高级微观经济学

第 6 讲：非合作博弈基础

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本讲内容

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简要历史

博弈论研究的是多个决策者在策略互动 (strategic interactions) 环境下的决策行为

- 博弈论的思想源远流长：中国古代的兵法
- 经济的例子伴随着近代工业化厂商竞争行为的发展而出现：
  - Antoine A. Cournot，1838，数量竞争
  - Joseph L. F. Bertrand，1883，价格竞争
  - Francis Y. Edgeworth，1889，严格的模型 (+ 产量限制)
- John von Neumann & Oskar Morgenstern, 1944

  Theory of Games and Economic Behavior

  奠定现代博弈论的基础

博弈论的两大分支

非合作博弈 (non-cooperative game)

- 注重从单个决策者的策略选择出发，详细规定各决策者策略互动的过程和影响，一般通过均衡概念把个体最优化决策加总，从而预测博弈结果
- 决策者互不交流单独决策，但对均衡有一致预期

合作博弈 (cooperative game)

- 从决策者策略互动可能的结果出发，并规定一个合适的“解”所应满足的性质——特别是对决策者联盟 (coalition) 具有如“稳定”或“公平”的性质，从而预测博弈结果
- 决策者交流协商，寻求合作

现代发展：Nobel 奖历史

1969 至 2023，55 次 Nobel 奖，93 名得主中有 17 名的主要得奖工作在博弈论及其直接应用方面，仅次于宏观经济学 (共 20 名，1990 年后共 10 名)

- 1994 John C. Harsanyi, John F. Nash, Reinhard Selten
- 1996 James A. Mirrlees, William S. Vickrey
- 2005 Robert J. Aumann, Thomas C. Schelling
- 2007 Leonid Hurwicz, Eric S. Maskin, Roger B. Myerson
- 2012 Alvin E. Roth, Lloyd S. Shapley
- 2014 Jean Tirole
- 2016 Oliver Hart, Bengt Holmström
- 2020 Paul Milgrom, Robert Wilson

Shapley 的贡献

It is a truth universally acknowledged that much that is fundamental and beautiful in the field of Game Theory has been shaped, and nurtured over the years, by Lloyd Shapley. One need merely name topics where his work was seminal and path-breaking, and served to define entire areas of research: the value (with finite and continuum player sets), core, voting games and power indices, stochastic games, repeated games, matching, potential games, market games in coalitional and strategic form, the convergence phenomenon for perfectly competitive economies (core and value in the coalitional setting, and non-cooperative equilibrium in the strategic), convex games, fictitious play, etc.

Dubey and Tauman, 2012

本节内容

非合作博弈的策略形式和扩展形式

策略形式 (strategic form)，亦称正规形式 (normal form)

- 参与者集合 $I = \{1, \ldots, I\}$
- 参与者 i 的策略集 S_i；$S = S_1 \times \cdots \times S_I$
- 参与者 i 的收益函数 $u_i : S \to \mathbb{R}$
- 所有参与者同时做出策略选择，形成 $s = (s_1, \ldots, s_I) \in S$
- $\Gamma = (I, S, (u_i)_{i \in I})$ 称为一个博弈的策略形式

扩展形式 (extensive form)

- 策略形式 + 信息结构 (information structure)
- 信息结构可以用树状结构表示；博弈树 (game tree)

一个 EF 对应一个 SF；一个 SF 可对应多个 EF

策略形式博弈与 Nash 均衡

给定策略形式博弈 $\Gamma = (I, S, (u_i)_{i \in I})$

- 若一个策略组 (strategy profile) $s = (s_i, \ldots, s_I)$ 满足一定条件，则称为 Γ 的解
  - 严格占优 (strict dominance，剔除严格劣势策略)，重复严格占优 (iterative SD)，可理性化 (rationalizability)，相关均衡 (correlated equilibrium)
- 应用最广的还是 Nash 均衡：策略组 s 若满足

$$s_i \in \operatorname*{argmax}_{t \in S_i} u_i(s|_i t)$$

对任意 $i \in I$ 成立，则称其为 Nash 均衡

  - 均衡策略：给定所有其他人选择这组策略，没有参与者有动力改变目前的策略选择

混合策略

- 策略形式博弈最基本的情形是有限纯策略：所有 S_i 均有限；此时 S_i 中每个备选策略称为纯策略
- 有限纯策略基本的问题是 Nash 均衡可能不存在；但若使用 von Neumann & Morgenstern 引入的混合策略 (mixed strategy)，则存在性不是问题
- 给定有限纯策略集 S_i，一个混合策略 σ_i 是 S_i 上的一个概率分布；混合策略的集合记为

$$\Sigma_i = \left\{\sigma_i : \sum_{s_i \in S_i} \sigma_i(s_i) = 1, \sigma_i(s_i) \geq 0\right\}.$$

- 总假设各个参与者的混合策略是相互独立的

混合策略下的 Nash 均衡

- 给定混合策略组 $\sigma = (\sigma_i)_{i\in I} \in \Sigma = \Sigma_1 \times \cdots \times \Sigma_I$，则 i 的期望收益为

$$U_i(\sigma) = \mathbb{E}^{\sigma} u_i = \sum_{s\in S} \prod_{j\in I} \sigma_j(s_j) u_i(s),$$

  $\prod_{j\in I} \sigma_j(s_j) = \sigma_1(s_1)\cdots\sigma_I(s_I)$ 表示纯策略组 $s = (s_j)_{j\in I}$ 出现的概率 (σ_j 互相独立)

- 混合策略组 $\sigma \in \Sigma$ 称为策略博弈 $\Gamma = (I, S, (u_i)_{i\in I})$ 的一个 (混合策略) Nash 均衡，若对所有 $i \in I$ 有

$$\sigma_i \in \operatorname*{argmax}_{\tau\in\Sigma_i} U_i(\sigma|_i\tau),$$

  其中 $\sigma|_i\tau$ 表示混合策略下 i 的单方面偏离

纯策略与混合策略示例

- 考虑如下 2×2 策略形式博弈，左/右数字是 P1/P2 的收益

P2

		L	R
P1	U	1, 0	0, 1
	D	0, 1	1, 0

- 该博弈没有纯策略 Nash 均衡，但有唯一的混合策略 Nash 均衡 $\sigma_1^*(U) = \sigma_2^*(L) = 0.5$
- 令 $p = \sigma_1(U), q = \sigma_2(L)$，则有

$$U_1(\sigma) = pq + (1-p)(1-q), \qquad U_2(\sigma) = p(1-q) + (1-p)q$$

进一步可求解最优回应对应 $\beta_1(\sigma), \beta_2(\sigma)$

Nash 均衡的存在性

定理 1

若策略形式博弈 $\Gamma = (I, S, (u_i)_{i \in I})$ 中每个参与者的纯策略集 S_i 均有限，则一定存在混合策略 Nash 均衡

证明.

$U_i(\sigma)$ 连续，关于 σ_i 线性，故拟凹；又可直接验证 Berge 最大值定理条件得到满足，故 $\beta_i(\sigma) = \operatorname{argmax}_{\tau \in \Sigma_i} U_i(\sigma|_i \tau)$ 是 $\Sigma \rightrightarrows \Sigma_i$ 的上半连续、紧致、凸对应，进而 $\beta(\sigma) = (\beta_1(\sigma), \ldots, \beta_I(\sigma))$ 是 $\Sigma \rightrightarrows \Sigma$ 的上半连续、紧致、凸对应，最后由 Kakutani 不动点定理知存在 $\sigma^* \in \beta(\sigma^*)$ □

Nash (1950)：一页纸论文

This follows from the arguments used in a forthcoming paper.[13] It is proved by constructing an "abstract" mapping cylinder of λ and transcribing into algebraic terms the proof of the analogous theorem on CW-complexes.

* This note arose from consultations during the tenure of a John Simon Guggenheim Memorial Fellowship by MacLane.

[2] Whitehead, J. H. C., "Combinatorial Homotopy I and II," *Bull. A.M.S.*, **55**, 214–245 and 453–496 (1949). We refer to these papers as CH I and CH II, respectively.

[3] By a complex we shall mean a connected CW complex, as defined in §5 of CH I. We do not restrict ourselves to finite complexes. A fixed 0-cell $e^0 \epsilon K^0$ will be the base point for all the homotopy groups in K.

[4] MacLane, S., "Cohomology Theory in Abstract Groups III," *Ann. Math.*, **50**, 736–761 (1949), referred to as CT III.

[5] An (unpublished) result like Theorem 1 for the homotopy type was obtained prior to these results by J. A. Zilber.

[6] CT III uses in place of equation (2.4) the stronger hypothesis that λB contains the center of A, but all the relevant developments there apply under the weaker assumption (2.4).

[7] Eilenberg, S., and MacLane, S., "Cohomology Theory in Abstract Groups II," *Ann. Math.*, **48**, 326–341 (1947).

[8] Eilenberg, S., and MacLane, S., "Determination of the Second Homology . . . by Means of Homotopy Invariants," these PROCEEDINGS, **32**, 277–280 (1946).

[9] Blakers, A. L., "Some Relations Between Homology and Homotopy Groups," *Ann. Math.*, **49**, 428–461 (1948), §12.

[10] The hypothesis of Theorem C, requiring that ν^{-1} (1) not be cyclic, can be readily realized by suitable choice of the free group X, but this hypothesis is not needed here (cf. [6]).

[11] Eilenberg, S., and MacLane, S., "Homology of Spaces with Operators II," *Trans. A.M.S.*, **65**, 49–99 (1949); referred to as HSO II.

[12] $C(\tilde{K})$ here is the $C(K)$ of CH II. Note that $\tilde{K}$ exists and is a CW complex by (N) of p. 231 of CH I and that $p^{-1}K^n = \tilde{K}^n$, where p is the projection $p:\tilde{K} \to K$.

[13] Whitehead, J. H. C., "Simple Homotopy Types." If $W = 1$, Theorem 5 follows from (17:3) on p. 155 of S. Lefschetz, *Algebraic Topology*, (New York, 1942) and arguments in §6 of J. H. C. Whitehead, "On Simply Connected 4-Dimensional Polyhedra" (*Comm. Math. Helv.*, **22**, 48–92 (1949)). However this proof cannot be generalized to the case $W \neq 1$.

EQUILIBRIUM POINTS IN N-PERSON GAMES

BY JOHN F. NASH, JR.*

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Communicated by S. Lefschetz, November 16, 1949

One may define a concept of an n-person game in which each player has a finite set of pure strategies and in which a definite set of payments to the n players corresponds to each n-tuple of pure strategies, one strategy being taken for each player. For mixed strategies, which are probability

distributions over the pure strategies, the pay-off functions are the expectations of the players, thus becoming polylinear forms in the probabilities with which the various players play their various pure strategies.

Any n-tuple of strategies, one for each player, may be regarded as a point in the product space obtained by multiplying the n strategy spaces of the players. One such n-tuple counters another if the strategy of each player in the countering n-tuple yields the highest obtainable expectation for its player against the $n - 1$ strategies of the other players in the countered n-tuple. A self-countering n-tuple is called an equilibrium point.

The correspondence of each n-tuple with its set of countering n-tuples gives a one-to-many mapping of the product space into itself. From the definition of countering we see that the set of countering points of a point is convex. By using the continuity of the pay-off functions we see that the graph of the mapping is closed. The closedness is equivalent to saying: if $P_1, P_2, \ldots$ and $Q_1, Q_2, \ldots, Q_n, \ldots$ are sequences of points in the product space where $Q_n \to Q$, $P_n \to P$ and Q_n counters P_n then Q counters P.

Since the graph is closed and since the image of each point under the mapping is convex, we infer from Kakutani's theorem[1] that the mapping has a fixed point (i.e., point contained in its image). Hence there is an equilibrium point.

In the two-person zero-sum case the "main theorem"[2] and the existence of an equilibrium point are equivalent. In this case any two equilibrium points lead to the same expectations for the players, but this need not occur in general.

* The author is indebted to Dr. David Gale for suggesting the use of Kakutani's theorem to simplify the proof and to the A. E. C. for financial support.

[1] Kakutani, S., *Duke Math. J.*, **8**, 457–459 (1941).

[2] Von Neumann, J., and Morgenstern, O., *The Theory of Games and Economic Behaviour*, Chap. 3, Princeton University Press, Princeton, 1947.

REMARK ON WEYL'S NOTE "INEQUALITIES BETWEEN THE TWO KINDS OF EIGENVALUES OF A LINEAR TRANSFORMATION"*

BY GEORGE POLYA

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Communicated by H. Weyl, November 25, 1949

In the note quoted above H. Weyl proved a Theorem involving a function $\varphi(\lambda)$ and concerning the eigenvalues α_i of a linear transformation A and those, κ_i, of A^*A. If the κ_i and $\lambda_i = |\alpha_i|^2$ are arranged in descending order,

经典的例子

- 囚徒悖论：

	否认	招供
否认	5, 5	0, 10
招供	10, 0	2, 2

  均衡为 (招供, 招供) 是 Pareto 无效的

- 可以理解为一种策略外部性 (strategic externality)：自利个体的行为导致整体无效率
  - 古话：以邻为壑
  - Dubey (1986) 证明，Nash 均衡几乎都是 Pareto 无效的

本节内容

扩展形式博弈的正式描述

- 参与者集合 N；行动集合 A
- 节点集 (历史集，树) H 满足：(i) 有一个初始点 h_0；(ii) $H \setminus \{h_0\}$ 中的节点形如 $h = (a_1, \ldots, a_k)$，即历史行动决定当前节点；(iii) 若 $(a_1, \ldots, a_k) \in H \setminus \{h_0\}$，则 $(a_1, \ldots, a_{k-1}) \in H \setminus \{h_0\}$，即当前节点有唯一的前一步节点
- 自然 (nature) 在初始点 h_0 从 $A(h_0) \subset A$ 中按分布 π 随机的选择一个行动 a_1；博弈终点集为 E
- 对决策节点集 $H \setminus (E \cup \{h_0\})$ 中每一点 h，指定一个参与者 $\iota(h)$ 做出决策
- 决策节点集划分为一组互不相交的信息集；每个信息集 I 满足：若 $h, h' \in I$ 则有 $\iota(h) = \iota(h')$ 且 $A(h) = A(h')$
- 参与者 i 有终点效用函数 $u_i : E \to \mathbb{R}$

信息集与参与者策略

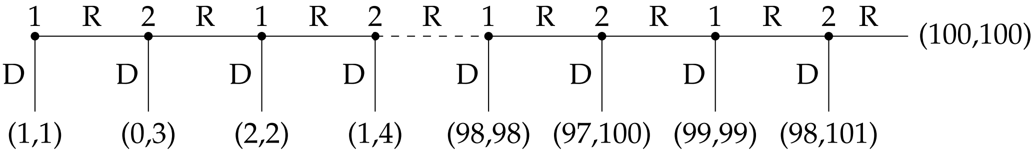
- 把信息集 I 对应的参与者记为 $\iota(I)$；$\iota(I)$ 知道他处于 I 包含的某一节点 h 中，并且需要从行动集 $A(h)$ 中做出选择
- 参与者 i 的纯策略：在每一个需要其决策的信息集上选择一个相应的行动；相应的可定义混合策略
- 所有参与者的策略 $\sigma = (\sigma_1, \ldots, \sigma_N)$ 共同决定了 (部分) H 以及 (部分) E 上的一个概率分布；按照这个分布可以计算各参与者的期望效用 $\mathbb{E}^\sigma u_i$
- 如果所有的信息集都是单点集 (singleton)，那么称这个博弈为完美信息 (perfect information) 博弈；除此之外，称为不完美信息 (imperfect information) 博弈

倒向归纳和子博弈 Nash 均衡

- 对于完美信息博弈，可以使用倒向归纳法 (backward induction) 获得一个解，并且这个解是 Nash 均衡
  - 例子：Stackelberg 博弈
- 对于特定的不完美信息博弈，仍可使用“倒推”的思想
- 如果一个信息集 h 是单点集，且其后节点所在的信息集中所有节点均源自 h，则从 h 开始的博弈称为子博弈 (subgame)
- 子博弈完美 (subgame perfect) Nash 均衡：限制在所有子博弈上仍然构成 Nash 均衡的策略组

倒推法可能产生反直觉结果

考虑下面的“蜈蚣”博弈 (centipede game, cf. Rosenthal, 1981)：



黑色节点上方 1、2 表示第一、二参与者，括号里前一数字表示 1 的收益，后一数字表示 2 的收益；这个博弈中两人轮流行动，从 1 开始，分别选择向右继续 (R) 还是向下停止 (D)，一共进行 200 轮：如果某人选择继续而下一轮中对方选择停止，那么前一个人会损失 1 单位收益而后一个人会增加 2 单位收益，以此类推

唯一 Nash 均衡违反直觉，不“合理”

本节内容

Cournot 数量竞争

- 考虑两家企业，生产同质产品，成本函数为 cy_i，$i = 1, 2$，即两企业边际成本均为 $c > 0$
- 产品需求函数为 $d = D(p) = A - p$，$A > c$，相应的反需求函数为 $p = A - d$
- 两个企业采取数量竞争的形式：具体而言，若两个企业的产量决策为 y_1, y_2，则市场价格为 $p = A - y_1 - y_2$
- Cournot 竞争博弈：两个企业同时选择产量水平
- 此时唯一的 Nash 均衡产量水平为 y_1^c, y_2^c，则总产出 $y^c = y_1^c + y_2^c$ 小于垄断企业产量选择 y^m
- 拓展：多个厂商 (市场集中度 HHI 的理论基础)，固定成本，非对称生产技术 (边际成本)，产能投资，一般的需求函数形式

Stackelberg 数量竞争

- 继续考虑上述两家企业
- 假设企业 1 首先选择产量 y_1，而企业 2 在观察到 y_1 之后选择产量 y_2
    - Stackelberg 博弈的策略形式与 Cournot 博弈的策略形式完全相同，差别在于扩展形式，即行动时点及参与者的信息
- 此时有唯一的 Nash 均衡产量水平 y_1^s, y_2^s，且 $y_1^s > y_2^s$，即存在领先者优势
    - 领先者优势 (leader's advantage)：预见到对手的策略选择，从而确保自己的策略更占优
- 拓展：Cournot 博弈的拓展全部适用；以及更多的动态、不对称信息拓展，如阻止竞争对手市场进入的极限定价 (limit pricing)
    - 一个系列重要的拓展在宏观经济学和动态政治经济学建模：动态的政策/政治互动，可以用动态 Stackelberg (重复) 博弈来刻画

Bertrand 价格竞争

- 考虑两个企业进行价格而非数量竞争
- 假设两个企业同时选择价格水平 p_1, p_2：若 $p_i < p_j$，则 i 获得所有市场需求；若两个企业出价相等 $p_1 = p_2$，则均分此时的市场需求
  - Bertrand 博弈与 Cournot 博弈同样是同时行动(simultaneous move) 博弈，即其扩展形式与策略形式完全相同
  - 如果改为 Stackelberg 式的扩展形式，一个厂商先选价格，后一个厂商再选，结果有何差别？
- 该价格竞争博弈有唯一的 Nash 均衡 $p_1^b = p_2^b = c$，此时均衡产出 $y^b = y_1^b + y_2^b$ 高于 y^m, y^c, y^s
  - 此时又称为边际成本定价，按照古典经济学的观点，认为达到了社会有效 (socially efficient) 产出水平
- 拓展：Cournot 博弈拓展同样适用，而不对称信息下的价格竞争，形成了一整套的拍卖 (auction) 理论

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