A Note on Steady State Financial Friction in Banking

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Abstract

In this note, we provide an analytical characterization of the steady state financial friction in the framework of Gertler and Karadi (2011). We obtain three results. First, the steady state financial friction is purely determined by the parameters of the banking sector. Second, there are two steady state values of bank leverage, one stable and the other unstable. Third, we identify the necessary and sufficient condition for the existence of a unique positive steady state spread, which corresponds to either stable or unstable leverage.

Keywords: Financial friction; banking sector; bank leverage; credit spread; steady state

JEL: E30; E44; E50

1. Introduction

Considerable effort has been devoted to a better understanding of frictions in the financial sector from a macroeconomic perspective, especially since the onset of the 2007–08 financial crisis. Among many theories developed so far, the framework of Gertler and Karadi (2011, henceforth GKa) has proved to be a workhorse in quantitative works on the banking friction. Building on the framework, subsequent works have investigated the quantitative implications of the friction and addressed important policy questions,

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such as quantitative easing and macro-prudential policies.[1]

Despite its wide usage in the literature, there is still a lack of understanding on the analytical properties of the banking friction in GKa. In this note, we provide a complete characterization of the banking friction in the deterministic steady state. And to our best knowledge, this is the first work to provide such a characterization. The analytical results we derive reveal three important properties of the steady state friction. First, the steady state spread is fully pinned down, in conjunction with the bank leverage, by two equations, one for individual bank incentive constraint and the other for aggregate bank net worth constraint. Since the two equations do not involve any aggregate variables outside the banking sector, the steady state spread and leverage are independent from those variables as well. Second, we show that in the steady state, the incentive constraint implies a mapping from spread to bank leverage. For a given spread, bank leverage can take two values in general, where one is stable and the other is unstable under a perturbation. Third, we identify the necessary and sufficient condition for the existence of a unique positive spread in steady state, and the condition under which the spread corresponds to either stable or unstable leverage.

The paper is organized as follows. Section 2 describes the banking setup. Section 3 analyzes the model and presents our main results. Section 4 concludes.

2. Model

2.1. Basic setup

In the framework of GKa, a banker with net worth n_t survives each period with probability $\theta \in (0,1)$, and exits with probability $1-\theta$ taking n_t as payoff. Upon surviving, the banker raises deposit b_t to fund asset holdings $Q_t s_t$, where Q_t denotes the price of the asset, so that

$$Q_t s_t = n_t + b_t.$$

After raising funds, the banker can either divert the asset or collect future returns. The choice is unobservable and leads to a moral hazard problem. In the case of diverting, the banker simply walks away with a fraction $\lambda \in [0,1)$ of assets $Q_t s_t$ in hands. Alternatively, the banker obtains a gross asset return

[1] A partial list includes Gertler and Kiyotaki (2010), Gertler et al. (2012), Gertler and Karadi (2013), Dedola et al. (2013), Gertler and Kiyotaki (2015), Bocola (2016), Liu (2016), and Paoli and Paustian (2017).

R_{kt+1} next period. After paying a gross deposit rate R_{t+1}, the banker has a net worth of

$$n_{t+1} = R_{kt+1} Q_t s_t - R_{t+1} b_t.$$

Combining those two equations, the transition law of bank net worth is

$$n_{t+1} = (R_{kt+1} - R_{t+1}) Q_t s_t + R_{t+1} n_t. \tag{1}$$

Given the stochastic discount factor $\{\beta^j \Lambda_{t,t+j}\}_{j \geq 1}$ and the initial net worth n_t, the banker chooses asset positions $\{s_{t+j}\}_{j \geq 0}$ to maximize the expected discount value of future payoffs

$$V_t = \mathbb{E}_t \sum_{j=0}^{\infty} \beta^{j+1} \Lambda_{t,t+j+1} \theta^j (1-\theta) n_{t+j+1}, \tag{2}$$

subject to the net worth transition law and the incentive constraint

$$n_{t+j+1} = (R_{kt+j+1} - R_{t+j+1}) Q_{t+j} s_{t+j} + R_{t+j+1} n_{t+j},$$
$$V_{t+j} \geq \lambda Q_{t+j} s_{t+j},$$

for all $j \geq 0$. By imposing the incentive constraint, asset diverting is ruled out in the equilibrium.

The setup is closed with a transition law of the aggregate banking net worth:

$$N_{t+1} = \theta[(R_{kt+t} - R_{t+1}) Q_t S_t + R_t N_t] + \omega Q_{t+1} S_t, \tag{3}$$

where $\omega/(1-\theta)$ is the fraction of the total asset from the exiting bankers distributed to each of new bankers.

2.2. Recursive solution

Following GKa, by (1) and (2), the banker's problem has the following recursive representation:

$$V_t = \mathbb{E}_t \beta \Lambda_{t,t+1} \{(1-\theta)[(R_{kt+1} - R_{t+1}) Q_t s_t + R_{t+1} n_t] + \theta V_{t+1}\}.$$

As shown by GKa, V_t is linear in $Q_t s_t$ and n_t. Let $V_t = \mu_t Q_t s_t + \nu_t n_t$, where μ_t and ν_t are parameters to be solved. From the recursion of V_t, we have

$$\mu_t = \mathbb{E}_t \beta \Lambda_{t,t+1} [(1-\theta)(R_{kt+1} - R_{t+1}) + \theta \mu_{t+1} x_{t+1}], \tag{4}$$
$$\nu_t = \mathbb{E}_t \beta \Lambda_{t,t+1} [(1-\theta) R_{t+1} + \theta \nu_{t+1} z_{t+1}]. \tag{5}$$

where $x_{t+1} = Q_{t+1} s_{t+1}/(Q_t s_t)$ and $z_{t+1} = n_{t+1}/n_t$.

We focus on the solution of the banking sector in which the incentive constraints are always binding, i.e, $V_t = \lambda Q_t s_t$. This leads to

$$\mu_t Q_t s_t + \nu_t n_t = \lambda Q_t s_t,$$

and implies

$$\phi_t = \frac{\nu_t}{\lambda - \mu_t}, \tag{6}$$

where $\phi_t = Q_t s_t / n_t$ denotes the bank leverage. Since all banks are symmetric, μ_t and ν_t are the same across the banking sector, and thus ϕ_t is also the banking sector leverage.

With (6), the law of individual bank net worth transition becomes $n_{t+1} = [(R_{kt+1} - R_{t+1})\phi_t + R_{t+1}]n_t$, so that

$$z_{t+1} = n_{t+1}/n_t = (R_{kt+1} - R_{t+1})\phi_t + R_{t+1}, \tag{7}$$

$$x_{t+1} = Q_{t+1}s_{t+1}/Q_t s_t = (\phi_{t+1}/\phi_t)z_{t+1}. \tag{8}$$

3. Analysis

3.1. Incentive constraint

We use the same notation, suppressing time subscripts, for the steady state values. Let $\Delta R = R_k - R$ denote the credit spread. We start by considering the incentive constraint facing an individual bank, i.e., the conditions corresponding to (4)–(8):

$$\begin{aligned}
\mu &= \beta[(1-\theta)\Delta R + \theta x \mu], \\
\nu &= \beta[(1-\theta)R + \theta z \nu], \\
\phi &= \frac{\nu}{\lambda - \mu}, \\
z &= \Delta R \phi + R, \\
x &= z,
\end{aligned}$$

as $\Lambda = 1$ in the steady state. Substitution of z and x into μ and ν yields:

$$\begin{aligned}
\mu &= \frac{\beta(1-\theta)\Delta R}{1 - \beta\theta(\Delta R \phi + R)}, \\
\nu &= \frac{\beta(1-\theta)R}{1 - \beta\theta(\Delta R \phi + R)}.
\end{aligned}$$

Note that the substitution is meaningful if $1 > \beta\theta(\Delta R\phi + R) = \theta(\beta\Delta R\phi + 1)$, as $\beta R = 1$. This is satisfied if ϕ is not too large for a given ΔR.[2]

Furthermore, substitution of μ and ν into ϕ gives

$$\phi = \frac{1-\theta}{\lambda(1-\theta-\beta\theta\Delta R\phi) - \beta(1-\theta)\Delta R} \equiv G(\phi), \tag{9}$$

which summarizes the incentive constraint facing an individual bank by a single function between ΔR and ϕ. It is easy to verify that the denominator of (9) is positive if $\Delta R < \lambda/\beta$, and below we shall only consider ΔR within this range. Given ΔR, we can transform (9) as a quadratic equation of ϕ:

$$F(\phi) \equiv \lambda\beta\theta\Delta R\phi^2 - (1-\theta)(\lambda - \beta\Delta R)\phi + (1-\theta) = 0. \tag{10}$$

Observing $F(0) = 1-\theta > 0$ and

$$F(1) = \lambda\beta\theta\Delta R + (1-\theta)(1-\lambda+\beta\Delta R) > 0,$$

it follows that whenever the two roots of (10) are real, they are both greater than 1. The discriminant associated with $F(\phi)$ equals to

$$(1-\theta)^2(\lambda-\beta\Delta R)^2 - 4\lambda\beta\theta\Delta R(1-\theta) = (1-\theta)^2\left[\lambda^2 + \beta^2\Delta R^2 - 2\frac{1+\theta}{1-\theta}\lambda\beta\Delta R\right]. \tag{11}$$

For the existence of a steady state ϕ, the expression in the brackets has to be non-negative. Treating the expression as a quadratic function of ΔR, the associated discriminant is always greater than 0 as

$$4\left(\frac{1+\theta}{1-\theta}\right)^2\lambda^2\beta^2 - 4\lambda^2\beta^2 = 4\lambda^2\beta^2\frac{4\theta}{(1-\theta)^2} > 0,$$

so there are always two roots, and (11) is non-negative only for

$$\Delta R \le \Delta R_l \equiv \frac{\lambda}{\beta}\frac{1-\sqrt{\theta}}{1+\sqrt{\theta}}, \quad \text{or} \quad \Delta R \ge \Delta R_r \equiv \frac{\lambda}{\beta}\frac{1+\sqrt{\theta}}{1-\sqrt{\theta}}. \tag{12}$$

Since $\Delta R_l < \lambda/\beta < \Delta R_r$, we do not need to consider the case of $\Delta R \ge \Delta R_r$. For $\Delta R < \Delta R_l$, (9) has two roots greater than 1. The next proposition shows that they have distinct stability properties.

[2] To be precise, this requires that $\phi < (1-\theta)/(\beta\theta\Delta R)$, a condition satisfied by any possible steady state ϕ, as shown below.

Proposition 1. *Given $\Delta R \geq 0$, if there are two values of steady state leverage ϕ^* and ϕ^{**} with $\phi^* < \phi^{**}$, then only ϕ^* is stable under a perturbation.*

Proof. From (9), it is straightforward to verify that $G(\phi)$ is negative for $\phi > \bar{\phi} \equiv (1-\theta)(\lambda-\beta\Delta R)/(\beta\lambda\theta\Delta R)$, and is positive, increasing, and convex for $\phi < \bar{\phi}$. Hence whenever $G(\phi)$ intersects with the 45° line ϕ at two points $\phi^* < \phi^{**} < \bar{\phi}$, it crosses the line at ϕ^* from above, goes underneath from ϕ^* to ϕ^{**}, and crosses the line at ϕ^{**} from below again.

Consider three cases of a small perturbation around the two candidate values of the steady state leverage.

(i) If a banker chooses an asset s^0 so that $\phi^0 = Qs^0/n \in (\phi^*, \phi^{**})$, then the banker would find it profitable to divert asset, since

$$\frac{\nu}{\lambda-\mu} = G(\phi^0) < \phi^0 = \frac{Qs^0}{n} \Rightarrow V^0 = \mu Q s^0 + \nu n < \lambda Q s^0.$$

Realizing the possibility of diverting, depositors would renegotiate with the bank to reduce the leverage to ϕ^*, otherwise they can simply withdraw deposits from the bank. Hence ϕ^0 drops to ϕ^*.

(ii) If s^0 is such that $\phi^0 > \phi^{**}$, or equivalently $G(\phi^0) > \phi^0$, then the banker would find it profitable to further increase s^0 without violating the incentive constraint, leading ϕ^0 to be further away from ϕ^{**}.

(iii) Lastly, if $\phi^0 < \phi^*$, the banker would utilize the unexploited leverage capacity and increase ϕ^0 to ϕ^*.

This demonstrates that the candidate leverage ϕ^{**} is unstable to a small perturbation, while ϕ^* is stable. □

We conclude that the *stable* leverage corresponds to the smaller root of (10):

$$\phi^* = \frac{1-\theta}{2\lambda\beta\theta} \cdot \frac{\lambda - \beta\Delta R - \sqrt{\beta^2\Delta R^2 - 2\frac{1+\theta}{1-\theta}\lambda\beta\Delta R + \lambda^2}}{\Delta R}, \tag{13}$$

whereas the *unstable* leverage corresponds to the larger root:

$$\phi^{**} = \frac{1-\theta}{2\lambda\beta\theta} \cdot \frac{\lambda - \beta\Delta R + \sqrt{\beta^2\Delta R^2 - 2\frac{1+\theta}{1-\theta}\lambda\beta\Delta R + \lambda^2}}{\Delta R}. \tag{14}$$

We remark that $\phi^* < \phi^{**} < (1-\theta)/(\beta\theta\Delta R)$,[3] which confirms the validity of the substitution for μ and ν at the beginning of this subsection.

As a numerical example, consider the calibration of GKa. The authors choose $\beta = 0.99$, $\theta = 0.972$, and $\lambda = 0.381$, targeting an annual spread of $4 \times \Delta R = 1\%$ with leverage of 4. It can be verified that $\phi = 4$ is the smaller root given by (13), whereas the larger root is around 7.4. Figure 1 provides an illustration.

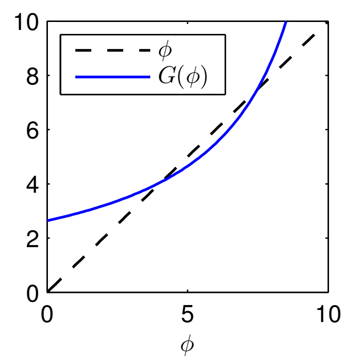

Figure 1: Bank incentive constraint under GKa's calibration

3.2. Net worth constraint

Unlike the individual bank net worth and asset, the aggregate net worth and asset are constant in the steady state. As a consequence, the steady state version of (3) is $N = \theta(\Delta R\phi + R)N + \omega QS$. With $R = 1/\beta$, we obtain the second equation between the steady state leverage and spread:

$$\phi = \frac{1-\theta/\beta}{\theta\Delta R + \omega}. \tag{15}$$

Evidently, $1-\theta/\beta$ should be greater than 0, so that we require $\theta < \beta$. Intuitively, individual banks have to exit the market quickly enough, otherwise they will accumulate net worth indefinitely, preventing the existence of a proper steady state. Given the parameterization of GKa with $\omega = 0.002$, the net worth constraint also implies $\phi = 4$.

It is clear that the steady state incentive constraint and net worth constraint give two equations with two unknowns, i.e., spread and leverage,

[3]It is straightforward to check that $\sqrt{\beta^2\Delta R^2 - 2\frac{1+\theta}{1-\theta}\lambda\beta\Delta R + \lambda^2} < \lambda - \beta\Delta R$, therefore $\phi^{**} < (1-\theta)/(\beta\theta\Delta R)\cdot(1-\beta\Delta R/\lambda) < (1-\theta)/(\beta\theta\Delta R)$.

and thus fully pin down the steady state of the banking sector. Since no aggregate variable outside the banking sector is involved, we conclude that the steady state spread and leverage are independent from the steady state variables in rest of the economy.

3.3. Spread and leverage

We first consider the case of stable leverage. Combining (13) and (15) gives the equation of the steady state spread ΔR:

$$\frac{1-\theta/\beta}{\theta\Delta R+\omega}=\frac{1-\theta}{2\lambda\beta\theta}\cdot\frac{\lambda-\beta\Delta R-\sqrt{\beta^2\Delta R^2-2\frac{1+\theta}{1-\theta}\lambda\beta\Delta R+\lambda^2}}{\Delta R}. \tag{16}$$

Focusing on the case of $\Delta R>0$, we rewrite the above equation as

$$\Omega(\Delta R)\equiv\frac{(1-\theta/\beta)\Delta R}{\theta\Delta R+\omega}= \frac{1-\theta}{2\lambda\beta\theta}\left(\lambda-\beta\Delta R-\sqrt{\beta^2\Delta R^2-2\frac{1+\theta}{1-\theta}\lambda\beta\Delta R+\lambda^2}\right)\equiv\Phi(\Delta R). \tag{17}$$

To investigate the existence of a steady state with positive spread, we first note that for $\Delta R\in(0,\Delta R_l]$, $\Omega(\Delta R)$ is positive, increasing, and concave, with

$$\Omega'(0)=\frac{1-\theta/\beta}{\omega}>0 \quad\text{and}\quad \Omega''(\Delta R)=-\frac{2(1-\theta/\beta)\omega\theta}{(\theta\Delta R+\omega)^3}<0,$$

as $\theta/\beta<1$ by assumption. Furthermore, $\Phi(\Delta R)$ is positive, increasing, and convex, with

$$\Phi'(0)=\frac{1}{\lambda},\qquad \lim_{\Delta R\to\Delta R_l}\Phi'(\Delta R)=\infty,$$

and

$$\Phi''(\Delta R)\propto\frac{4\theta}{(1-\theta)^2}>0.$$

Since $\Omega(0)=\Phi(0)=0$, it follows that (17) has a unique solution greater than 0 if and only if

$$\frac{1-\theta/\beta}{\omega}=\Omega'(0)>\Phi'(0)=\frac{1}{\lambda}, \tag{18}$$

and $\Omega(\Delta R_l)\le\Phi(\Delta R_l)$, or equivalently,

$$\frac{\sqrt{\theta}(1-\sqrt{\theta}/\beta)}{1+\sqrt{\theta}}\le\frac{\omega}{\lambda}. \tag{19}$$

It is straightforward to verify that the parameterization in GKa implies $4\Delta R_l = 1.09\%$, and the steady state spread is $4\Delta R = 1\%$ annually.

Next, consider the case of unstable leverage. The equation characterizing the steady state spread takes a slightly different form:

$$\Omega(\Delta R) \equiv \frac{(1-\theta/\beta)\Delta R}{\theta\Delta R + \omega} = \frac{1-\theta}{2\lambda\beta\theta}\left(\lambda - \beta\Delta R + \sqrt{\beta^2\Delta R^2 - 2\tfrac{1+\theta}{1-\theta}\lambda\beta\Delta R + \lambda^2}\right) \equiv \Psi(\Delta R), \quad (20)$$

where we continue to focus on the case of $\Delta R > 0$.

Similar to the analysis under the stable leverage, it is straightforward to show that $\Psi(\Delta R)$ is positive and concave over $(0, \Delta R_l]$, with

$$\Psi'(0) = -\frac{1}{\lambda\theta}, \quad \lim_{\Delta R \to \Delta R_l} \Psi'(\Delta R) = -\infty,$$

and

$$\Psi''(\Delta R) \propto -\frac{4\theta}{(1-\theta)^2} < 0.$$

Since $\Omega(0) = 0 < \Phi(0)$, it is clear that a unique solution to (20) exists if and only if $\Omega(\Delta R_l) \geq \Psi(\Delta R_l)$, or equivalently

$$\frac{\sqrt{\theta}(1-\sqrt{\theta}/\beta)}{1+\sqrt{\theta}} \geq \frac{\omega}{\lambda}. \quad (21)$$

We can further combine the two cases, by observing that $\Phi(\Delta R_l) = \Psi(\Delta R_l)$, and (19) is just the opposite of (21). The following proposition summarizes the results:

Proposition 2. *There exists a unique, positive steady state spread, if and only if*

$$1 - \theta/\beta > \omega/\lambda.$$

Moreover, when

$$\sqrt{\theta}(1-\sqrt{\theta}/\beta)/(1+\sqrt{\theta}) \leq \omega/\lambda,$$

the corresponding steady state leverage is stable; on the contrary, when

$$\sqrt{\theta}(1-\sqrt{\theta}/\beta)/(1+\sqrt{\theta}) \geq \omega/\lambda,$$

the corresponding leverage is unstable.

Figure 2 gives an illustration of the proposition. The left panel shows the steady state spread under GKa's parameterization, where $\omega = 0.002$. Clearly, Ω crosses Φ only, implying a steady state associated with the stable leverage. The right panel has the same parameterization as the GKa except for a smaller $\omega = 0.0001$. In this case, Φ and Ψ remain the same, but Ω shifts upward and intersects with Ψ instead of Φ, leading to a steady state featuring unstable leverage.

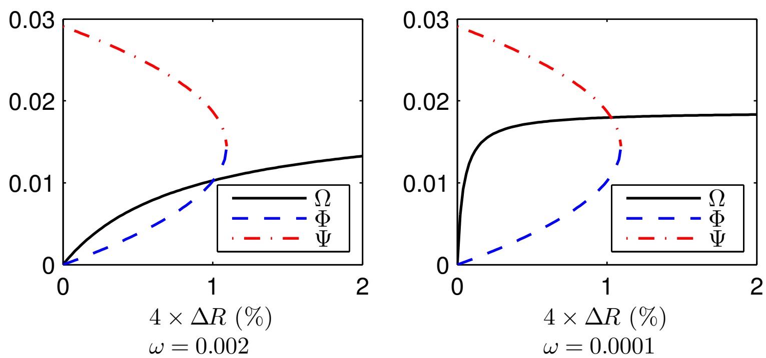

Figure 2: Steady state spreads under different ω

As a final remark, we discuss the remaining case where $1 - \theta/\beta \leq \omega/\lambda$. First, when $1-\theta/\beta = \omega/\lambda$, Ω and Φ are tangent to each other at $\Delta R = 0$, and it can be easily verified by l'Hospital law that $\Delta R = 0$ is the unique solution to (16). Indeed, this is also the only case where the steady state spread is zero *with* a binding incentive constraint. Second, when $1 - \theta/\beta < \omega/\lambda$, we have a steady state of a non-binding incentive constraint and a zero spread.[4] The result follows from the fact that a positive spread is inconsistent with a non-binding incentive constraint, while a binding constraint is incompatible with any steady state spread under the stated condition.

4. Conclusion

Besides the theoretical interests on their own, the analytical results in this note can be readily used to guide calibration works building on the GKa

[4]Since the incentive constraint ceases to be binding, the steady state leverage is determined solely by the aggregate net worth constraint. Moreover, as already known in the literature, even if $1 - \theta/\beta \geq \omega/\lambda$, the zero spread steady state with non-binding incentive constraint exists as well, a situation where financial friction is not effective.

framework. Moreover, the results on the steady state financial friction can shed light on the effects of various financial policies, such as the QE type government credit intervention, bank equity injection, and bank funding cost subsidy. In some preliminary work, we show that all these policies can help reduce the steady state spread, hence yield a first order impact for financial stabilization.

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