t\}$ denotes the productivity shock and is the only source of uncertainty in this economy,

[24] The first explicity application of Muth's rational expectations hypothesis in an equilibrium model is in Lucas and Prescott (1971), which studies an inventory management problem under stochastic demand over time. As proponents, Lucas and Prescott (1971, footnote 9) state: "We can think of no objection to this [rational expectations] assumption which is not better phrased as an objection to our hypothesis that the stochastic component of demand has a *regular*, *stationary* structure." [Our emphasis.]

[25] For comprehensive surveys of rational expectations approach with its application in macroeconomics, see Shiller (1978), Barro (1981) and Taylor (1985). For critical discussions of various issues, both conceptual and methodological, of rational expectations, see Kantor (1979), the symposium issue in *Journal of Money, Credit and Banking* with an introduction by McCallum (1980), and McCallum (1982). Subsequently, rational expectations approach has became a standard paradigm in macroeconomic modeling, as exemplified by RBC and New Keynesian models in the following two decades. See Sargent (1996) for a particularly crispy overview of the development of macroeconomics both before and after the rational expectations revolution, which also posits Lucas (1972) at the central place.

while $c_t(s^t)$ is the time t consumption plan and $r_t(s^t)$ is time t rent price of capital. In this example, the emergence of the operator $\mathbb{E}_t$, defined as taking expectation of s_{t+1} conditional on s^t, is a natural consequence of household's utility maximization problem. One remark follows: The assumption of S being a Polish space, stated in section 2, is used here to guarantee the conditional expectation to be well-defined in a precise sense. The key point is under this assumption, $\mathbb{P}(s_{t+1} \in B|s^t)$ is a regular conditional distribution, regardless of the probabilistic nature of $\{s_t\}$ (see Durrett 2010, Ch. 4). Note here we do not assume $\{s_t\}$ to be a Markovian process. This technical condition was first made clear by Kreps (1977) (for a slightly more general case), but had been used before without explicit acknowledgement, see Jordan (1976).

We use the term *perfect foresight* to emphasize the expectation aspect of this approach. As already seen, perfect foresight necessarily implies expectations being fulfilling. However, we do not recommend to use *fulfilled expectations equilibrium* interchangeably with *perfect foresight equilibrium*, as occasionally seen in the literature, since one could use fulfilled expectations in much wider setups, e.g. see Jordan (1976) for a self-fulfilling model in a temporary equilibrium model. A underlying (even subtler, to some extent) difference between these two terms lies in the observation that, for perfect foresight expectations, agents are rendered in an ex ante manner the ability of holding correct, though unrestricted exogenously, expectations of the actual market outcomes in the future; whereas for fulfilled expectations, only a mild condition of the ex post coincidence of expectations and market outcomes is required, which is essentially equivalent to asking for expectations being representable as a fixed point of some function, e.g. honoring $\phi(p^*) = p^*$ where ϕ maps expectations to market outcomes, regardless whether the function is determined as part of the equilibrium or given exogenously.

More precisely, the function ϕ, stating the structural relation of expectations and equilibrium outcomes, is determined endogenously under agents' optimizing behaviors and market arrangements, e.g. market clearings, in a perfect foresight equilibrium; in contrast, for expectations to be fulfilling only, the structural relation ϕ can also be given exogenously as in the adaptive expectations approach,[26] and one can simply regard this ϕ as part of the fixed data of the economy, and still solve for the equilibrium which continually delivers fulfilled expectations. However, in doing this, as showed by Sidrauski (1967) for solving equilibrium prices of money, one has to face the difficulty raised by Jorgenson (1967) on the presence of multiple mutually inconsistent price concepts embodied in the model. The only way to get round of this obstacle, is to use perfect foresight approach instead as showed by Brock (1974). This distinction also helps to clarify the claim made by Muth (1961, p. 316) regarding the ineffectiveness of a "public prediction," formulated and analyzed by Grunberg and Modigliani (1954), upon the economy. Since those authors

[26] Although, this is typically done in another direction, i.e. by assuming $p^e = \varphi(p)$ exogenously, which can then be transformed to deduce the structural relation in our presentation by setting $\phi = \varphi^{-1}$.

assume directly that the price expectation is given by a fixed function of current price and a public prediction.[27] Nonetheless, the use of merely *fulfilled expectation* is justified in our point of view, as the discussion above should indicate.

4.4 Revealing Equilibrium

In the perfect foresight equilibrium we assume that all agents have the same information structure. An obvious extension of this assumption leads us to consider a model with heterogenous, or differential, information. Formally, in this subsection we assume that for different agent a, its information structure $\mathcal{I}^a$ may well be different from others.[28] Following the exposition in previous subsection, both trading plans $z_t^a = \{z_{t\tau}^a\}$ and price system $p_t = \{p_{t\tau}\}$ are still assumed to be functions from S to corresponding spaces, with two slight modifications that $z_{t\tau}^a$ be $\mathcal{I}^{at+\tau}$-measurable — if no additional information beyond $\mathcal{I}^a$ is available to a — and $p_{t\tau}$ be $\mathcal{P}^{t+\tau}$-measurable.[29]

This type of model was first proposed by Radner (1966) under a fairly general setup with sequential markets and production. About the same time, a formal extension of Arrow-Debreu model with production to cope with differential information was outlined by Radner (1968), in which markets are complete and all trading takes place at time 0, well before the beginning of

[27] Yet this paper is regarded as the first attempt on demonstrating fulfilled expectations by using a fix point theorem. At this place, it is also valuable, as an appropriate historic anecdote, to quote from Sargent (1996, footnote 2) to highlight the intellectual ingenuity of Professor Muench:

> [At that time] most of us were inadequately trained. In a 1971 meeting at the Minneapolis Fed. Neil Wallace and I tried to convince Thomas Muench that an infinite regress problem would render it impossible to construct a macroeconomic model along the lines of Tobin's 1955 'Dynamic Macroeconomic Model' which attributed to investors correct knowledge of *all* derivatives of the price level. I recall how I didn't know what to make of Muench's innocent query: 'Have either of you heard about fixed point theorems being applied to differential equations?' We hadn't, and neither had we understood how to adapt Grunberg and Modigliani's (1954) argument.

Muench's query just hits the key point. The crucial adaption in need is first assume there exists such a function (equilibrium prices), then let the model (various laws in physics, or equilibrium conditions in economics) pins down the equation that the proposed function should satisfy, and lastly employ a fixed point theorem to guarantee the existence of such a function.

[28] In this setup, in general there is no requirement for prior belief $\mathbb{Q}^a$ being identical across a. However, typically objective probability $\mathbb{P}$ is assumed instead, and note differential information already induces different conditional probability (belief) for each agent.

[29] For the former point, see the discussion in section 2, especially the brief explanation for a finite states case. This condition represents the idea of informational feasibility. For the latter one, it was first stated explicit by Kreps (1977, p. 36). The justification is simply that the aggregate trade plan $z_{t\tau}$ is at most $\mathcal{P}^{t+\tau}$-measurable, i.e. $z_{t\tau}$ must be at least jointly verifiable. See discussion below.

history. However, two distinct features arise when information is heterogeneous.

First, as pointed out in Radner (1968, p. 50), even if the market structure is complete, i.e. all state contingent commitments are possible a priori, the actual trades which effectively happen ex post are significantly restricted due the differential information held by agents. More precisely, consider a one period example with finite agents, i.e. $\mathcal{T} = 1$ and $|A| < \infty$, in which complete futures markets exist at time 0. Let the trade plan chosen by each agent be z^a. Then according to the conventional market clearing condition $\sum_{a \in A} z^a(s) = 0$ for m-almost $s \in S$, we have

$$z^a(s) = -\sum_{b \neq a} z^b(s).$$

Since z^a is $\mathcal{I}^a$-measurable on the one hand, and the right side is at most $\mathcal{P}^{-a} \equiv \bigvee_{b \neq a} \mathcal{I}^b$-measurable on the other hand.[30] Therefore, z^a is $\mathcal{I}^a \bigvee \mathcal{P}^{-a}$-measurable at most, which is necessarily smaller than $\mathcal{S}$, that is only a smaller set of markets will be in effective ex post. Eventually, using the argument elaborated in Radner (1982, pp. 948–949), one could reasonably argue that the effective markets should only be those induced by the common information $\mathcal{C}$ of all agents, which could be much smaller than $\mathcal{S}$. As a result, differential information actually implies a sort of market incompleteness.

Second, as noticed at the beginning of this subsection, the equilibrium price system p defined in the usual way typically would be $\mathcal{P}$-measurable, at least in principle, given the fact that the aggregate trade plan is $\mathcal{P}$-measurable. Since the $\mathcal{P}$ is in general finer than all $\mathcal{I}^a$, it suggests that some information is *revealed* to agents by the equilibrium price, which is typically not available had agents known only the (ex ante) given information in $\mathcal{I}^a$.

To elaborate this point further, let's consider again the one-period finite agents example discussed above, however replace the time 0 futures markets by the time 1 spot markets.[31] Thus, agents form expectations about market prices at time 0. Now, given an (commonly) expected price system p, the information conveyed by p is the σ-field generated by p, i.e. $\sigma(p)$. Given this additional information $\sigma(p)$, which is in general not fully contained in every $\mathcal{I}^a$, agents would have incentives to renegotiate on the trading contracts concluded before, for the information available to a is enlarged to $\sigma(\mathcal{I}^a, p)$, i.e. the σ-field generated by $\mathcal{I}^a$ and $\sigma(p)$. This effect is exemplified if agents' preferences conform the expected utility hypothesis, since the posterior beliefs can be updated from the prior beliefs by incorporating the information revealed by the (expected) price

[30] For example, if z^b and z^c are $\mathcal{I}^b$- and $\mathcal{I}^c$-measurable respectively, then we can (at most) claim that $z^b + z^c$ is measurable with respect to the *pooled information* induced by $\mathcal{I}^b$ and $\mathcal{I}^c$, i.e. $\mathcal{I}^b \bigvee \mathcal{I}^c$, provided that z^b and z^c be non-degenerate, e.g. be constant over S or be *step* functions.

[31] The choice of spot markets as the setup does not affect our analysis. It results only in a separate budget constraint for each state. See Green (1973, 1977), Grossman (1977, 1981) and Bray (1981) for analyses in a setup of futures markets.

system. As made clear in the extensive discussion by Grossman (1981) based on a two period example, the revealing property would impair any equilibrium price defined in the conventional way to prevail in the markets ex post.

This phenomenon has already been noticed by Radner (1966, sec. 7) which was originally published in French.[32] And in the same paper a new equilibrium concept is proposed to deal with the difficulty caused by the information revealed by prices (Radner, 1966, pp. 49–50). To simplify the notation and underscore the essence, let us further confine our discussion to the case of an exchange economy with common belief $\mathbb{P}$ over S. Let e^a denote a's endowment, which is required to be $\mathcal{I}^a$-measurable. Then if the trade plan is z^a, the corresponding consumption plan is $c^a = e^a - z^a$. Also let $u^a(c(s), s)$ denote a's utility for a particular consumption bundle in state $s \in S$, i.e. utility is state-dependent, and assume expected utility hypothesis applies here. Then, following the exposition in Allen (1986), the equilibrium proposed by Radner consists of an expected price system p and a trade plan z^a of agent a such that: (i) z^a solves the optimization problem

$$\begin{aligned} \max_{y^a} \quad & \mathbb{E}[u^a(e^a - y^a, \cdot)|\sigma(\mathcal{I}^a, p)](s) \\ \text{s.t.} \quad & p(r) \cdot y^a(r) = 0, \text{ for } m\text{-almost } r \in S \\ & \text{and } y^a \text{ is } \sigma(\mathcal{I}^a, p)\text{-measurable} \end{aligned}$$

for m-almost $s \in S$ and each $a \in A$; and (ii) markets clear $\sum_a z^a = 0$ for m-almost $s \in S$. The interpretation is straightforward: Taking into account of the information revealed by equilibrium price, optimal trade plans still clear the markets; or in Radner's own words (1966, sec. 6) "there exists a general equilibrium of the markets if the price structure at date 2 determines an information structure such that the decision at date 1 bring about the given price structure at date 2."

Two remarks is immediate: First, note the maximizing operator works for each $s \in S$, hence one is looking for a function subject to the two stated constraints which maximizes the utility function in almost all states of the economy. Thus this formulation differs from Allen (1981b, 1982) who adopts a state-wise formulation in defining the optimal trade plan, i.e. for each s, choose $y^a(s)$ optimally subject to $p(s)y^a(s) = 0$. These two formulations are essentially equivalent, as the optimal trade plan defined in the latter way can be showed to be $\sigma(\mathcal{I}^a, p)$-measurable (Allen,

[32] It is not clear whether the original French version of this paper was published in 1966 or 1967. According to the record in JSTOR (`http://www.jstor.org/stable/20075410`), the French version was published in 1966. However according to the bibliographic materials in Radner (1972, 1979, 1982, 1991), the paper was published in 1967. The English translation was never published, and only distributed as a technical report of UC Berkeley (filed in April 1967) to a few institutes in the U.S. However in the title remark of the technical report, it states that the original French version "will appear in a forthcoming issue of *Cahiers du Séminaire d'Économétrie*." One possibility is that the manuscript was originally written in French in 1966 and later on (in the same year) published in Pairs, while due to delay in the translating process, the English translation appeared in 1967.

1981b, appendix). Second, if the state-dependent utility is replaced by the ordinary utility, then $\mathbb{E}[u^a(e^a - y^a)|\sigma(\mathcal{I}^a, p)] = u^a(e^a - y^a)$ provided that $u^a(\cdot)$ is Borel measurable, for under this mild condition, $u^a(e^a - y^a)$ is $\mathcal{I}^a$-measurable. Employing state-dependent utility seems to be a common device in studying revealing equilibrium.

In Radner's (1966) original formulation, the information structures of agents are purely expressed by different partitions of the finite states of the world, and no formal role of probability (belief) is introduced in the analysis, except mentioning one possible representation of preference by expected utility (p. 39). The main effect of the information revealed by prices is the augmentation of trading opportunities. Recall that a trade plan z^a is at most $\mathcal{I}^a$-measurable without the revealed information of p whereas $\sigma(\mathcal{I}^a, p)$-measurable with the revealed information. However, this is exactly what is needed to refine agents' conditional belief given $\mathbb{P}$ over S. In this regard, the claim made by Grossman (1981, endnote 4) is not quite accurate: "[The equilibrium concept] is also suggested in Radner (1966). Radner did not model the idea that traders use current prices to revise their current demands and did not define an equilibrium of this process." On the contrary, the novelty of taking into account of the information revealed by prices is evident in Radner's original formulation regardless being expressed in a more fundamental level.

Although this equilibrium concept differs from the Arrow-Debreu equilibrium drastically, Radner (1966) did not propose a specific name for it. Indeed this paper is largely overlooked in 1970s, probably because it was published in French. A consequence of this is that the same equilibrium concept was re-introduced independently by Lucas (1972) and Green (1973).[33] However, neither Lucas nor Green proposed a specific name for the equilibrium concept. Later on, in Kreps (1977) and Green (1977), this equilibrium was termed as fulfilled expectations equilibrium and informational equilibrium respectively, yet none of both were widely used subsequently. In Radner (1979), for the first time, this equilibrium concept was formally named as *rational expectations equilibrium*, which has been widely accepted in the micro literature in 1980s, exemplified by the special issue in *Journal of Economic Theory* with an introduction by Jordan and Radner (1982). Nonetheless, we prefer to name this concept by *revealing equilibrium*, which serves better in both demonstrating the nature of this type of equilibrium and differentiating itself from the more widely adopted concept of rational expectations equilibrium discussed in the previous subsection.

Lucas's (1972) model is cast in an overlapping generations framework, in which younger generation needs to extract out of equilibrium price information about the monetary and demand shocks not observable directly. While motivated by Phelps (1970), especially the idea of *island economy*, Lucas' model is designed specifically to deliver the neutrality result for an expectational Phillips curve. Despite being attributed as triggering rational expectations revolution (McCallum,

[33]This claim about the priority has been made repeatedly by Radner, see Radner (1982, p. 996) and Radner (1991, endnote 19).

1982), the equilibrium in Lucas' paper is actually revealing equilibrium but not the typical rational expectations equilibrium. In contrast, Green's (1973) paper is motivated by Hirshleifer (1971) and extends the analysis to the efficiency problem brought about by the information conveyed through prices in a market economy. This has further intrigued a considerable amount of studies in the following year, of which the framework was replaced by a simpler and analytically more tractable one developed in a series papers by Grossman (1976, 1977, 1978) and Grossman and Stiglitz (1976), and eventually led to the famous conclusion on the impossibility of informational efficiency elucidated by Grossman and Stiglitz (1980).

The specific nature of informational efficiency of economies with differential information is directly linked to the "amount" of information revealed by prices in the equilibrium, which in another direction, is related to the existence problem of revealing equilibrium. In the overlapping generations model of Lucas (1972), the existence of a (unique) revealing equilibrium is demonstrated by applying contraction mapping theorem to a two period dynamic programming problem. However, in a general setup, as already noticed by Radner (1966, p. 51), the continuity of agents' demand functions will in general be destroyed due to the information revealed by prices, and this observation is confirmed via concrete examples constructed by Kreps (1977) and Green (1977), in which no revealing equilibrium exists. However, following an observation pointed out by Rothschild (Radner, 1982, p. 998), Radner (1979) successfully established the *generic existence* of revealing equilibrium for exchange economy possessing finite states.[34] The proof appeals a particular property of revealing equilibrium, that a *full communication equilibrium* is necessarily a revealing equilibrium, while the existence of full communication equilibrium follows directly existence results of classical Arrow-Debreu type economy. This same trick was also employed independently by Grossman (1976, 1978) yet in more restricted setups.

To spell out some more details, a full communication equilibrium is defined in an identical way to the revealing equilibrium except that agents optimize their utilities conditional on $\mathcal{P}$, i.e. the pooled information which contains $\sigma(\mathcal{I}^a, p)$ as a sub σ-field. However, as a result of appealing to full communication equilibrium, the revealing equilibrium has the property that equilibrium prices actually reveal all information, i.e. $\sigma(p) = \mathcal{P}$. In other words, the equilibrium price function $p : S \to \Delta$ is one-to-one, where Δ denotes the corresponding price simplex. We shall call this kind of equilibrium *fully revealing equilibrium* (FRE) for obvious reason.

In Radner's setup, it is quite conceivable to have equilibrium price function being one-to-one, since S is a finite set whereas Δ is continuum. One could reasonably anticipates that the FRE exists

[34]When all economies under consideration are parameterized by a certain space, where a point represents all data of a particular economy (preferences, endowments, etc.), then generic existence of equilibria means for almost all economies (in a certain sense) contained in the space, an equilibrium exists for each one. This conceptual device was proposed by Debreu (1970), and has became a central theme in general equilibrium analyses ever since.

whenever the state space S is "smaller" than the price space Δ under certain measurement. Allen's work (1981b, 1982) shows that the measurement needed is dimensionality. In Allen (1981b), it is showed under certain regularity conditions FRE exists generically if $2\dim S + 1 < \dim \Delta$, and the equilibrium price function $p(\cdot)$ is one-to-one for every $s \in S$. A stronger result is further proved in Allen (1982) that under additional (though mild) regularity conditions, FRE exists generically if $\dim S < \dim \Delta$, and in this case the equilibrium function $p(\cdot)$ is one-to-one m-a.s. Regarding to the higher dimensional case $\dim S > \dim \Delta$, Jordan (1983, thm. 2.12) shows that FRE is no where dense, i.e. generically no FRE exists; moreover, FRE exists only for very restricted class of utility functions, including risk-neutral, CARA and CRRA (Jordan, 1983, thm. 2.10). This non-existential result can be partially mitigated by requiring only ϵ-*revealing*, i.e. $||s - s'|| < \epsilon$ if $p(s) = p(s')$, which is showed to exist generically by construction (Jordan, 1982, thm. 2.4). However, the artificially constructed equilibrium price function turns out to be very complicated, not even continuous (it is actually a two-to-one function). The remaining case where $\dim S = \dim \Delta$ has been filled by a counter example explained in Jordan and Radner (1982), which shows no revealing equilibrium exists, and the result is robust under perturbation of the economy, thus generic existence fails as well. In sum, the qualitative results established by these authors under prototype setups indicate that fully revealing or close to fully revealing prices is a necessary feature of the equilibrium in markets with differential information.[35]

The informational efficiency property of fully revealing equilibrium is not only evident but actually too strong in the sense that even a completely *uninformed* agent, i.e. $\mathcal{I}^a = \{\varnothing, S\}$, ends up with full information $\mathcal{P}$, including all private information possessed by those informed agents, by merely observing the equilibrium prices in the realized spot market. Of course, fully revealing equilibrium could be viewed as a theoretic foundation of the *efficient market hypothesis* in the finance literature (Fama, 1970). However, acute conceptual difficulties of revealing equilibrium arise wherever it is fully revealing. Firstly, as pointed out by Beja (1976), whenever prices are fully revealing, then under an explicitly specified trading system relating trading orders and market clearing prices, the revealed information is basically useless (defined in a precise fashion). Secondly, as argued eloquently in Grossman and Stiglitz (1976, 1980), fully revealing equilibrium leads to a conundrum whenever information is costly to acquire: Since equilibrium prices reveal all information, no one has incentive to purchase/gather costly information, on the one side; but if no one actually acts to acquire those information, then in the equilibrium no information can be revealed, on the other side. Thirdly, the intuition that equilibrium prices can only be observed in an ex post manner, so that agents should not be able to condition their choice on information revealed by prices, leads to the critique by Dubey et al. (1987) about the setup of the revealing equilibrium, in which a model of sequentially revelation of information via prices is proposed and analyzed.

[35] See Allen (1986) for a survey of works in early 1980s, and Allen and Jordan (1998) for a more recent survey of revealing equilibrium.

Resorting to *partially revealing equilibrium*, in which equilibrium price function p is not one-to-one over a non-negligible set of S, has been considered as an desirable way of circumvent, at least the first two, difficulties raised above, e.g. (Allen, 1986, p. 9).[36] However, it is in general not easy to construct example of genuinely partially revealing equilibrium in a general setup, since it could well be the case that $\sigma(\mathcal{I}^a, p) = \mathcal{P}$ for all a even as $p(\cdot)$ is not generically one-to-one, i.e. the combination of private information and information revealed by prices gives rise to full information (Allen, 1981a). Moreover, the same discontinuity of agents' demand functions conditional on prices mentioned earlier prevents the utilization of fixed point theorem to guarantee the existence of equilibrium. One class of works include Allen (1983, 1985c,b,a) and Anderson and Sonnenschein (1982, 1985), in which the strict definition of revealing equilibrium is slightly modified to allow for various imperfectness either in price expectations or in market clearings, such that demand functions become continuous. Another class of works, initialized by Grossman (1977) and Grossman and Stiglitz (1980) and recently followed by DeMarzo and Skiadas (1998) and Hu and Qin (2013), confines to more restricted models with noisy price observations, of which partially prevailing equilibrium can be solved in close form. In contrast, a third class of works, including Ausubel (1990) and Pietra and Siconolfi (2008), succeeds in constructing explicitly partially revealing equilibrium in a comparatively much more general setup. A forth class of works, including recent works of Polemarchakis and Siconolfi (1993), Rahi (1995) and Citanna and Villanacci (2000), combines revealing equilibrium models with incomplete markets and establishes generically partial revelation results. A fifth class of works, initialized by McAllister (1990) and followed by Dutta and Morris (1997), Pietra and Siconolfi (1997) and Krebs (2001), extends the point price expectations to expected distributions over the price space and demonstrates the generical existence of partially revealing equilibrium, in which a large number of equilibrium information structures can be revealed by prices.[37]

The significance of partially revealing equilibrium is not confined to a non-trivial property of informational efficiency, but it has also profound effects in the classic welfare property of market models. In a pioneering work based on Lucas' (1972) model, which is perhaps the most well known example of genuinely partially revealing equilibrium, Muench (1977) performed detailed welfare analysis using a variant of standard Pareto optimality criterion and demonstrated the non-

[36] In contrast, Balder and Yannelis (2009) proposes a completely different equilibrium concept, rested on Bayesian-Nash equilibrium used in game theory, for market equilibrium with differential information. Moreover, in Cornet and De Boisdeffre (2002, 2009) and subsequently De Boisdeffre (2005, 2007, 2009), the price-information-revelation mechanism, which assumes agents know the price function, is replaced by a different mechanism of information revelation based on an extension of arbitrage analysis from homogeneous information framework to heterogeneous information framework, and the corresponding existence results are established. See the special issue of *Economic Theory* and the introduction by Glycopantis et al. (2009).

[37] See De Boisdeffre (2011) for a model in the similar fashion which deals with the standard incomplete markets model.

trivial welfare consequences caused by the unusual information structure. Later on, a systematic welfare analysis of the revealing equilibrium models was carried out by Laffont (1985), which actually confirmed the founding of Muench on the possibility of welfare improvement by public intervention when the equilibrium is partially revealing.

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