Lecture Notes: Global Games

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Introduction: Morris and Shin 1998

- Study a general class of binary choice coordination games
- Under complete information, this class of games admit multiple equilibria
- However, adding small heterogeneous information delivers a unique equilibrium

Multiple equilibria under common knowledge

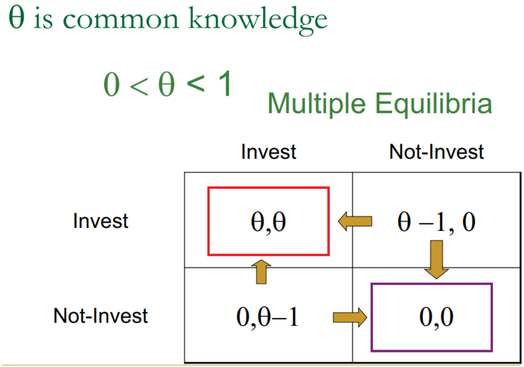

Figure 1: Common Knowledge

Model setting: attacking game

- There is a measure one continuum of agents, indexed by $i \in [0, 1]$
- Each agent i chooses to attack or not attack:
  - $a_i = 0$ if not attack
  - $a_i = 1$ if attack
- The payoff from not attacking is normalized to zero.
- The payoff from attacking is $1 - c$ if the status quo is abandoned and a 'regime change ' occurs, and is $-c$ otherwise, with $c \in (0, 1)$.
- The status quo is abandoned and 'regime change' occurs iff $A > \theta$
  - A denotes the mass of agents attacking
  - $\theta \in R$ is an exogenous fundamental parameterizing the strength of the regime

Payoff of agent

- Payoff of the agent

$$U(a_i, A, \theta) = a_i(\mathbf{1}_{A>\theta} - c), \tag{1}$$

where $\mathbf{1}_{A>\theta}$ is an indicator of regime change, equal to 1 if $A > \theta$ and zero otherwise.

- Payoffs can be written as

	$\mathbf{1}_{A>\theta} = 1$	$\mathbf{1}_{A>\theta} = 0$
$a_i = 1$	$1 - c$	$-c$
$a_i = 0$	0	0

- The actions of agents are strategic complements.

Complementarity

- It pays off for an agent to attack iff the status quo collapses
- The status quo collapses iff a sufficiently large fraction of agents attack
- The coordination motive is the key feature of the model
- The incentive to attack increases with the mass of agents attacking

Common knowledge benchmark

- Assume θ is known
- The best response of any agent is

$$BR(A,\theta) = \begin{cases} 1, & \text{if } A > \theta \\ 0, & \text{if } A \leq \theta \end{cases} \tag{2}$$

- Let $\underline{\theta} = 0$ and $\bar{\theta} = 1$. Under common knowledge, we have the following
  1. For $\theta < \underline{\theta}$, fundamentals are week, and $a_i = 1$ is a dominant strategy
  2. For $\theta > \bar{\theta}$, fundamentals are strong, and $a_i = 0$ is a dominant strategy.
- Now consider $\theta \in (\underline{\theta}, \bar{\theta})$, there are multiple equilibria: both $A = 1$ and $A = 0$ are equilibria.
  - Each equilibrium is sustained by self-fulfilling expectations

Interpretation and applications

- Self-fulfilling currency crises (Obstfeld, 1986)
  - Central bank is interested in maintaining a currency peg
  - A large number of speculators, with finite wealth, deciding whether to attack the currency or not.
- Self-fulfilling bank runs (Diamond and Dybvig, 1983)
  - Depositors decide whether or not to withdraw their deposits
  - θ represents the liquid resources available to the bank

Incomplete and asymmetric information

- Assume θ is not common knowledge
- Agents have a common prior over θ, let it be improper uniform over the real line
- Each agent receives an exogenous private signal

$$x_i = \theta + \xi_i \tag{3}$$

and an exogenous public signal

$$z = \theta + \epsilon \tag{4}$$

where $\xi_i \sim N(0, \sigma_x^2)$ is idiosyncratic noise and $\epsilon \sim N(0, \sigma_z^2)$ is a common error.

  - Let $\alpha_x = 1/\sigma_x^2$ and $\alpha_z = 1/\sigma_z^2$ denote the precisions of the private and public signals, respectively.

Symmetric Bayesian equilibrium definition

An equilibrium is a strategy $a(x, z)$ and an aggregate attack $A(\theta, z)$ such that

$$a(x, z) \in argmax \mathbb{E}[U(a, A(\theta, z), \theta)|x, z]$$

$$A(\theta, z) = \int a(x, z)\phi(\sqrt{\alpha_x}(x - \theta))dx$$

where $\phi(\cdot)$ is the PDF of the standard Normal. Technical note: $x_i \sim N(\theta, \sigma_x^2)$ implies $\frac{x_i - \theta}{\sigma_x} = \sqrt{\alpha_x}(x - \theta) \sim N(0, 1)$.

Equilibrium analysis

- We consider monotone (or threshold) equilibria: equilibria in which $a(x, z)$ is monotonic in x.
- **Attack decision:** in a monotone equilibrium, for any realization of z, there is a threshold $x^*(x)$ such that agents attack iff
$$x \leq x^*(z)$$
- **Regime switch condition:** by implication, the aggregate size of the attack is decreasing in θ, so that there is also a threshold $\theta^*(z)$ such that the status quo is abandoned iff
$$\theta \leq \theta^*(z)$$
- A monotone equilibrium is therefore identified by the threshold functions of x^* and θ^*.

Stpe 1: Characterize θ^* for a given x^*

- For a given realizations of θ and z, the aggregate size of attack is given by the mass of agents who receive signals $x \leq x^*$. Thus

$$A(\theta, z) = \Phi(\sqrt{\alpha_x}(x^*(z) - \theta)) \tag{5}$$

  where $\Phi(\cdot)$ is the CDF of the standard Normal.

- Notice $A(\theta, z)$ is decreasing in θ, so that regime change occurs iff $\theta < \theta^*(x)$ where $\theta^*(z)$ is the unique solution to

$$A(\theta^*(z), z) = \theta^*(z) \Longleftrightarrow \Phi(\sqrt{\alpha_x}[x^*(z) - \theta^*(z)]) = \theta^*(z)$$

- Solving this for $x^*(z)$ we obtain

$$x^*(z) = \theta^*(z) + \frac{1}{\sqrt{\alpha_x}}\Phi^{-1}(\theta^*(z)) \tag{6}$$

Stpe 1: Characterize θ^* for a given x^*

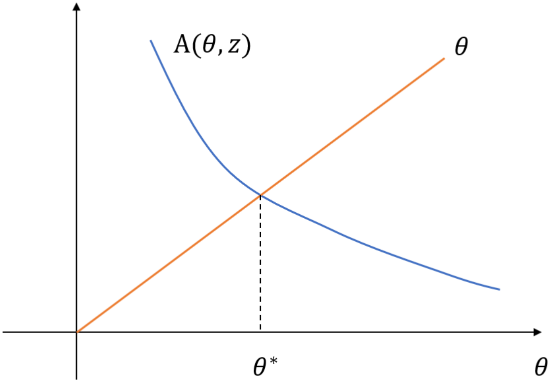

Figure 2: Threshold value θ^*

Step 2: Characterize x^* for given θ^*

- Given that regime change occurs iff $\theta \leq \theta^*(z)$, the payoff of an agent is

$$\mathbb{E}[U(a, A(\theta, z), \theta)|x, z] = a(Pr[\theta \leq \theta^*(z)|x, z] - c) \qquad (7)$$

- Given his signal, the posterior of the agent is

$$\theta|x, z \sim N(\frac{\alpha_x}{\alpha_x + \alpha_z}x + \frac{\alpha_z}{\alpha_x + \alpha_z}z, \frac{1}{\alpha_x + \alpha_z}) \qquad (8)$$

Let $\alpha \equiv \alpha_x + \alpha_z$ denote the precision of this posterior.

- The posterior probability of regime change is

$$\begin{aligned} Pr[\theta \leq \theta^*(z)|x, z] &= \Phi\left(\sqrt{\alpha}\left(\theta^*(z) - \frac{\alpha_x}{\alpha_x + \alpha_z}x - \frac{\alpha_z}{\alpha_x + \alpha_z}z\right)\right) \\ &= 1 - \Phi\left(\sqrt{\alpha}\left(\frac{\alpha_x}{\alpha_x + \alpha_z}x + \frac{\alpha_z}{\alpha_x + \alpha_z}z\right) - \theta^*(z)\right) \end{aligned}$$

which is decreasing in x.

Step 2: Characterize x^* for given θ^*

It follows that the agents attacks iff $x \leq x^*(z)$ solves indifferent condition

$$0 = a(Pr[\theta \leq \theta^*(z)|x, z] - c) \tag{9}$$

This implies

$$Pr[\theta \leq \theta^*(z)|x, z] = c \tag{10}$$

Thus we obtain

$$\Phi\left(\sqrt{\alpha}\left(\frac{\alpha_x}{\alpha_x + \alpha_z}x^*(z) + \frac{\alpha_z}{\alpha_x + \alpha_z}z\right) - \theta^*(z)\right) = 1 - c \tag{11}$$

which solves the unique $x^*(z)$.

Stpe 2: Characterize x^* for a given θ^*

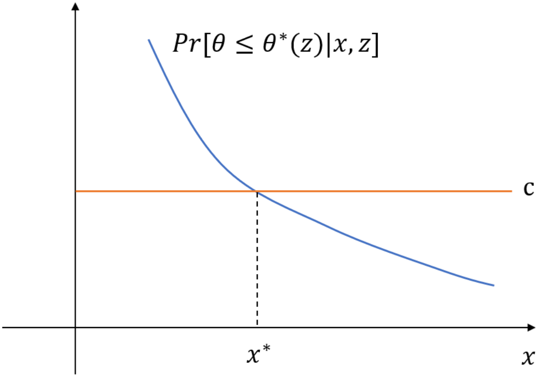

Figure 3: Threshold value x^*

Step 3: Combine two equilibrium conditions

Combine (6) and (11) to get one equilibrium condition.
Substituting (6) into (11) we get

$$\Phi\left(\sqrt{\alpha}\left(\frac{\alpha_x}{\alpha}\left[\theta^*(z)+\frac{1}{\sqrt{\alpha_x}}\Phi^{-1}(\theta^*(z))\right]+\frac{\alpha_z}{\alpha}z\right)-\theta^*(z)\right)=1-c$$

$$\frac{\alpha_x}{\alpha}\left[\theta^*(z)+\frac{1}{\sqrt{\alpha_x}}\Phi^{-1}(\theta^*(z))\right]+\frac{\alpha_z}{\alpha}z-\theta^*(z)=\frac{1}{\sqrt{\alpha}}\Phi^{-1}(1-c)$$

$$\frac{\alpha_z}{\alpha}(z-\theta^*(z))+\frac{\sqrt{\alpha_x}}{\alpha}\Phi^{-1}(\theta^*(z))=\frac{1}{\sqrt{\alpha}}\Phi^{-1}(1-c)$$

Finally, the one equilibrium condition becomes

$$\frac{\alpha_z}{\sqrt{\alpha_x}}(z-\theta^*(z))+\Phi^{-1}(\theta^*(z))=\sqrt{\frac{\alpha}{\alpha_x}}\Phi^{-1}(1-c) \qquad (12)$$

Equilibrium

Proposition 1

A monotone equilibrium in this game is characterized by thresholds $\theta^*(z)$ and $x^*(z)$ such that

(i) $\theta^*(z)$ *is given by*

$$G(\theta^*(z), z) = g \tag{13}$$

where $g = \sqrt{\frac{\alpha_x+\alpha_z}{\alpha_x}}\Phi^{-1}(1-c)$ *is a constant, and*

$$G(\theta, z) = \frac{\alpha_z}{\sqrt{\alpha_x}}(z-\theta) + \Phi^{-1}(\theta)$$

(ii) $x^*(z)$ *is given by*

$$x^*(z) = \theta^*(z) + \frac{1}{\sqrt{\alpha_x}}\Phi^{-1}(\theta^*(z))$$

Existence of equilibrium

- We establish existence of equilibrium by considering the properties of function G.
- For every $z \in \mathbb{R}$, $G(\theta, z)$ is continuous in θ.

$$G(\underline{\theta}, z) = \frac{\alpha_z}{\sqrt{\alpha_x}}(z - 0) + \Phi^{-1}(0) = -\infty$$
$$G(\bar{\theta}, z) = \frac{\alpha_z}{\sqrt{\alpha_x}}(z - 1) + \Phi^{-1}(1) = +\infty$$

- Thus, there exists a solution and any solution satisfies $\theta^*(z) \in (\underline{\theta}, \bar{\theta})$.

Equilibirum: uniqueness or multiplicity?

Note that

$$\frac{\partial G(\theta, z)}{\partial \theta} = -\frac{\alpha_z}{\sqrt{\alpha_x}} + \frac{1}{\phi(\Phi^{-1}(\theta))}$$

We know that $max_{\omega \in \mathbb{R}} \phi(\omega) = \frac{1}{\sqrt{2\pi}}$, thus $min \frac{1}{\phi(\Phi^{-1}(\theta))} = \sqrt{2\pi}$.
(Technical note: $\phi(\omega) = \frac{1}{\sqrt{2\pi}} exp \frac{1}{2}\omega^2)$

1. If $\frac{\alpha_z}{\sqrt{\alpha_x}} < \sqrt{2\pi}$, then $\frac{\partial G(\theta,z)}{\partial \theta} > 0$. Unique solution to (13).
2. If $\frac{\alpha_z}{\sqrt{\alpha_x}} > \sqrt{2\pi}$, then G is non-monotonic in θ. There is an interval $z \in (\underline{z}, \bar{z})$ such that (13) admits multiple solutions to $\theta^*(z)$ whenever, $z \in (\underline{z}, \bar{z})$, and a unique solution otherwise.

We conclude that monotone equilibrium is unique iff

$$\frac{\alpha_z}{\sqrt{\alpha_x}} < \sqrt{2\pi}$$

Equilibrium characterization

Proposition 2 (Morris and Shin)

There always exists a monotone equilibrium. It is unique if and only if private noise is small enough relative to the public noise,

$$\frac{\sigma_x}{\sigma_z^2} \leq \sqrt{2\pi}$$

Otherwise, there is an interval of z such that there are three different pairs (x^*, θ^*) that define monotone equilibria.

Limits

Proposition 3 (Morris and Shin Limit)

As either

(i) $\sigma_x \to 0$, *for given* σ_z, *or*

(ii) $\sigma_z \to \infty$, *for given* σ_x

there is a unique monotone equilibrium in which regime change occurs iff $\theta \leq \hat{\theta}$ *where* $\hat{\theta} \equiv 1 - c \in (\underline{\theta}, \bar{\theta})$.

Proof.

Take the limit of both sides of (13). □

Discontinuity around perfect information

- We know that when information is perfect ($\sigma_x = 0$) there exists multiple equilibria and any regime outcome is possible.
- However, for an arbitrarily small perturbation away from perfect information, the regime outcome is uniquely pinned down.
- Crises, then, defined as situations displaying high non-fundamental volatility, cannot be addressed in the limit as private information becomes arbitrarily precise ($\sigma_x \to 0$), since there the regime outcome is dictated only by fundamentals, that is, θ.
- Furthermore, note that the outcome is only a function of θ, and independent of z, which means that all volatility is fundamentals driven.
- In conclusion, Morris and Shin show us that in these coordination games, multiplicity is the unintended consequences of common knowledge.

Reference

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