Computational Methods of Heterogeneous Agent Models

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Contents

Overview

- The representative agent framework has become the standard tool for modern macroeconomics.
  - Agents are different with regard to many characteristics: abilities, education, age, wealth holdings et al.
  - Simple aggregation may sometimes not be possible or lead to wrong implications.
  - Policy and welfare questions that analyze the redistribution of income among agents

A Simple Heterogeneous-Agent Model with Aggregate Certainty

- Simple model: households, production, and the government.
- We consider heterogeneity at the household level only consider idiosyncratic risk: unemployed or employed.
- Stationary equilibrium of the economy: the distribution of the state variable, the aggregate wage and the aggregate interest rate are all constant, while the employment status and the wealth level of the individual households vary

Households

- Households differ only with regard to their employment status and their asset holdings.

$$E_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t\right) \tag{1}$$

$$u\left(c_t\right) = \frac{c_t^{1-\eta}}{1-\eta}, \quad \eta > 0 \tag{2}$$

- At time zero, the agent knows his beginning-of-period wealth a_0 and his employment status $\epsilon_0 \in \{e, u\}$. Agents are endowed with one indivisible unit of time in each period. ϵ is assumed to follow a first-order Markov chain.

$$\pi\left(\epsilon' \mid \epsilon\right) = \text{Prob}\left\{\epsilon_{t+1} = \epsilon' \mid \epsilon_t = \epsilon\right\} = \begin{pmatrix} p_{uu} & p_{ue} \\ p_{eu} & p_{ee} \end{pmatrix} \tag{3}$$

Households

- There are no private insurance markets against unemployment and unemployed agents only receive unemployment compensation from the government. Budget constraint:

$$a_{t+1} = \begin{cases} (1+(1-\tau)r_t)\,a_t + (1-\tau)w_t - c_t & \text{if } \epsilon = e \\ (1+(1-\tau)r_t)\,a_t + b_t - c_t & \text{if } \epsilon = u \end{cases} \tag{4}$$

- Household problem

$$\begin{aligned} \mathscr{L} = E_0 \textstyle\sum_{t=0}^{\infty} \{\beta^t\,[u\,(c_t) + \lambda_t\,(1_{\epsilon_t=u} b_t + (1+(1-\tau)r_t)\,a_t \\ +1_{\epsilon_t=e}(1-\tau)w_t - a_{t+1} - c_t)]\} \end{aligned} \tag{5}$$

$$\frac{u'\,(c_t)}{\beta} = E_t\left[u'\,(c_{t+1})\,(1+(1-\tau)r_{t+1})\right] \tag{6}$$

- The solution is given by the policy function $c\,(\epsilon_t, a_t)$, next-period asset holdings $a_{t+1} = a'\,(\epsilon_t, a_t)$

Production and Government

- Firms are owned by the households and maximize profits with respect to their labor and capital demand

$$Y_t = N_t^{1-\alpha} K_t^{\alpha}, \quad \alpha \in (0,1) \tag{7}$$

- In a market equilibrium, factors are compensated according to their marginal products and profits are zero:

$$\begin{aligned} r_t &= \alpha \left(\frac{N_t}{K_t}\right)^{1-\alpha} - \delta \\ w_t &= (1-\alpha)\left(\frac{K_t}{N_t}\right)^{\alpha} \end{aligned} \tag{8}$$

- Government expenditures consist of unemployment compensation B_t which are financed by a tax on income. The government budget is assumed to balance in every period

$$B_t = T_t \tag{9}$$

Stationary Equilibrium

- In a stationary equilibrium, the aggregate variables and the factor prices are constant. Furthermore, the distribution of assets is constant for both the employed and unemployed agents, and the numbers of employed and unemployed agents are constant, too.
- Two distribution functions, $F(e, a)$ and $F(u, a)$. The corresponding density functions: $f(e, a)$ and $f(u, a)$. The individual state space consists of the sets $(\epsilon, a) \in \mathcal{X} = \{e, u\} \times [a_{\min}, \infty)$
- Recursive representation of the consumer' s problem

$$V(\epsilon, a) = \max_{c, a'} \left[u(c) + \beta E \left\{ V \left(\epsilon', a' \right) \mid \epsilon \right\} \right]$$

subject to the budget constraint, the government policy $\{b, \tau\}$, and the stochastic process of ϵ

Definition

- A stationary equilibrium for a given government policy parameter b is a value function $V(\epsilon, a)$, individual policy rules $c(\epsilon, a)$ and $a'(\epsilon, a)$, a time-invariant density of $x = (\epsilon, a) \in \mathcal{X}$, $f(e, a)$ and $f(u, a)$, time-invariant $\{w, r\}$, and a vector of aggregates K, N, C, T, and B such that:
- 1. Factor inputs, consumption, tax revenues, and unemployment compensation are obtained aggregating over households:

$$K = \sum_{\epsilon \in \{e,u\}} \int_{a_{\min}}^{\infty} af(\epsilon, a)da, N = \int_{a_{\min}}^{\infty} f(e, a)da$$

$$C = \sum_{\epsilon \in \{e,u\}} \int_{a_{\min}}^{\infty} c(\epsilon, a)f(\epsilon, a)da, T = \tau(wN + rK), B = (1 - N)b$$

- 2. $c(\epsilon, a)$ and $a'(\epsilon, a)$ are optimal decision rules and solve the household decision problem.

Definition

- 3. Factor prices are equal to the factors' marginal productivities.
- 4. The goods market clears:

$$N^{1-\alpha}K^{\alpha} + (1-\delta)K = C + K' = C + K$$

- 5. The government budget is balanced: $T = B$.
- 6. The distribution of the individual state variable (ϵ, a) is stationary:

$$F\left(\epsilon', a'\right) = \sum_{\epsilon \in \{e,u\}} \pi\left(\epsilon' \mid \epsilon\right) F\left(\epsilon, a^{-1}\left(\epsilon, a'\right)\right)$$

for all $(\epsilon', a') \in \mathcal{X}$. Here, $a^{-1}\left(\epsilon, a'\right)$ denotes the inverse of the function $a'(\epsilon, a)$ with respect to its first argument a. Accordingly, the distribution over states $(\epsilon, a) \in \mathcal{X}$ is unchanging.

Definition

$$\begin{aligned}\text{Prob}\left(a_{t+1}=a', s_{t+1}=s'\right) = \sum_{a_t}\sum_{s_t} \text{Prob}\left(a_{t+1}=a' \mid a_t=a, s_t=s\right) \\ \cdot \text{Prob}\left(s_{t+1}=s' \mid s_t=s\right) \cdot \text{Prob}\left(a_t=a, s_t=s\right)\end{aligned}$$

Or

$$\lambda_{t+1}\left(a', s'\right) = \sum_{a}\sum_{s} \lambda_t(a,s)\,\text{Prob}\left(s_{t+1}=s' \mid s_t=s\right) \cdot \mathcal{I}\left(a', s, a\right)$$

where we define the indicator function $\mathcal{I}\left(a', a, s\right)=1$ if $a'=g(a,s)$, and 0 otherwise. The indicator function $\mathcal{I}\left(a', a, s\right)=1$ identifies the time t states a, s that are sent into a' at time $t+1$. The preceding equation can be expressed as

$$\lambda_{t+1}\left(a', s'\right) = \sum_{s}\sum_{\{a:a'=g(a,s)\}} \lambda_t(a,s)\mathcal{P}\left(s, s'\right)$$

Stationary Equilibrium of a Heterogeneous-Agent Economy

a) Households are allocated uniformly on the unit interval [0,1] and are of measure one. The individual household maximizes

$$V(\epsilon, a) = \max_{c,a'} \left[\frac{c^{1-\eta}}{1-\eta} + \beta E \left\{ V\left(\epsilon', a'\right) \mid \epsilon \right\} \right]$$

s.t.

$$a' = \begin{cases} (1+(1-\tau)r)a + (1-\tau)w - c & \epsilon = e \\ (1+(1-\tau)r)a + b - c & \epsilon = u \end{cases}$$

$$a \geq a_{\min}$$

$$\pi\left(\epsilon' \mid \epsilon\right) = \operatorname{Prob}\left\{\epsilon_{t+1} = \epsilon' \mid \epsilon_t = \epsilon\right\} = \begin{pmatrix} p_{uu} & p_{ue} \\ p_{eu} & p_{ee} \end{pmatrix}$$

b) Distribution of (ϵ, a) is stationary and aggregate capital K, aggregate consumption C, and aggregate employment N are constant.

Stationary Equilibrium of a Heterogeneous-Agent Economy

c) Factors prices are equal to their respective marginal products:

$$r = \alpha \left(\frac{N}{K}\right)^{1-\alpha} - \delta$$

$$w = (1-\alpha)\left(\frac{K}{N}\right)^{\alpha}$$

d) The government budget balances: $B = T$.

e) The aggregate consistency conditions hold:

$$K = \sum_{\epsilon \in \{e,u\}} \int_{a_{\min}}^{\infty} af(\epsilon, a)da, N = \int_{a_{\min}}^{\infty} f(e, a)da$$

$$C = \sum_{\epsilon \in \{e,u\}} \int_{a_{\min}}^{\infty} c(\epsilon, a)f(\epsilon, a)da, T = \tau(wN + rK), B = (1-N)b$$

Algorithm 1: Computation of the stationary equilibrium

Step 1: Compute the stationary employment N.
Step 2: Make initial guesses of the aggregate capital stock K and the tax rate τ.
Step 3: Compute the wage rate w and the interest rate r.
Step 4: Compute the household's decision functions.
Step 5: Compute the stationary distribution of assets for the employed and unemployed agents.
Step 6: Compute the capital stock K and taxes T that solve the aggregate consistency conditions.
Step 7 : Compute the tax rate τ that solves the government budget.
Step 8: Update K and τ and return to step 2 if necessary.

Algorithm 1

- Step 1, we compute the stationary employment N
  $N_t = p_{ue}(1 - N_{t-1}) + p_{ee}N_{t-1}$
- Three different kinds of methods are presented in order to compute the invariant distribution $F(\epsilon, a)$
  - Compute the distribution function on a discrete number of grid points over the assets.
  - Monte-Carlo simulations by constructing a sample of households and tracking them over time
  - A specific functional form of the distribution function will be assumed and we will use iterative methods to compute the approximation.

Algorithm 2: Computation of the Invariant Distribution Function

Step 1: Place a grid on the asset space $\mathcal{A} = \{a_1 = a_{\min}, a_2, \ldots$ $a_m = a_{\max}\}$ such that the grid is finer than the one used to compute the optimal decision rules.
Step 2: Choose an initial piecewise distribution function $F_0(\epsilon = e, a)$ and $F_0(\epsilon = u, a)$ over the grid. The vectors have m rows each.
Step 3 : Compute the inverse of the decision rule $a'(\epsilon, a)$
Step 4: Iterate on

$$F_{i+1}\left(\epsilon', a'\right) = \sum_{\epsilon=e,u} \pi\left(\epsilon', \epsilon\right) F_i\left(a'^{-1}\left(\epsilon, a'\right), \epsilon\right)$$

on grid points (ϵ', a').
Step 5: Iterate until F converges.

Algorithm 2: Computation of the Invariant Distribution Function

- In step 1, we choose an equidistant grid with $m = 600$ points on $[-2; 3,000]$ for the computation of the distribution function.
- In step 2 , we initialize the distribution function with the equal distribution so that each agent has the steady-state capital stock of the corresponding representative agent economy.
- In step 3 , we compute the inverse of the policy function $a'(\epsilon, a)$, $a = a^{-1}\left(\epsilon, a_j\right)$, over the chosen grid with $j = 1, \ldots, m$.
- In step 4, 1) If $a^{-1}\left(\epsilon, a_j\right) < a_{\min} \, F\left(\epsilon, a_j\right) = 0$, and 2) if $a'^{-1}\left(\epsilon, a_j\right) \geq a_{\max}, F\left(\epsilon, a_j\right) = g(\epsilon)$. In addition, we normalize the number of all agents equal to one and multiply $F_{i+1}\left(e, a'\right)$ and $F_{i+1}\left(u, a'\right)$ by $0.92/F_{i+1}\left(e, a_{\max}\right)$ and $0.08/F_{i+1}\left(u, a_{\max}\right)$, respectively.

Algorithm 1: Step 6

- We assume that the distribution of wealth a is uniform in any interval $\left[a_{j-1}, a_j\right]$. Thus, with the denotation $\Delta = F\left(\epsilon, a_j\right) - F\left(\epsilon, a_{j-1}\right)$, we have

$$\int_{a_{j-1}}^{a_j} af(\epsilon, a)da = \int_{a_{j-1}}^{a_j} a\frac{\Delta}{a_j - a_{j-1}}da = \frac{1}{2}\frac{a^2\Delta}{a_j - a_{j-1}}\Bigg|_{a_{j-1}}^{a_j} = \frac{1}{2}\left(F\left(\epsilon, a_j\right) - F\left(\epsilon, a_{j-1}\right)\right)\left(a_j + a_{j-1}\right)$$

- The aggregate capital can be computed as follows:

$$K = \sum_{\epsilon\in\{e,u\}} \int_{a_{\min}}^{\infty} af(\epsilon, a)da$$

$$\approx \sum_{\epsilon}\left(\sum_{j=2}^{m}\left(F\left(\epsilon, a_j\right) - F\left(\epsilon, a_{j-1}\right)\right)\frac{a_j + a_{j-1}}{2} + F\left(\epsilon, a_1\right)a_1\right)$$

Algorithm 1

- We start with an initial number of 500 iterations i over $F_i(.)$ which we increase by 500 in each iteration to 25,000 iterations in the iteration $q = 50$ over the capital stock. In the first iterations over the capital stock, we do not need a high accuracy in the computation of the invariant distribution. It saves computational time to increase the accuracy as we get closer to the solution for the aggregate capital stock.
- Not always converges! initial distribution, number of simulations. a lot of trial and error!
- Convergence of the distributions only occurs after a substantial number of iterations well in excess of several thousands. Computation of the stationary equilibrium of a heterogeneous-agent economy is extremely time-consuming

Algorithm 1

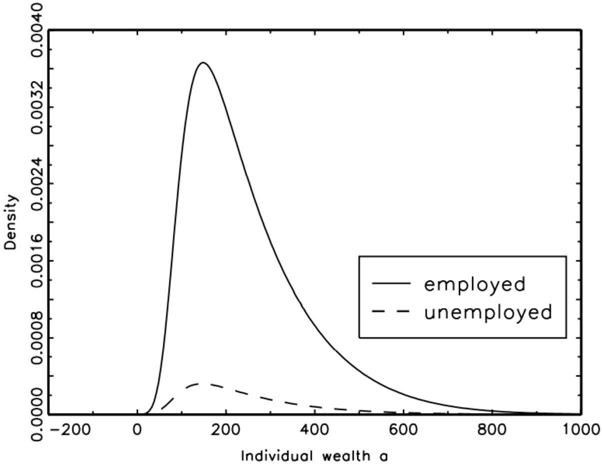


Figure 7.3: Invariant Density Function of Wealth

Algorithm 3: Computation of the Invariant Density Function

Step 1: Place a grid on the asset space $\mathcal{A}=\{a_1=a_{\min}, a_2, \ldots$ $a_m=a_{\max}\}$ such that the grid is finer than the one used to compute the optimal decision rules.
Step 2: Set $i=0$. Choose inilial discrele densily $f_0(\epsilon=e, a$ and $f_0(\epsilon=u, a)$ over that grid. The two vectors have m rows each.
Step 3: Set $f_{i+1}(\epsilon, a)=0$ for all ϵ and a. i) For every $a \in \mathcal{A}$, $\epsilon \in\{e, u\}$, compute the optimal next-period wealth $a_{j-1} \leq a'=a'(\epsilon, a)<a_j$ and ii for all $a' \in \mathcal{A}$ and $\epsilon' \in\{e, u\}$ the following sums:

$$f_{i+1}\left(\epsilon', a_{j-1}\right)=\sum_{\epsilon=e,u} \sum_{\substack{a \in \mathcal{A} \\ a_{j-1} \leq a'(\epsilon, a)}} \pi\left(\epsilon' \mid \epsilon\right) \frac{a_j-a'}{a_j-a_{j-1}} f_i(\epsilon, a)$$

$$f_{i+1}\left(\epsilon', a_j\right)=\sum_{\epsilon=e,u} \sum_{\substack{a \in \mathcal{A} \\ a_{j-1}<a'(\epsilon, a)}} \pi\left(\epsilon' \mid \epsilon\right) \frac{a'-a_{j-1}}{a_j-a_{j-1}} f_i(\epsilon, a)$$

Step 4: Iterate until f converges.

Algorithm 3: Computation of the Invariant Density Function

- If the optimal next-period capital stock happens to lie between a_{j-1} and a_j, $a_{j-1} < a' < a_j$, we simply assume that the next-period capital stock will be a_j with probability $\left(a' - a_{j-1}\right) / \left(a_j - a_{j-1}\right)$ and a_{j-1} with the complementary probability $\left(a_j - a'\right) / \left(a_j - a_{j-1}\right)$.

Algorithm 4: Computation of $F(\epsilon, a)$ by Monte-Carlo Simulation

Step 1: Choose a sample size N.
Step 2 : Initialize the sample. Each household $i = 1, \ldots, N$ is assigned an initial wealth level a_0^i and employment status ϵ_0^i
Step 3: Compute the next-period wealth level $a'\left(\epsilon^i, a^i\right)$ for all $i = 1, \ldots, N$
Step 4: Use a random number generator to obtain $\epsilon^{i'}$ for all $i = 1, \ldots, N$
Step 5: Compute a set of statistics from this sample. We choose the mean and the standard deviation of a and ϵ.
Step 6 : Iterate until the distributional statistics converge.

Algorithm 5: Function Approximation

- We approximate the distribution function by a flexible functional form with a finite number of coefficients.
- The class of exponential functions for the n th order approximation of the wealth holdings

$$\begin{aligned} F(\epsilon, a) &= 0 && a < a_{\min} \\ F(\epsilon, a) &= \rho_0^\epsilon \int_{-\infty}^{a} e^{\rho_1^\epsilon x^1 + \ldots + \rho_n^\epsilon x^n} dx && a \geq a_{\min} \end{aligned}$$

- $\rho^\epsilon = \left(\rho_0^\epsilon, \rho_1^\epsilon, \rho_2^\epsilon\right), \epsilon \in \{e, u\}$ that correspond to μ^ϵ and $(\sigma^\epsilon)^2$, we have to solve the following set of non-linear equations:

$$g(\epsilon) = \rho_0^\epsilon \int_{-\infty}^{a_{\max}} e^{\rho_1^\epsilon a + \rho_2^\epsilon a^2} da$$

$$\mu^\epsilon = \rho_0^\epsilon \int_{-\infty}^{a_{\max}} \max\left(a, a_{\min}\right) e^{\rho_1^\epsilon a + \rho_2^\epsilon a^2} da$$

$$(\sigma^\epsilon)^2 = \rho_0^\epsilon \int_{-\infty}^{a_{\max}} \left(\max\left(a, a_{\min}\right) - \mu^\epsilon\right)^2 e^{\rho_1^\epsilon a + \rho_2^\epsilon a^2} da$$

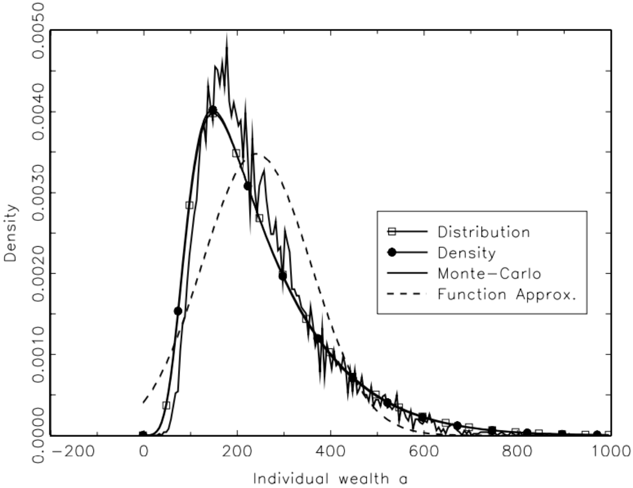
Algorithm 5: Function Approximation

Step 1: Choose initial moments μ^ϵ and $(\sigma^\epsilon)^2$, compute ρ^ϵ of the exponential distribution by solving the non-linear equation problem
Step 2: Compute the moments of the next-period

$$\begin{aligned}
\mu^{e'} &= \pi(e \mid e)\rho_0^e \int_{-\infty}^{a_{\max}} \max\left(a'(a,e), a_{\min}\right) e^{\rho_1^e a + \rho_2^e a^2} da \\
&+ \pi(e \mid u)\rho_0^u \int_{-\infty}^{a_{\max}} \max\left(a'(a,u), a_{\min}\right) e^{\rho_1^u a + \rho_2^u a^2} da \\
\left(\sigma^{e'}\right)^2 &= \pi(e \mid e)\rho_0^e \int_{-\infty}^{a_{\max}} \left(\max\left(a'(a,e), a_{\min}\right) - \mu^e\right)^2 e^{\rho_1^e a + \rho_2^e a^2} da \\
&+ \pi(e \mid u)\rho_0^u \int_{-\infty}^{a_{\max}} \left(\max\left(a'(a,u), a_{\min}\right) - \mu^u\right)^2 e^{\rho_1^u a + \rho_2^u a^2} da
\end{aligned}$$

and compute the parameters $\rho^\epsilon, \epsilon \in \{e, u\}$, corresponding to the computed next-period moments μ' and σ'^2.
Step 3: Iterate until the moments μ^ϵ and σ^ϵ converge.

Algorithm Comparison



Algorithm Comparison

Table 7.1

	Invariant Distribution	Invariant Density	Monte Carlo	Exponential Function n=2
Mean	243.7	243.7	243.4	246.6
Runtime	5:45	4:05	15:14	3:52
Iterations	51	51	54	63

Notes: Run time is given in hours:minutes on an Intel Pentium(R) M, 319 MHz computer. Iterations are over the aggregate capital stock K.

Applications 1: The Risk-Free Rate in Economies with Heterogeneous Agents and Incomplete Insurance

- Two different phenomena have been observed in financial markets during the last hundred years:
  - 1) the low risk-free rate, During the last 100 years, the average real return on US Treasury Bills has been about one percent.
  - 2) the large equity premium. The average real return on US stocks has been six percent higher.

Applications 1: The Risk-Free Rate in Economies with Heterogeneous Agents and Incomplete Insurance

- MEHRA and PRESCOTT(1985) show that the representative agent model can only explain the large equity premium and the low risk-free rate if the typical investor is implausibly risk averse.
- KOCHERLAKOTA(1996) argues that one of the three assumptions of the representative-agent model needs to be abandoned in order to explain the two puzzles:
  - 1) the standard utility function,
  - 2) complete markets, and
  - 3) costless trading.

Applications 1: The Risk-Free Rate in Economies with Heterogeneous Agents and Incomplete Insurance

- The Exchange Economy. HUGGETT (1993) considers a simple exchange economy without production.
- The endowment set $\mathcal{E} - \{e_h, e_l\}$. The endowment (or employment) process follows a first-order Markov process with transition probability $\pi(e' \mid e) = \text{Prob}(e_{t+1} = e' \mid e_t = e) > 0$ for $e', e \in \mathcal{E}$.
- The agent maximizes expected discounted utility:

$$E_0\left[\sum_{t=0}^{\infty} \beta^t u(c_t)\right], u(c) = \frac{c^{1-\eta}}{1-\eta}$$

- Agents may hold a single asset. The budget constraint of the household is $c + a'q = a + e, \quad$ where $a' \geq \bar{a}$

$$v(e, a; q) = \max_{c, a'} u(c) + \beta \sum_{e'} \pi(e' \mid e) v(e', a'; q')$$

Applications 1: The Risk-Free Rate in Economies with Heterogeneous Agents and Incomplete Insurance

- Our definition of the stationary equilibrium allows for a positive mass of households at the borrowing constraint $\bar{a}$, $F(e, \bar{a}) \geq 0$. For $a \geq \bar{a}$, the distribution function $F(e, a)$ is associated with a density function $f(e, a)$.
- In a stationary equilibrium, furthermore, markets clear so that the average credit balance is equal to zero.

Applications 1: Definition

- A stationary equilibrium for the exchange economy is a vector $(c(e,a), a'(e,a), q, F(e,a))$ satisfying:
  1. $c(e,a)$ and $a'(e,a)$ are optimal decision rules given q.
  2. Markets clear:

$$\sum_e \left(\int_{\bar{a}}^{\infty} c(e,a)f(e,a)da + c(e,\bar{a})F(e,\bar{a})\right) \\ = \sum_e \left(\int_{\bar{a}}^{\infty} ef(e,a)da + eF(e,\bar{a})\right) \\ \sum_e \left(\int_{\bar{a}}^{\infty} a'(e,a)f(e,a)da + a'(e,\bar{a})F(e,\bar{a})\right) = 0$$

  3. $F(e,a)$ is a stationary distribution:

$$F\left(e', a'\right) = \pi\left(e' \mid e_h\right) F\left(e_h, a_h\right) + \pi\left(e' \mid e_l\right) F\left(e_l, a_l\right)$$

  for all $a' \in \mathcal{A}$ and $e' \in \{e_l, e_h\}$ and with $a' = a'\left(e_h, a_h\right) = a'\left(e_l, a_l\right)$

Applications 1: The Risk-Free Rate in Economies with Heterogeneous Agents and Incomplete Insurance

- Analogous to the Algorithm 1. First, we make an initial guess of the interest rate r and compute the policy functions. Second, we compute the stationary equilibrium and the equilibrium average asset holdings. Finally, we update the interest rate and return to the first step, if necessary.

Applications 1: The Risk-Free Rate in Economies with Heterogeneous Agents and Incomplete Insurance

Table 7.2

Credit limit $\bar{a}$	Interest rate r	price q
-2	-1.27%	1.0129
-4	0.196%	0.9983
-6	0.507%	0.9949
-8	0.627%	0.9938

- In conclusion, we find that incomplete insurance (against the risk of a negative endowment shock) and credit constraints help to explain that the empirically observed risk-free rate of return is lower than the one found in standard representative-agent models.

Applications 2: Heterogeneous Productivity and Income Distribution

This section is organized as follows.

- First, empirical facts from the US and the German economy with regard to the distribution of wealth and income are reviewed.
- Second, we discuss the standard way of introducing income heterogeneity into heterogeneous agent models.
- Finally, we present a model with income heterogeneity and compute the endogenous invariant wealth distribution.
- We also analyze the steady-state effects of a fiscal policy reform that consists of a switch from a flat-rate income tax to a consumption tax.

Applications 2: Empirical Facts on the Income and Wealth Distribution

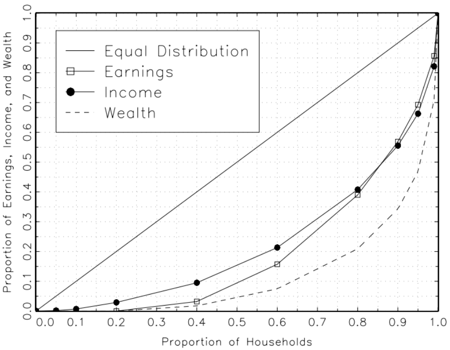


Figure 7.9: Lorenz Curve of US Wealth, Income, and Earnings in 1992

- Gini coefficients of income, earnings, and wealth equal to 0.57, 0.63, and 0.78, respectively.

Applications 2: Heterogeneous Productivity and Income Distribution

- One crucial aspect for the analysis of redistributive effects of economic policy is the consideration of mobility.
- Redistribution comes at the expense of efficiency.
- If we also consider income mobility, the welfare effect of such a policy is reduced further. The reason is simple: income poor agents may move up the income hierarchy and will also be harmed by higher taxes and a reduction in the efficiency of the economy in the future.
- Therefore, if we consider the redistributive effects of an economic policy in a heterogeneous-agent model, mobility is a crucial ingredient.

Applications 2: The US earnings mobility

$$\text{1984 Quintile}\quad \overset{\text{1989 Quintile}}{\begin{bmatrix} 0.858 & 0.116 & 0.014 & 0.006 & 0.005 \\ 0.186 & 0.409 & 0.300 & 0.071 & 0.034 \\ 0.071 & 0.120 & 0.470 & 0.262 & 0.076 \\ 0.075 & 0.068 & 0.175 & 0.465 & 0.217 \\ 0.058 & 0.041 & 0.055 & 0.183 & 0.663 \end{bmatrix}}$$

Applications 2: Modelling income heterogeneity

- The individual's earnings y_t^i are stochastic or that labor productivity ϵ_t^i is stochastic.
  - In the first case, labor income is an exogenous variable
  - In the latter case, agents may still be able to vary their labor supply so that labor income $y_t^i = \epsilon_t^i w_t n_t^i$, which is the product of individual productivity ϵ, wage w_t, and labor time n_t^i is endogenous.
- Some computable general equilibrium models with income heterogeneity and exogenous labor supply have used a regression to the mean process for log-labor earnings. Individual earnings y_t follow the process: $\ln y_t - \overline{\ln y} = \rho \left(\ln y_{t-1} - \overline{\ln y}\right) + \eta_t$ where $\eta_t \sim N\left(0, \sigma_\eta^2\right)$.
- Having specified the log earnings process as an AR(1)-process, we need to discretize the process for computational purpose. The earnings process can easily be approximated with a finite-state Markov chain.

Applications 2: Modelling income heterogeneity

- Assume productivity ϵ to follow a first-order Markov chain with conditional transition probabilities given by:

$$\pi\left(\epsilon' \mid \epsilon\right) = \text{Prob}\left\{\epsilon_{t+1} = \epsilon' \mid \epsilon_t = \epsilon\right\}$$

where $\epsilon, \epsilon' \in \mathcal{E} = \left\{\epsilon^1, \ldots, \epsilon^{n\epsilon}\right\}$. The productivities $\epsilon \in \mathcal{E} = \left\{\epsilon^1, \ldots, \epsilon^{n\epsilon}\right\}$ are chosen to replicate the discretized distribution of hourly wage rates which, in our model, are proportional to productivity.
$\left\{\epsilon^2, \epsilon^3, \epsilon^4, \epsilon^5\right\} = \{0.4476, 0.7851, 1.0544, 1.7129\}$

Applications 2: Modeling the Distributional Effects of Income Tax Reforms

- Households are heterogeneous with regard to their employment status, their productivity ϵ^j, and their wealth $k^j, \vec{j} \in [0,1]$
  Individual productivity $\epsilon^j \in \mathcal{E} = \{0, 0.4476, 0.7851, 1.0544, 1.7129\}$
- Budget constraint

$$k_{t+1}^j = (1+r)k_t^j + w_t n_t^j \epsilon_t^j - (1+\tau_c)\, c_t^j - \tau_y y_t^j + 1_{\epsilon=\epsilon^1} b_t$$

- Household j, which is characterized by productivity ϵ_t^j and wealth k_t^j in period t, maximizes his intertemporal utility with regard to consumption c_t^j and labor supply n_t^j

$$E_0 \sum_{t=0}^{\infty} \beta^t u\left(c_t^j, 1-n_t^j\right), u\left(c_t, 1-n_t\right) = \frac{c_t^{1-\eta}}{1-\eta} + \gamma_0 \frac{\left(1-n_t\right)^{1-\gamma_1}}{1-\gamma_1}$$

Applications 2: Production and Government

- Firms are owned by the households and maximize profits with respect to their labor and capital demand.

$$Y_t = N_t^{1-\alpha} K_t^{\alpha}$$

Factors are compensated according to their marginal products and profits are zero:

$$r_t = \alpha \left(\frac{N_t}{K_t} \right)^{1-\alpha} - \delta$$

$$w_t = (1-\alpha) \left(\frac{K_t}{N_t} \right)^{\alpha}$$

- We will compare the employment and distribution effects of two tax systems with equal tax revenues: (i) a flat-rate income tax structure and (ii) only a consumption tax $(\tau_y = 0)$
The government budget is balanced in every period

$$G_t + B_t = T_t$$

Applications 2: Definition

- A stationary equilibrium for a given set of government policy parameters is a value function $V(\epsilon, k)$, individual policy rules $c(\epsilon, k), n(\epsilon, k)$, and $k'(\epsilon, k)$, a time-invariant distribution $F(\epsilon, k)$ of the state variable $(\epsilon, k) \in \mathcal{E} \times [0, \infty)$, time-invariant relative prices of labor and capital $\{w, r\}$, and a vector of aggregates K, N, B, T, and C such that:
  1. Factor inputs, consumption, tax revenues, and unemployment compensation are obtained aggregating over households:

$$K = \sum_{c \in \mathcal{E}} \int_0^\infty kf(\epsilon, k) dk, N = \sum_{c \in \mathcal{E}} \int_0^\infty \epsilon n(\epsilon, k) f(\epsilon, k) dk$$

$$C = \sum_{c \in \mathcal{S}} \int_0^\infty c(\epsilon, k) f(\epsilon, k) dk, T = \tau_y \left(K^\alpha N^{1-\alpha} - \delta K \right) + \tau_c C$$

$$B = \int_0^\infty bf\left(\epsilon_1, k\right) dk$$

Applications 2: Definition

2. $c(\epsilon, k), n(\epsilon, k)$, and $k'(\epsilon, k)$ are optimal decision rules and solve the household decision problem

$$V(\epsilon, k) = \max_{c,n,k'} \left[u(c, 1-n) + \beta E \left\{ V\left(\epsilon', k'\right) \mid \epsilon \right\} \right]$$

subject to the budget constraint, the tax policy, and the stochastic mechanism

3. Factor prices are equal to the factors' marginal productivities.

4. The goods market clears:

$$F(K, L) + (1-\delta)K = C + K' + G = C + K + G$$

5. The government budget is balanced: $G + B = T$.

6. The distribution of the individual state variables is constant:

$$F\left(\epsilon', k'\right) = \sum_{e \in \mathcal{E}} \pi\left(\epsilon' \mid \epsilon\right) F(\epsilon, k)$$

for all $k' \in [0, \infty)$ and $\epsilon' \in \mathcal{E}$ and with $k' = k'(\epsilon, k)$.

Applications 2: Computation

The solution algorithm for the benchmark case with a flat-rate income tax is described by the following steps:
1. Make initial guesses of the aggregate capital stock K, aggregate employment N, the consumption tax τ_c, and the value function $V(\epsilon, k)$
2. Compute the wage rate w, the interest rate r, and unemployment compensation b.
3. Compute the household's decision functions $k'(\epsilon, k), c(\epsilon, k)$, and $n(\epsilon, k)$
4. Compute the steady-state distribution of assets.
5. Compute K, N, and taxes T that solve the aggregate consistency conditions.
6. Compute the consumption tax τ_c that solves the government budget.
7. Update K, N, and τ_c, and return to step 2 if necessary.

Applications 2: Results

Table 7.3

Tax Policy	K	N	$\bar{n}$	r	Gini $w\epsilon n$	Gini k	$\sigma_n/\bar{n}$	$\sigma_{\epsilon n}/N$
τ_y	2.70	0.251	0.324	3.88%	0.317	0.406	0.367	0.691
τ_c	3.24	0.249	0.323	3.01%	0.316	0.410	0.366	0.685

Notes: τ_y refers to the case of a flat-rate income tax and τ_c to the case where the income tax rate is zero and the consumption tax rate τ_c is increased such that the government budget balances.

Thank You!